On Graphs Satisfying a Local Ore-Type Condition

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Abstract

For an integer i, a graph is called an L_i-graph if, for each triple of vertices u, v, w with $d(u,v) = 2$ and $w \in N(u) \cap N(v)$, $d(u) + d(v) \geq |N(u) \cup N(v) \cup N(w)| - i$. Asratian and Khachatrian proved that connected L_0-graphs of order at least 3 are hamiltonian, thus improving Ore's Theorem. All $K_{1,3}$-free graphs are L_1-graphs, whence recognizing hamiltonian L_1-graphs is an NP-complete problem. The following results about L_1-graphs, unifying known results of Ore-type and known results on $K_{1,3}$-free graphs, are obtained. Set $\mathcal{K} = \{\, G \mid K_{p,p+1} \subseteq G \subseteq K_p \vee \overline{K_{p+1}}$ for some $p \geq 2 \,\}$ ($\vee$ denotes join). If G is a 2-connected L_1-graph, then G is 1-tough unless $G \in \mathcal{K}$. Furthermore, if G is a connected L_1-graph of order at least 3 such that $|N(u) \cap N(v)| \geq 2$ for every pair of vertices u, v with $d(u,v) = 2$, then G is hamiltonian unless $G \in \mathcal{K}$, and every pair of vertices x, y with $d(x,y) \geq 3$ is connected by a Hamilton path. This result implies that of Asratian and Khachatrian. Finally, if G is a connected L_1-graph of even order, then G has a perfect matching.

Keywords: hamiltonian graph, Hamilton path, local condition, Ore-type condition, $K_{1,3}$-free graph, 1-tough graph, perfect matching

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1 Introduction

We use Bondy & Murty [6] for terminology and notation not defined here and consider finite simple graphs only.

A classical result on hamiltonian graphs is the following.

Theorem 1 (Ore [11])
If G is a graph of order $n \geq 3$ such that $d(u) + d(v) \geq n$ for each pair of nonadjacent vertices u, v, then G is hamiltonian.

In Asratian[1] & Khachatrian [7], Theorem 1 was improved to a result of local nature, Theorem 2 below. For an integer i, we call a graph an *L_i-graph* (L for *l*ocal) if, for each triple of vertices u, v, w with $d(u, v) = 2$ and $w \in N(u) \cap N(v)$,

$$d(u) + d(v) \;\geq\; |N(u) \cup N(v) \cup N(w)| - i,$$

or, equivalently (see [7]),

$$|N(u) \cap N(v)| \;\geq\; |N(w) \setminus (N(u) \cup N(v))| - i.$$

Theorem 2 ([7])
If G is a connected L_0-graph of order at least 3, then G is hamiltonian.

Clearly, Theorem 2 implies Theorem 1.

Almost all of the many existing generalizations of Theorem 1 only apply to graphs G with large edge density ($|E(G)| \geq \text{constant} \cdot |V(G)|^2$) and small diameter ($o(|V(G)|)$). An attractive feature of Theorem 2 is that it applies to infinite classes of graphs G with small edge density ($\Delta(G) \leq$ constant) and large diameter ($\geq \text{constant} \cdot |V(G)|$) as well. One such class is provided in [7]. For future reference also, we here present a similar class. For positive integers p, q, define the graph $G_{p,q}$ of order $p\,q$ as follows: its vertex set is $\bigcup_{i=1}^{q} V_i$, where $V_1, \ldots, V_q$ are pairwise disjoint sets of cardinality p; two vertices of $G_{p,q}$ are adjacent if and only if they both belong to $V_i \cup V_{i+1}$ for some $i \in \{1, \ldots, q-1\}$, or to $V_1 \cup V_q$. Considering a fixed integer $p \geq 2$, we observe that $G_{p,q}$, being an L_{2-p}-graph, is hamiltonian by Theorem 2 unless $p = 2$ and $q = 1$; furthermore, $G_{p,q}$ has maximum degree $3\,p-1$ for $q \geq 3$, and diameter $\lfloor \frac{q}{2} \rfloor = \lfloor \frac{1}{2p}\,|V(G_{p,q})| \rfloor$ for $q \geq 2$.

