Some Results on an Edge Coloring Problem of Folkman and Fulkerson

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Abstract. In 1968, Folkman and Fulkerson posed the following problem: Let \( G \) be a graph and let \( (n_1, \ldots, n_t) \) be a sequence of positive integers. Does there exist a proper edge coloring of \( G \) with colors \( 1, 2, \ldots, t \) such that precisely \( n_i \) edges receive color \( i \), for each \( i = 1, \ldots, t \)? If such a coloring exists then the sequence \( (n_1, \ldots, n_t) \) is called color-feasible for \( G \).

Some sufficient conditions for a sequence to be color-feasible for a bipartite graph where found by Folkman and Fulkerson, and de Werra.

In this paper we give a generalization of their results for bipartite graphs. Furthermore, we find a set of color-feasible sequences for an arbitrary simple graph. In particular, we describe the set of all sequences which are color-feasible for a connected simple graph \( G \) with \( \Delta(G) \geq 3 \), where every pair of vertices of degree at least 3 are non-adjacent.

Keywords: Graphs; Edge colorings; Color-feasible sequences

1. Introduction

We use Bondy and Murty [7] for terminology and notation not defined here. Let \( V(G) \) and \( E(G) \) denote, respectively, the vertex set and edge set of a graph \( G \). For each vertex \( u \) of \( G \) let \( N_G(u) \) denote the set of vertices adjacent to \( u \) and \( d_G(u) \) denote the degree of \( u \). The maximum vertex-degree of \( G \) is denoted by \( \Delta(G) \). An edge \( t \)-coloring or simply \( t \)-coloring of \( G \) is a mapping \( f : E(G) \rightarrow \{1, \ldots, t\} \). If \( e \in E(G) \) and \( f(e) = k \) then we say that the edge \( e \) is colored \( k \). A \( t \)-coloring of \( G \) is called proper if no pair of adjacent edges receives the same color. The minimum number \( t \) for which there exists a proper \( t \)-coloring of \( G \) is called the chromatic index of \( G \) and is denoted by \( \chi'(G) \). A graph is simple if it has no loops and no two of its edges join the same pair of vertices.

In 1968, Folkman and Fulkerson [10] posed and investigated the following problem:

**Problem 1.** Let $G$ be a graph with $q$ edges, and let $(n_1, \ldots, n_t)$ be a sequence of non-increasing positive integers, $\sum_{i=1}^{t} n_i = q$. Does there exist a proper $t$-coloring of $G$ in which precisely $n_i$ edges receive color $i$, for each $i = 1, \ldots, t$?

If such a coloring exists then the sequence $(n_1, \ldots, n_t)$ is called color-feasible for $G$.

If $G$ is a bipartite graph with bipartition $(V_1, V_2)$ where $V_1 = \{x_1, \ldots, x_n\}$ and $V_2 = \{y_1, \ldots, y_m\}$, then $G$ can be represented by an integral $n \times m$ matrix $B = (b_{ij})$ where $b_{ij}$ is the number of edges in $G$ with endvertices $x_i$ and $y_j$, $1 \leq i \leq n, 1 \leq j \leq m$. Then Problem 1 has the following two reformulations:

**Problem 1a.** When can a matrix $B$ with nonnegative integer entries be written as a sum

$$B = P_1 + P_2 + \ldots + P_t,$$

where each $P_i$ is a permutation matrix of size $n_i$, that is, $P_i$ has at most one 1 in each row and column and contains $n_i$ 1’s?

**Problem 1b.** In a school, there are $n$ classes $C_1, \ldots, C_n$ and $m$ teachers $T_1, \ldots, T_m$. Given that the number of one hour lectures which teacher $T_j$ must give to class $C_i$, is $b_{ij}$. Also the number $n_h$ of classrooms available at period $h$ is given, for $h = 1, \ldots, t$. The problem is to determine whether there exists a timetable of $t$ periods, so that each class receives all its teaching corresponding to the matrix $B = (b_{ij})$, precisely $n_h$ classrooms are used in each hour $h$ and no class or teacher is involved in more than one lecture at a time?

A general way for investigation of Problem 1 was suggested by Folkman and Fulkerson.