We define the family $\mathcal{K}$ of graphs by

$$\mathcal{K} \;=\; \{\, G \mid K_{p,p+1} \subseteq G \subseteq K_p \vee \overline{K_{p+1}} \text{ for some } p \geq 2 \,\},$$

where $\vee$ is the join operation. The class of extremal graphs for Theorem 1, i.e., nonhamiltonian graphs G such that $d(u) + d(v) \geq |V(G)| - 1 \geq 2$ for each pair of nonadjacent vertices u, v, is $\mathcal{K} \cup \{\, K_1 \vee (K_r + K_s) \mid r, s \geq 1 \,\}$ (see, e.g. Skupień [13]). We point out here that the class of extremal graphs for Theorem 2, i.e., nonhamiltonian L_1-graphs of order at least 3, is far less restricted. If G and H are graphs, then G is called *H-free* if G has no induced subgraph isomorphic to H. The following observation was first made in Asratian & Khachatrian [2].

[1]In [7] the last name of the first author was transcribed as Hasratian.

Proposition 3 ([2])
Every $K_{1,3}$-free graph is an L_1-graph.

Proof Let u, v, w be vertices of a $K_{1,3}$-free graph G such that $d(u,v) = 2$ and $w \in N(u) \cap N(v)$. Then $|N(w) \setminus (N(u) \cup N(v))| \leq 2$ and $|N(u) \cap N(v)| \geq 1$, implying that G is an L_1-graph. ■

In Bertossi [4] it was shown that recognizing hamiltonian line graphs, and hence recognizing hamiltonian $K_{1,3}$-free graphs is an NP-complete problem. Hence the same is true for recognizing hamiltonian L_1-graphs, and there is little hope for a polynomial characterization of the extremal graphs for Theorem 2.

The study of L_1-graphs in subsequent sections was motivated by the interesting fact that the class of L_1-graphs contains all $K_{1,3}$-free graphs as well as all graphs satisfying the hypothesis of Theorem 1 (even with n replaced by $n-1$). The nature of the investigated properties of L_1-graphs is reflected by the titles of Sections 2, 3 and 4. The proofs of the obtained results are postponed to Section 5.

2 Toughness of L_1-graphs

Let $\omega(G)$ denote the number of components of a graph G. A graph G is *t-tough* if $|S| \geq t \cdot \omega(G-S)$ for every subset S of $V(G)$ with $\omega(G-S) > 1$. Clearly, every hamiltonian graph is 1-tough. Hence the following result implies Theorem 1 (for $n \geq 11$).

Theorem 4 (Jung [8])
If G is a 1-tough graph of order $n \geq 11$ such that $d(u) + d(v) \geq n - 4$ for each pair of nonadjacent vertices u, v, then G is hamiltonian.

By analogy, one might expect that Theorem 2 could be strengthened to the assertion that 1-tough L_4-graphs of sufficiently large order are hamiltonian. However, our first result shows that the problem of recognizing hamiltonian graphs remains NP-complete even within the class of 1-tough L_1-graphs. (Recall that the problem is NP-complete for L_1-graphs, and hence for 2-connected L_1-graphs.)

Theorem 5
If G is a 2-connected L_1-graph, then either G is 1-tough or $G \in \mathcal{K}$.

By Proposition 3, Theorem 5 extends the case $k = 2$ of the following result.

Theorem 6 (Matthews & Sumner [10])
Every k-connected $K_{1,3}$-free graph is $\frac{k}{2}$-tough.

In view of Theorem 6 we note that there exist 1-tough L_1-graphs of arbitrary connectivity that are not $(1+\varepsilon)$-tough for any $\varepsilon > 0$. For example, consider the graphs $K_{p,p}$ and $K_p \vee \overline{K_p}$, and the graphs obtained from $K_{p,p}$ and $K_p \vee \overline{K_p}$ by deleting a perfect matching ($p \geq 3$).

3 Hamiltonian properties of L_1-graphs

If u, v, w are vertices of an L_0-graph such that $d(u, v) = 2$ and $w \in N(u) \cap N(v)$, then $N(w) \setminus (N(u) \cup N(v)) \supseteq \{u, v\}$, and hence $|N(u) \cap N(v)| \geq |N(w) \setminus (N(u) \cup N(v))| \geq 2$. Thus our next result implies Theorem 2.