Let $D_q$ denote the set of non-increasing sequences of positive integers which sum to $q$. For two sequences $P = (p_1, \ldots, p_m)$ and $N = (n_1, \ldots, n_t)$ from $D_q$, the sequence $P$ is said to majorise $N$, written $P \succeq N$, if $m \leq t$ and $\sum_{i=1}^{r} p_i \geq \sum_{i=1}^{r} n_i$, for each $r = 1, \ldots, m - 1$. Clearly the majorisation relation can be viewed as a partial order on the set $D_q$.

**Theorem 1.1([10]).** If a sequence $P \in D_q$ is color-feasible for $G$ then every sequence $N \in D_q$ for which $P \succeq N$ is also color-feasible for $G$.

The proof of Theorem 1.1 in [10] provides a polynomial algorithm for constructing a coloring corresponding to $N$ if a coloring corresponding to $P$ is given.
The sequences in the set $D_q$ can be classified in the following way [18]. With each non-increasing sequence $N = (n_1, \ldots, n_t)$ we associate a new sequence $s(N) = (s(0), s(1), \ldots, s(l))$, where $s(0) = 0$ and $s(i + 1) = \max\{j : 1 + n_j \geq n_{s(i)+1}\}$, for $i = 0, 1, \ldots, l - 1$. For example, if $N = (9, 8, 8, 7, 4, 3)$ then $s(N) = (0, 3, 4, 6)$. In fact the integer $l$ in the definition of $s(N)$ is the minimum number of disjoint subsequences, in each of which any two members differ by at most 1. If the sequence $s(N)$ consists of $l$ positive members then we call $N$ an $l$-step sequence.

The next result follows from Theorem 1.1.

**Corollary 1.2** ([10,18]). A 1-step sequence $N \in D_q$ of length $t$ is color-feasible for a graph $G$ with $q$ edges if and only if $\chi'(G) \leq t$.

Since $\chi'(G) = \Delta(G)$ for a bipartite graph $G$, we obtain the following result.

**Corollary 1.3** A 1-step sequence $N \in D_q$ of length $t$ is color-feasible for a bipartite graph $G$ with $q$ edges if and only if $\Delta(G) \leq t$.

It is known that the problem of deciding whether $\chi'(G) \leq t$ is NP-complete even in the case when $G$ is a simple $t$-regular graph [13,14]. Therefore, Corollary 1.2 implies that Problem 1 is NP-complete in general case.

However, given a little more notation, we can formulate a simple necessary condition for the color-feasibility of a sequence $N$.

Let $k$ be a positive integer. An edge subset $F \subseteq E(G)$ is called a $k$-matching of $G$ if each vertex of $G$ is incident with at most $k$ edges of $F$. A $k$-matching of maximum cardinality is called a maximum $k$-matching of $G$. We shall denote by $q_k(G)$ the number of edges in a maximum $k$-matching of $G$. Note that the number $q_k(G)$ can be found in polynomial time [5].

Let $D_q(G)$ denote the set of all sequences $(n_1, \ldots, n_t)$ in $D_q$ which satisfy the following condition:

$$\begin{align*}
t &\geq \Delta(G), \\
|E(G)| &= \sum_{i=1}^{t} n_i, \\
\text{and} \\
q_k(G) &\geq \sum_{i=1}^{k} n_i, \quad \text{for } k = 1, \ldots, \Delta(G) - 1.
\end{align*}$$

Clearly, this condition can be checked in polynomial time.

**Property 1.4.** If a sequence $N \in D_q$ is color-feasible for $G$, then $N \in D_q(G)$.

In some cases this necessary condition, $N \in D_q(G)$, is also sufficient for color-feasibility of $N$ for $G$.

**Theorem 1.5.** ([18]) Let $G$ be a bipartite graph with $q$ edges. Then a 2-step sequence $N \in D_q$ is color-feasible for a bipartite graph $G$ if and only if $N \in D_q(G)$. 
Corollary 1.6. ([10]) Let $G$ be a bipartite graph and $N = (n_1, \ldots, n_t)$ be a sequence in $D_q$ with $n_1 = \cdots = n_k > n_{k+1} = \cdots = n_t$. Then $N$ is color-feasible for $G$ if and only if $N \in D_q(G)$.

Some other properties of the set of color-feasible sequences for a bipartite graph $G$ can be found in [1,2,8,10,15,18,19,20].

Corollary 1.3 and Theorem 1.5 imply that Problem 1 is solved polynomially if $G$ is a bipartite graph and $N$ is a 1- or 2-step sequence. However, even for a bipartite graph $G$ Problem 1 is NP-complete if $N$ is a 3-step sequence. (Indeed, Problem 1 is NP-complete even if $G$ is bipartite, $\Delta(G) = 3$ and $N = (n_1, n_2, n_3)$ (see [4,12])).

Let $G$ be a graph with $\Delta = \Delta(G)$ and, for each integer $k$, let $G_k$ denote the subgraph induced by the set of vertices of degree at least $k$. B. Fournier [11] proved that if $G$ is a simple graph and the subgraph $G_\Delta$ has no edges then $\chi'(G) = \Delta(G)$. Berge and Fournier [6] observed that the same proof actually gives the following broader result, which was stated earlier by Lovasz and Plummer ([16], 7.4.3).

Proposition 1.7 If $G$ is a simple graph where the subgraph $G_\Delta$ is acyclic, then $\chi'(G) = \Delta(G)$.

In this paper we use similar ideas for investigation of Problem 1. Let $\Delta_1(G)$ denote the minimum integer $k$ such that the subgraph $G_{k+1}$ is acyclic. In particular, if the subgraph $G_\Delta$ contains a cycle, then $\Delta_1(G) = \Delta(G)$. Furthermore, let $\Delta_2(G)$ denote the minimum integer $k$ such that the subgraph $G_{k+1}$ contains no edges. It is clear that $\Delta_1(G) \leq \Delta_2(G) \leq \Delta(G)$. In this terminology Proposition 1.7 can be reformulated in the following way: if $\Delta_1(G) < \Delta(G)$ then $\chi'(G) = \Delta(G)$.

We say that the number $\Delta_i(G)$, $i \in \{1, 2\}$, is a threshold for a sequence $N \in D_q$ with $s(N) = (s(0), s(1), \ldots, s(l))$ if $s(1) > \Delta_i(G)$.

The following results are obtained in this paper.

1. We give a sufficient condition for an $l$-step sequence $N \in D_q$ with $l \geq 2$ to be color-feasible for a bipartite graph. This result implies Theorem 1.5.

2. We investigate Problem 1 for an arbitrary simple graph $G$:
   a) We prove that all sequences with threshold $\Delta_2(G)$ in the set $D_q(G)$ are color-feasible for $G$. We also prove that if $G$ is connected and $\Delta_2(G) = 2 < \Delta(G)$, that is, every pair of vertices of degree at least 3 are non-adjacent, then all sequences in $D_q(G)$ are color-feasible for $G$.
   b) We show that all 1- and 2-step sequences with threshold $\Delta_1(G)$ in the set $D_q(G)$ are color-feasible for $G$. 
Note that we described in [3] a polynomial algorithm to solve the following problem: are all 3-step sequences with threshold $\Delta_1(G)$ in the set $D_q(G)$ color-feasible for $G$? By using this algorithm we can, in particular, determine for an arbitrary tree $G$: are all 3-step sequences in $D_q(G)$ color-feasible for $G$? For a tree with bounded degrees there exists a polynomial algorithm to determine all color-feasible sequences (see [21]).

2. Color-feasibility for bipartite graphs

Let $G$ be a graph with $|E(G)| = q$ and $\Delta(G) = \Delta$, and let $H(G) = (h_1, \ldots, h_{\Delta})$ be a sequence where $h_1 = q_1(G)$ and $h_i = q_{i+1}(G) - q_i(G)$, for $i = 1, \ldots, \Delta - 1$. If $H(G)$ is non-increasing, that is, $H(G) \in D_q(G)$ then the condition $N \in D_q(G)$ is equivalent to the condition $H(G) \succeq N$.

**Remark 2.1.** It follows from Theorem 1.1 that if $H(G)$ is color-feasible for $G$ then $D_q(G)$ is the set of all color-feasible sequences for $G$.