Theorem 7
Let G be a connected L_1-graph of order at least 3 such that $|N(u) \cap N(v)| \geq 2$ for every pair of vertices u, v with $d(u, v) = 2$. Then each of the following holds.

(a) Either G is hamiltonian or $G \in \mathcal{K}$.

(b) Every pair of vertices x, y with $d(x, y) \geq 3$ is connected by a Hamilton path of G.

An immediate consequence of Theorem 7 (a) is the following.

Corollary 8 (Asratian, Ambartsumian & Sarkisian [1])
Let G be a connected L_1-graph such that $|N(u) \cap N(v)| \geq 2$ for every pair of vertices u, v with $d(u, v) = 2$. Then G contains a Hamilton path.

The lower bound 3 on $d(x, y)$ in Theorem 7 (b) cannot be relaxed. For example, consider for $p \geq 2$ the graphs $K_{p,p}$ and $K_p \vee \overline{K_p}$, and for $p \geq 4$ the graphs obtained from $K_{p,p}$ and $K_p \vee \overline{K_p}$ by deleting a perfect matching. Each of these graphs satisfies the hypothesis of Theorem 7, but contains pairs of vertices at distance 1 or 2 that are not connected by a Hamilton path.

By Proposition 3, Theorem 7 (a) has the following consequence also.

Corollary 9 (see, e.g., Shi Ronghua [12])
Let G be a connected $K_{1,3}$-free graph of order at least 3 such that $|N(u) \cap N(v)| \geq 2$ for every pair of vertices u, v with $d(u, v) = 2$. Then G is hamiltonian.

An example of a graph that is hamiltonian by Theorem 7, but not by Theorem 2 or Corollary 9, is the graph obtained from $G_{3,q}$ ($q \geq 3$) by deleting the edges of a cycle of length q, containing exactly one vertex of V_i for $i = 1, \ldots, q$.

Although Theorem 7 implies Theorem 2, in Section 5 we also present a direct proof of Theorem 2 as a simpler alternative for the algorithmic proof in Asratian & Khachatrian [7].

4 Perfect matchings of L_1-graphs

Our last result is the following.

Theorem 10
If G is a connected L_1-graph of even order, then G has a perfect matching.

The graph $K_{p,p+2}$ ($p \geq 1$) is a connected L_2-graph of even order without a perfect matching. Thus Theorem 10 is, in a sense, best possible.

Corollary 11 (Las Vergnas [9], Sumner [14])
If G is a connected $K_{1,3}$-free graph of even order, then G has a perfect matching.

Corollary 12 (see, e.g., Bondy & Chvátal [5])
If G is a graph of even order $n \geq 2$ such that $d(u) + d(v) \geq n - 1$ for each pair of nonadjacent vertices u, v, then G has a perfect matching.

5 Proofs

We successively present proofs of Theorems 5, 7, 2 and 10, but first introduce some additional notation.

Let G be a graph. For $S \subseteq V(G)$, $N_G(S)$, or just $N(S)$ if no confusion can arise, denotes the set of all vertices adjacent to at least one vertex of S. For $v \in V(G)$, we write $N_G(v)$ instead of $N_G(\{v\})$.

Let C be a cycle of G. We denote by $\overrightarrow{C}$ the cycle C with a given orientation, and by $\overleftarrow{C}$ the cycle C with the reverse orientation. If $u, v \in V(C)$, then $u\overrightarrow{C}v$ denotes the consecutive vertices of C from u to v in the direction specified by $\overrightarrow{C}$. The same vertices, in reverse order, are given by $v\overleftarrow{C}u$. We use u^+ to denote the successor of u on $\overrightarrow{C}$ and u^- to denote its predecessor.

Analogous notation is used with respect to paths instead of cycles.

In the proofs of Theorems 5 and 7 we will frequently use the following key lemma.