It is known [18] that if $G$ is a bipartite graph then $H(G) \in D_q$. The following criterion for feasibility of $H(G)$ was found by de Werra [18]: Let $G$ be a bipartite graph with $\Delta(G) = \Delta$ and let $s(H(G)) = (s(0), s(1), \ldots, s(l)), l \geq 2$. Then $H(G)$ is color-feasible for $G$ if and only if there exist edge subsets $F_1, F_2, \ldots, F_l$ such that $F_1 \subseteq F_2 \subseteq \ldots \subseteq F_l$ and $F_j$ is a maximum $s(j)$-matching of $G$, for each $j = 1, \ldots, l$.

An immediate corollary (not stated in an explicit form in [18]) is a criterion (Proposition 2.2) for an arbitrary sequence $N$ to be color-feasible for a bipartite graph $G$. The interest of the proof given in the present paper is that it is constructive: if $G$ and $N$ satisfy this criterion then a corresponding coloring is constructed in polynomial time.

**Proposition 2.2.** Let $G$ be a bipartite graph, and let $N = (n_1, \ldots, n_l)$ be a non-increasing sequence with $s(N) = (s(0), s(1), \ldots, s(l)), l \geq 2$. Then $N$ is color-feasible for $G$ if and only if there exist edge subsets $F_1, F_2, \ldots, F_l$ such that $F_1 \subseteq F_2 \subseteq \ldots \subseteq F_l$ and $F_j$ is an $s(j)$-matching with $\sum_{i=1}^{s(j)} n_i$ edges, for each $i = 1, \ldots, l$.

**Proof.** Suppose that $G$ has a proper $t$-coloring corresponding to $N$. Then the set $F_j$ consisting of edges colored $1, 2, \ldots, s(j)$ is an $s(j)$-matching of $G$, $j = 1, \ldots, l$, and $F_1 \subseteq F_2 \subseteq \ldots \subseteq F_l$.

Conversely, suppose that there exist edge subsets $F_1, \ldots, F_l$ satisfying the condition of Proposition 2.2. We will prove that the edges in $F_j$ can be properly colored with colors $1, \ldots, s(j)$ such that precisely $n_i$ edges are colored $i$, for $i = 1, \ldots, s(j)$. 

For \( j = 1 \) it follows from Corollary 1.3 because \((n_1, ..., n_{s(1)})\) is a 1-step sequence, \(|F_1| = \sum_{i=1}^{s(1)} n_i\) and \(F_1\) is an \(s(1)\)-matching. Suppose that the required coloring is already constructed for \(F_j\), \(1 \leq j < l\). We will color edges from \(F_{j+1}\setminus F_j\) with colors from the set \(C = \{1, 2, ..., s(j + 1)\}\).

Let \(e\) be an edge which has so far not been colored, and \(u\) and \(v\) be the ends of \(e\). If there is a color \(\alpha \in C\) for which there is no edge colored \(\alpha\) adjacent to \(e\), then use color \(\alpha\) to color \(e\). Otherwise, since \(F_{j+1}\) is an \(s(j + 1)\)-matching, and we have \(s(j + 1)\) colors, there are a color \(t_v \in C\) which is not used to color an edge incident with \(u\), and a color \(t_u \in C\), \(t_u \neq t_v\), which is not used to color an edge incident with \(v\). Consider a path \(P\) of maximum length with initial vertex \(u\) whose edges are alternatively colored \(t_u\) and \(t_v\). Clearly, \(P\) cannot pass through \(v\), otherwise \(E(P) \cup \{e\}\) forms an odd cycle in \(G\), which contradicts \(G\) being bipartite. Thus if we interchange the two colors \(t_u\) and \(t_v\) along \(P\), the color \(t_u\) will no longer be used on an edge adjacent to either vertex, and we can color \(e\) with \(t_u\).