Lemma 13
Let G be an L_1-graph, v a vertex of G and $W = \{ w_1, \ldots, w_k \}$ a subset of $N(v)$ of cardinality k. Assume G contains an independent set $U = \{ u_1, \ldots, u_k \}$ of cardinality k such that $U \cap (N(v) \cup \{v\}) = \emptyset$ and, for $i = 1, \ldots, k$, $u_i w_i \in E(G)$ and $N(u_i) \cap (N(v) \setminus W) = \emptyset$. Then $N(w_i) \setminus (N(v) \cup \{v\}) \subseteq N(u_i) \cup U$ $(i = 1, \ldots, k)$.

Proof Under the hypothesis of the lemma, we have

$$(1) \qquad N(u_i) \cap N(v) \; = \; N(u_i) \cap W \qquad (i = 1, \ldots, k),$$

and since U is an independent set,

$$(2) \qquad N(w_i) \setminus (N(u_i) \cup N(v)) \; \supseteq \; (N(w_i) \cap U) \cup \{v\} \qquad (i = 1, \ldots, k).$$

Since G is an L_1-graph, it follows that

$$(3) \qquad \begin{aligned} 0 \; &\leq \; \sum_{i=1}^{k} (|N(u_i) \cap N(v)| - |N(w_i) \setminus (N(u_i) \cup N(v))| + 1) \\ &= \; \sum_{i=1}^{k} |N(u_i) \cap N(v)| - \sum_{i=1}^{k} (|N(w_i) \setminus (N(u_i) \cup N(v))| - 1) \\ &\leq \; \sum_{i=1}^{k} |N(u_i) \cap W| - \sum_{i=1}^{k} |N(w_i) \cap U| \; = \; 0. \end{aligned}$$

(Note that both $\sum_{i=1}^{k} |N(u_i) \cap W|$ and $\sum_{i=1}^{k} |N(w_i) \cap U|$ represent the number of edges with one end in U and the other in W.) We conclude that equality holds throughout (2) and (3). In particular, (2) holds with equality, implying that

$$N(w_i) \setminus (N(u_i) \cup N(v) \cup \{v\}) \;=\; N(w_i) \cap U \;\subseteq\; U,$$

and hence

$$N(w_i) \setminus (N(v) \cup \{v\}) \;\subseteq\; N(u_i) \cup U \qquad (\, i = 1, \ldots, k\,). \qquad \blacksquare$$

Proof of Theorem 5 Let G be a 2-connected L_1-graph and assume G is not 1-tough. Let X be a subset of $V(G)$ of minimum cardinality for which $\omega(G - X) > |X|$. Since G is 2-connected, $|X| \geq 2$. Set $l = |X|$ and $m = \omega(G - X) - 1$, so that $m \geq l \geq 2$. Let $H_0, H_1, \ldots, H_m$ be the components of $G - X$.

In order to prove that $G \in \mathcal{K}$, we first show that

(4) for every nonempty proper subset S of X, $|\{\, i \mid N(S) \cap V(H_i) \neq \emptyset \,\}| \geq |S| + 2$.

Suppose $S \subseteq X$, $\emptyset \neq S \neq X$ and $|\{\, i \mid N(S) \cap V(H_i) \neq \emptyset \,\}| \leq |S| + 1$. Set $T = X \setminus S$. Then $\omega(G - T) \geq m + 1 - |S| \geq l + 1 - |S| = |T| + 1$. This contradiction with the choice of X proves (4).

We next show that

(5) if $v \notin X$ and $N(v) \cap X \neq \emptyset$, then $N(v) \supseteq X$.

Suppose $v \notin X$ and $N(v) \cap X \neq \emptyset$, but $N(v) \not\supseteq X$. Set $W = N(v) \cap X$ and $k = |W|$. Then $1 \leq k < l$. Let $w_1, \ldots, w_k$ be the vertices of W. By (4) and Hall's Theorem (see Bondy & Murty [6, page 72]), $N(W) \setminus X$ contains a subset $U = \{\, u_1, \ldots, u_k \,\}$ of cardinality k such that no two vertices of $U \cup \{v\}$ are in the same component of $G - X$ and $u_1 w_1, \ldots, u_k w_k \in E(G)$. By Lemma 13, we have $N(w_i) \setminus (N(v) \cup \{v\}) \subseteq N(u_i) \cup U$ $(\, i = 1, \ldots, k\,)$. But then $|\{\, i \mid N(W) \cap V(H_i) \neq \emptyset \,\}| \leq k + 1 = |W| + 1$. This contradiction with (4) proves (5).

Let x be a vertex in X and y_i a vertex of H_i with $N(y_i) \cap X \neq \emptyset$ $(\, i = 0, 1, \ldots, m\,)$. Set $Y = \{\, y_0, y_1, \ldots, y_m \,\}$. By (5), $N(y_i) \supseteq X$ for all i, implying that $N(x) \supseteq Y$. Since G is an L_1-graph, we obtain

$$\begin{aligned} 0 \;&\leq\; |N(y_i) \cap N(y_j)| - |N(x) \setminus (N(y_i) \cup N(y_j))| + 1 \\ &=\; |X| - |N(x) \setminus (N(y_i) \cup N(y_j))| + 1 \\ &\leq\; |X| - |Y| + 1 \;=\; l - m \;\leq\; 0 \qquad (\, i \neq j\,). \end{aligned} \tag{6}$$

Thus equality holds throughout (6). Hence $m = l$ and $N(x) \setminus (N(y_i) \cup N(y_j)) = Y$ whenever $i \neq j$. Consider a vertex y_h in Y. We have $|X| \geq 2$ and hence $|Y| \geq 3$, so there exist distinct vertices y_i, y_j with $y_h \neq y_i, y_j$. Since $N(x) \setminus (N(y_i) \cup N(y_j)) = Y$, we obtain $N(x) \cap V(H_h) = \{y_h\}$. Since G is 2-connected, it follows that $V(H_i) = \{y_i\}$ for all i, whence $G \in \mathcal{K}$. $\blacksquare$

Proof of Theorem 7 Let G satisfy the hypothesis of the theorem. Since $|N(u) \cap N(v)| \geq 2$ whenever $d(u, v) = 2$,

(7) G is 2-connected.

(a) Assuming G is nonhamiltonian, let $\overrightarrow{C}$ be a longest cycle of G and v a vertex in $V(G) \setminus V(C)$ with $N(v) \cap V(C) \neq \emptyset$. Set $W = N(v) \cap V(C)$ and $k = |W|$. Let $w_1, \ldots, w_k$ be the vertices of W, occurring on $\overrightarrow{C}$ in the order of their indices. Set $u_i = w_i^+$ $(i = 1, \ldots, k)$ and $U = \{u_1, \ldots, u_k\}$.

The choice of C implies that $U \cap (N(v) \cup \{v\}) = \emptyset$, U is an independent set, and

(8) $\quad N(u_i) \cap (N(v) \setminus W) \;=\; N(u_i) \cap N(v) \cap (V(G) \setminus V(C)) \;=\; \emptyset \qquad (i = 1, \ldots, k).$

Hence by Lemma 13,

(9) $\quad N(w_i) \setminus (N(v) \cup \{v\}) \;\subseteq\; N(u_i) \cup U \qquad (i = 1, \ldots, k).$

Noting that $k \geq 2$ by (8) and the fact that $|N(u_1) \cap N(v)| \geq 2$, we now prove by contradiction that

(10) $\quad u_i \;=\; w_{i+1}^- \qquad (i = 1, \ldots, k;$ indices mod $k)$.

Assume without loss of generality that $u_1 \neq w_2^-$, whence $w_2^- \notin U$. Then by (9), $w_2^- \in N(u_2)$. Since C is a longest cycle, $w_2^- w_3^- \notin E(G)$. Hence $u_2 \neq w_3^-$. Repetition of this argument shows that $u_i \neq w_{i+1}^-$ and $u_i w_i^- \in E(G)$ for all $i \in \{1, \ldots, k\}$. By assumption, $N(u_1) \cap N(v)$ contains a vertex $x \neq w_1$. By (8), $x \in V(C)$, say that $x = w_i$. But then the cycle $w_1 v w_i u_1 \overrightarrow{C} w_i^- u_i \overrightarrow{C} w_1$ is longer than C. This contradiction proves (10).

Since C is a longest cycle, there exists no path joining two vertices of $U \cup \{v\}$ with all internal vertices in $V(G) \setminus V(C)$. Hence by (10), $\omega(G - W) > |W|$. By (7) and Theorem 5, it follows that $G \in \mathcal{K}$.