Suppose that the proper coloring of the edges of \(F_{j+1}\) produced by this procedure is such that \(n'_i\) edges are colored \(i\), for each \(i = 1, ..., s(j + 1)\). We may assume (possibly after permuting the colors) that \(n'_1 \geq n'_2 \geq \cdots \geq n'_{s(j+1)}\). It is not difficult to see that the above procedure of coloring certainly guarantees that \(n'_i \geq n_i\), for each \(i = 1, ..., s(j)\). Let \(k(j+1)\) denote the maximal \(i\) with \(n'_i > 0\). Then \(s(j) < k(j+1) \leq s(j + 1)\) and \((n'_1, ..., n'_{k(j+1)}) \succeq (n_1, ..., n_{s(j+1)})\) because \(n_{1+s(j)} - n_{s(j+1)} \leq 1\).

If \((n'_1, ..., n'_{k(j+1)}) \neq (n_1, ..., n_{s(j+1)})\) then, by using the algorithm suggested in [10], we can polynomially transform the coloring of \(F_{j+1}\) corresponding to the sequence \((n'_1, ..., n'_{k(j+1)})\) to a proper \(s(j + 1)\)-coloring of the edges of \(F_{j+1}\) corresponding to the sequence \((n_1, n_2, ..., n_{s(j+1)})\). ■

The next auxiliary lemma is a corollary of a result of Berge (see [5]).

**Lemma 2.3.** Let \(s\) be a positive integer and \(F\) a subset of the set of edges in a graph \(G\). Then \(F\) is a maximum \(s\)-matching of \(G\) if and only if there is no path \(P\) such that edges of \(P\) are alternatively in \(F\) and \(E(G) \setminus F\), both the first and the last edge of \(P\) is in \(E(G) \setminus F\) and the number of edges of \(F\) incident with the origin of \(P\) as well as the number of edges incident with the terminus of \(P\) is less than \(s\).

**Theorem 2.4.** Let \(N\) be a sequence in \(D_q\) with \(s(N) = (s(0), s(1), ..., s(l)), l \geq 2\). Then \(N\) is feasible for a bipartite graph \(G\) with \(q\) edges if every pair of vertices of degree more than \(s(2)\) are non-adjacent in \(G\) and \(H(G) \geq N\).
Proof. Let $X_1$ be a maximum $s(1)$-matching of $G$. We shall construct edge subsets $X_2, \ldots, X_l$ in the following way: suppose that $X_1, \ldots, X_{i-1}$ have already been constructed ($2 \leq i \leq l$). If $s(i) \geq \Delta(G)$, put $X_i = E(G)$. Otherwise, at each vertex $u$ with $d_G(u) > s(i)$ delete precisely $d_G(u) - s(i)$ edges from $E(G) \setminus X_{i-1}$. The remaining edges of $E(G) \setminus X_{i-1}$ together with $X_{i-1}$ form the next edge subset $X_i$.

By our construction, the following property holds for each edge $uv \in E(G) \setminus X_i$: if $u$ is incident with less than $s(i)$ edges of $X_i$ then $d_G(u) \leq s(i)$, and $v$ is incident with exactly $s(i)$ edges of $X_i$.

If $u$ is incident with exactly $s(i)$ edges of $X_i$ then $d_G(u) > s(i)$ and, therefore, $d_G(v) \leq s(i)$ since every pair of vertices of degree more than $s(2)$ are non-adjacent in $G$. It means that $v$ is incident with less than $s(i)$ edges of $X_i$.

It is not difficult to check now that every alternating path $P = a_0a_1 \ldots a_{2r+1}$ relative to $X_i$ (i.e., whose edges are alternatively in $X_i$ and in $E(G) \setminus X_i$) with end edges in $E(G) \setminus X_i$ has the following property: if $a_0$ is incident with less than $s(i)$ edges of $X_i$ then each of the vertices $a_1, a_3, \ldots, a_{2r+1}$ is incident with exactly $s(i)$ edges of $X_i$, and each of the vertices $a_2, a_4, \ldots, a_{2r}$ is incident with less than $s(i)$ edges of $X_i$. We begin with the vertex $a_1$.

This property and Lemma 2.3 imply that $X_i$ is a maximum $s(i)$-matching of $G$. Clearly $X_1 \subseteq X_2 \subseteq \cdots \subseteq X_l$. From $X_1$ choose a subset of edges $F_1$ of size $\sum_{i=1}^{s(1)} n_i$, and from $X_j$ choose a subset $F_j$ of size $\sum_{i=1}^{s(j)} n_i$ containing $F_{j-1}$, for each $j = 2, \ldots, l$. This is possible since $H(G) \geq N$. Now the theorem follows from Proposition 2.2.