(b) Let x and y be vertices of G with $d(x, y) \geq 3$ and let $\overrightarrow{P}$ be a longest (x, y)-path. Assuming P is not a Hamilton path, let v be a vertex in $V(G) \setminus V(P)$ with $N(v) \cap V(P) \neq \emptyset$. Set $W = N(v) \cap V(P)$ and $k = |W|$. As in the proof of (a), we have $k \geq 2$. Let $w_1, \ldots, w_k$ be the vertices of W, occurring on $\overrightarrow{P}$ in the order of their indices. Since $d(x, y) \geq 3$, $w_1 \neq x$ or $w_k \neq y$. Assume without loss of generality that $w_k \neq y$. Set $u_i = w_i^+$ $(i = 1, \ldots, k)$ and $U = \{u_1, \ldots, u_k\}$.

Since P is a longest (x, y)-path, Lemma 13 can be applied to obtain

(11) $\quad N(w_i) \setminus (N(v) \cup \{v\}) \;\subseteq\; N(u_i) \cup U \qquad (i = 1, \ldots, k).$

We now establish the following claims.

(12) If $i < j$ and $u_j w_j^- \in E(G)$, then $u_i w_j \notin E(G)$.

Assuming the contrary, the path $x \overrightarrow{P} w_i v w_j u_i \overrightarrow{P} w_j^- u_j \overrightarrow{P} y$ contradicts the choice of P.

(13) $\quad w_1 \;=\; x.$

Assuming $w_1 \neq x$, we have $u_1 w_1^- \in E(G)$ by (11). As in the proof of (10), we obtain $u_i w_i^- \in E(G)$ for all $i \in \{1, \dots, k\}$ and $u_i w_j \in E(G)$ for some $j \in \{2, \dots, k\}$, contradicting (12).

(14) $u_i = w_{i+1}^- \qquad (i = 1, \dots, k-1)$.

Assuming the contrary, set $r = \min\{ i \mid u_i \neq w_{i+1}^- \}$. As in the proof of (10), we obtain $u_i w_i^- \in E(G)$ for all $i \in \{r+1, \dots, k\}$. Hence by (12), $u_i w_j \notin E(G)$ whenever $i \leq r$ and $j \geq r+1$. By Lemma 13, it follows that $N(w_i) \setminus (N(v) \cup \{v\}) \subseteq N(u_i) \cup \{u_1, \dots, u_r\}$ $(i = 1, \dots, r)$. Hence $u_{r+1} w_i \notin E(G)$ for $i \leq r$, implying that $\emptyset \neq (N(u_{r+1}) \cap N(v)) \setminus \{w_{r+1}\} \subseteq \{w_{r+2}, \dots, w_k\}$, contradicting (12).

(15) For every longest (x, y)-path Q, $V(G) \setminus V(Q)$ is an independent set.

It suffices to show that $N(v) \subseteq V(P)$. Suppose v has a neighbor $v_1 \in V(G) \setminus V(P)$. The choice of P implies $N(v_1) \cap (U \cup W) = \emptyset = N(v_1) \cap N(w_1) \cap (V(G) \setminus (V(P) \cup \{v\}))$. In particular, $d(v_1, w_1) = 2$ and hence $|N(v_1) \cap N(w_1)| \geq 2$. Using (14) and the assumption $d(x, y) \geq 3$, we conclude that v_1 and w_1 have a common neighbor z on $u_k^+ \overrightarrow{P} y^{--}$. By (11), $u_1 z \in E(G)$. Repeating the above arguments with P and v_1 instead of P and v, we obtain $v_1 y \in E(G)$ (since $v_1 x \notin E(G)$), and $v_1 z^{++} \in E(G)$. Now the path $x u_1 z \overleftarrow{P} w_2 v v_1 z^{++} \overrightarrow{P} y$ contradicts the choice of P.

(16) $N(u_i) \subseteq V(P) \qquad (i = 1, \dots, k-1)$.

Assuming $N(u_i) \not\subseteq V(P)$ for some $i \in \{1, \dots, k-1\}$, the path $x \overrightarrow{P} w_i v w_{i+1} \overrightarrow{P} y$ contradicts (15).

The above observations justify the following conclusions.

(17) If some longest (x, y)-path does not contain the vertex z, then either $zx \in E(G)$ or $zy \in E(G)$.

(18) If $\overrightarrow{Q}$ is any longest (x, y)-path, $z \notin V(Q)$, $q \in V(Q)$ and $zq \in E(G)$, then the vertices of $x \overrightarrow{Q} q$ (if $zx \in E(G)$) or $q \overrightarrow{Q} y$ (if $zy \in E(G)$) are alternately neighbors and nonneighbors of z.

Henceforth additionally assume P and v are chosen in such a way that

(19) $d(v)$ is as large as possible.

If $u_i x \in E(G)$ for all $i \in \{1, \dots, k-1\}$, then, considering the path $x \overrightarrow{P} w_i v w_{i+1} \overrightarrow{P} y$, (18) and (19) imply u_i has no neighbor on $u_k \overrightarrow{P} y$ $(i = 1, \dots, k-1)$. Together with (16) this implies $\omega(G - W) > |W|$. By (7) and Theorem 5 we conclude that $G \in \mathcal{K}$, contradicting the fact that G has diameter at least 3. Hence, for some $i \in \{2, \dots, k-1\}$, u_i is not adjacent to x. By (17), we obtain

(20) $u_i y \in E(G)$ for some $i \in \{2, \dots, k-1\}$.

Let $r = \min\{i \in \{2, \dots, k-1\} \mid u_i y \in E(G)\}$ and $s = \max\{i \in \{1, \dots, k-1\} \mid u_i x \in E(G)\}$. We first show

(21) $r > s$.

Assuming the contrary, consider the vertex w_s. Clearly, (18) implies $u_s w_j \in E(G)$ for all $j \in \{1,\ldots,s\}$. If $j \in \{1,\ldots,s\}$ and $u_j x \in E(G)$, then, considering the path $x\overrightarrow{P}w_j u_s \overleftarrow{P} w_{j+1} v w_{s+1} \overrightarrow{P} y$ and using (18) again, we obtain $u_j w_s \in E(G)$. Hence $N(x) \cap U \subseteq N(w_s)$. Clearly, (18) implies $N(y) \cap \{u_r,\ldots,u_{s-1}\} \subseteq N(w_s)$ and $u_r w_j \in E(G)$ for all $j \in \{r+1,\ldots,k\}$. If $j \in \{s,\ldots,k\}$ and $u_j y \in E(G)$, then, considering the path $x\overrightarrow{P}w_r v w_j \overleftarrow{P} u_r u_j^+ \overrightarrow{P} y$ and using (18) again, we obtain $u_j w_{r+1} \in E(G)$ and hence $u_j w_s \in E(G)$. Hence $N(y) \cap U \subseteq N(w_s)$. We conclude that $U \subseteq N(w_s)$. Hence $|N(w_s) \setminus (N(u_r) \cup N(v))| \geq k+1$, while $|N(u_r) \cap N(v)| \leq k-1$. This contradiction with the fact that G is an L_1-graph completes the proof of (21).

Let $j \in \{r,\ldots,k\}$. By (17) and (21), $u_j y \in E(G)$ and by (18), $u_j w_k \in E(G)$. Suppose $u_j w_r \notin E(G)$. Then, by (18), $u_j w_i \notin E(G)$ for all $i \in \{1,\ldots,r\}$. Hence $|N(u_j) \cap N(v)| \leq k-r$, while $|N(w_k) \setminus (N(u_j) \cup N(v))| \geq k-r+2$, a contradiction. Thus

(22) $u_j w_r \in E(G)$ for all $j \in \{r,\ldots,k\}$.

Now consider the path $x\overrightarrow{P}w_r v w_{r+1} \overrightarrow{P} y$, and let $p = \min\{i \in \{2,\ldots,r\} \mid u_r w_i \in E(G)\}$, $j \in \{p-1,\ldots,r-1\}$. By (17) and (21), $u_j x \in E(G)$ and by (18), $u_j w_p \in E(G)$. Suppose $u_j w_r \notin E(G)$. Then, by (18), $u_j w_i \notin E(G)$ for all $i \in \{r,\ldots,k\}$. Hence $|N(u_j) \cap N(u_r)| \leq r-p$, while $|N(w_p) \setminus (N(u_j) \cup N(u_r))| \geq r-p+3$, a contradiction. Thus

(23) $u_j w_r \in E(G)$ for all $j \in \{p-1,\ldots,r-1\}$.