Corollary 2.5([18]). Let $G$ be a bipartite graph with $q$ edges. A sequence $N = (n_1, \ldots, n_q) \in D_q$ with $s(N) = (s(0), s(1), s(2))$ is color-feasible for $G$ if and only if $H(G) \geq N$.

Proof. Clearly, $s(2) = t$. If $H(G) \geq N$ then $t \geq \Delta(G)$, and the conditions of Theorem 2.4 are trivially satisfied. Therefore $N$ is color-feasible for $G$.

Theorem 2.6. Let $G$ be a bipartite graph with bipartition $(V_1, V_2)$ where $d_G(x) \geq d_G(y)$ for each edge $(x, y)$ with $x \in V_1$ and $y \in V_2$. Then $D_q(G)$ is the set of all color-feasible sequences for $G$. Furthermore, the sequence $H(G) = (h_1, h_2, \ldots, h_{\Delta})$ satisfies the condition: $h_i = |\{x \in V_1 / d_G(x) \geq i\}|$ for each $i = 1, \ldots, \Delta$.

Proof. We will show that the edges of $G$ can be properly colored with colors $1, 2, \ldots, \Delta(G)$ such that edges incident to each vertex $x \in V_1$ are colored with colors
1, 2, ..., \(d_G(x)\). This implies that \(H(G)\) satisfies the above condition and is color-feasible for \(G\). Then, by Remark 2.1, \(D_q(G)\) is the set of all color-feasible sequences for \(G\).

Let \(V_1 = \{x_1, x_2, \ldots, x_n\}\) with \(d_G(x_1) \geq d_G(x_2) \geq \cdots \geq d_G(x_n)\). Suppose that the edges incident with each vertex \(x_i, i = 1, \ldots, k, k < n\), are already colored with the colors \(1, 2, \ldots, d_G(x_i)\), and the edges incident with the vertices \(x_{k+1}, \ldots, x_n\) are not colored yet. Let \(e_1, e_2, \ldots, e_{d(x_{k+1})}\) be the edges incident with \(x_{k+1}\). Then for each \(j = 1, \ldots, d(x_{k+1})\) in turn we do the following:

Consider the vertex \(y_j\) which is the end in \(V_2\) of the edge \(e_j\). Since \(d_G(x_{k+1}) \geq d(y_j)\) there is a color \(l\) such that \(1 \leq l \leq d_G(x_{k+1})\) and there are no edges incident with \(y_j\) colored \(l\). If \(l = j\), then color \(e_j\) with color \(j\).

Otherwise, consider a path \(P\) of maximum length with origin at \(y_j\) whose edges are alternatively colored \(j\) and \(l\). Clearly, by construction, this path must end in \(V_2\). Thus we may interchange the two colors along this path, to make color \(j\) available at \(y_j\), and color \(e_j\) with color \(j\). ■

The above conditions use the structure of a graph \(G\). Now we will give some other type of conditions for color-feasibility of a sequence of length 3.

**Definition.** Let \(N = (n_1, n_2, n_3)\) be a non-increasing sequence. We define the weight of \(N\), denoted \(w(N)\), by \(w(N) = 3n_1 + 2n_2 + n_3\).

**Theorem 2.7.** Let \(G\) be a bipartite graph with \(\Delta(G) = 3\), \(|E(G)| = q\) and \(H(G) = (h_1, h_2, h_3)\). Then every sequence \(N = (n_1, n_2, n_3) \in D_q\), satisfying \(H(G) \geq N\) and

\[
w(N) \leq w(H(G)) - \left\lfloor \frac{h_1 - h_2}{2} \right\rfloor - \left\lfloor \frac{h_2 - h_3}{2} \right\rfloor + 1,
\]

is color-feasible for \(G\).