By (22) and (23), $|N(w_r) \setminus (N(u_r) \cup N(v))| \geq k-p+3$, while $|N(u_r) \cap N(v)| \leq k-p+1$, our final contradiction. ■

An independent algorithmic proof of Theorem 7 (a), similar to the proof of Theorem 2 given in Asratian & Khachatrian [7], will appear in Asratian & Sarkisian [3].

We now use the arguments in the proof of Theorem 7 (a) to obtain a short direct proof of Theorem 2, as announced in Section 3.

Proof of Theorem 2 Let G be a connected L_0-graph with $|V(G)| \geq 3$. Assuming G is nonhamiltonian, define $\overrightarrow{C}$, v, W, k, $w_1,\ldots,w_k$, $u_1,\ldots,u_k$, U as in the proof of Theorem 7 (a). By the choice of C, all conditions in Lemma 13 are satisfied. Hence (1) and (2) hold. Since G is an L_0-graph, we obtain, instead of (3),

$$\begin{aligned} 0 &\leq \sum_{i=1}^{k} (|N(u_i) \cap N(v)| - |N(w_i) \setminus (N(u_i) \cup N(v))|) \\ &= \sum_{i=1}^{k} |N(u_i) \cap N(v)| - \sum_{i=1}^{k} |N(w_i) \setminus (N(u_i) \cup N(v))| \\ &\leq \sum_{i=1}^{k} |N(u_i) \cap W| - \sum_{i=1}^{k} (|N(w_i) \cap U| + 1) \; = \; -k \; < \; 0, \end{aligned}$$

an immediate contradiction. ■

Proof of Theorem 10 (by induction). Let G be a connected L_1-graph of even order. If $|V(G)| = 2$, then clearly G has a perfect matching. Now assume $|V(G)| > 2$ and every connected L_1-graph of even order smaller than $|V(G)|$ has a perfect matching. If G is a block, then by Theorem 5, the number of components, and hence certainly the number of odd components of $G-S$ does not exceed $|S|$, and we are done by Tutte's Theorem (see BONDY & MURTY [6, page 76]). Now assume G contains a cut vertex w. Let G_1 and G_2 be distinct components of $G-w$. For $i = 1, 2$, let u_i be a neighbor of w in G_i. Since $|N(u_1) \cap N(u_2)| = 1$ and G is an L_1-graph, we have $N(w) \setminus (N(u_1) \cup N(u_2)) = \{u_1, u_2\}$. In other words, every vertex in $N(w) \setminus \{u_1, u_2\}$ is adjacent to either u_1 or u_2. It follows that G_1 and G_2 are the only components of $G - w$ and, since u_i is an arbitrary neighbor of w in G_i,

(24) $G[\, N(w) \cap V(G_i)\,]$ is complete ($i = 1, 2$).

Since $|V(G)|$ is even, exactly one of the graphs G_1 and G_2, G_1 say, has odd order. Set $H = G[\, V(G_1) \cup \{w\}\,]$. We now show that G_2 and H are L_1-graphs.

Let x, y and z be vertices of G_2 such that $d_{G_2}(x, y) = 2$ and $z \in N_{G_2}(x) \cap N_{G_2}(y)$. By (24), $w \notin N_G(x) \cap N_G(y)$, implying that $N_{G_2}(x) \cap N_{G_2}(y) = N_G(x) \cap N_G(y)$. Furthermore, $N_{G_2}(z) \setminus (N_{G_2}(x) \cup N_{G_2}(y)) \subseteq N_G(z) \setminus (N_G(x) \cup N_G(y))$. Since G is an L_1-graph, it follows that G_2 is an L_1-graph.

A similar argument shows that H is an L_1-graph.

Since, moreover, the graphs G_2 and H have even order smaller than $|V(G)|$, each of them has a perfect matching. The union of the two matchings is a perfect matching of G. ■

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