**Proof.** The proposition is evident if \(H(G)\) is color-feasible. Suppose that \(H(G)\) is not color-feasible for \(G\). It follows from Corollary 1.3 and Theorem 1.5 that \(H(G)\) is a 3-step sequence. Therefore, \(h_1 - h_2 \geq 2\) and \(h_2 - h_3 \geq 2\). Define three sequences:

\[
N_0 = (h_1 - \left\lfloor \frac{h_1 - h_2}{2} \right\rfloor, h_2 + \left\lfloor \frac{h_2 - h_3}{2} \right\rfloor, h_3),
\]

\[
N_1 = (h_1, h_2 - \left\lfloor \frac{h_2 - h_3}{2} \right\rfloor, h_3 + \left\lfloor \frac{h_2 - h_3}{2} \right\rfloor),
\]

\[
N_2 = (h_1 - \left\lfloor \frac{h_1 - h_2}{2} \right\rfloor, h_2 + \left\lfloor \frac{h_2 - h_3}{2} \right\rfloor - \left\lfloor \frac{h_2 - h_3}{2} \right\rfloor, h_3 + \left\lfloor \frac{h_2 - h_3}{2} \right\rfloor).
\]

Clearly, \(N_0\) and \(N_1\) are 2-step sequences and \(H(G) \succ N_0, H(G) \succ N_1\). Then, by Theorem 1.5, \(N_0\) and \(N_1\) are color-feasible for \(G\). Furthermore, it is clear that \(w(N_2) = (3h_1 + 2h_2 + h_3) - \left\lfloor \frac{h_1 - h_2}{2} \right\rfloor - \left\lfloor \frac{h_2 - h_3}{2} \right\rfloor = w(H(G)) - \left\lfloor \frac{h_1 - h_2}{2} \right\rfloor - \left\lfloor \frac{h_2 - h_3}{2} \right\rfloor\).
Let $N = (n_1, n_2, n_3)$ be a sequence in $D_q$ satisfying $H(G) \geq N$ and $w(N) \leq w(N_2) + 1$. We will show that either $N_0 \geq N$ or $N_1 \geq N$. Suppose that this is not true. It means that $n_1 = h_1 - \left[ \frac{h_1 - h_2}{2} \right] + b_1$, where $0 < b_1 \leq \left[ \frac{h_1 - h_2}{2} \right]$, and $n_1 + n_2 = h_1 + h_2 - \left[ \frac{h_2 - h_3}{2} \right] + b_2$, where $0 < b_2 \leq \left[ \frac{h_2 - h_3}{2} \right]$. This implies that

$$n_2 = h_2 + \left[ \frac{h_1 - h_2}{2} \right] - \left[ \frac{h_2 - h_3}{2} \right] + b_2 - b_1, n_3 = h_3 + \left[ \frac{h_2 - h_3}{2} \right] - b_2.$$

But then $w(N) = 3n_1 + 2n_2 + n_3 = w(N_2) + 3b_1 + 2(b_2 - b_1) - b_2 = w(N_2) + b_2 + b_1 > w(N_2) + 1$, which contradicts the condition $w(N) \leq w(N_2) + 1$. Therefore, either $N_0 \geq N$ or $N_1 \geq N$. Then, by Theorem 1.1, $N$ is color-feasible for $G$. ■

The bound in Theorem 2.7 is sharp in the sense that for every $r \geq 1$ there exists a bipartite graph $G$ with $q = 3r + 6$ edges and maximum degree 3 such that every sequence $N = (n_1, n_2, n_3) \in D_q$, satisfying $H(G) \geq N$ and

$$w(N) \geq w(H(G)) - \left[ \frac{h_1 - h_2}{2} \right] - \left[ \frac{h_2 - h_3}{2} \right] + 2,$$

is not color-feasible for $G$.

![Fig. 1](image)

Consider, for example, the graph $G$ in Fig. 1. Clearly, $H(G) = (r + 4, r + 2, r)$. Furthermore, $H(G)$ is not color-feasible for $G$, because $G$ has the unique maximum matching $M_1 = \{e_1, e_2, ..., e_{r+4}\}$ and the graph $G - M_1$ has no matching of cardinality $r + 2$. Finally, by Theorem 2.7, every sequence of weight at least $w(H(G)) - 1$ is color-feasible for $G$.

3. Color-feasibility for arbitrary simple graphs

Let $G$ be a simple graph with $\Delta(G) = m$ where the subgraph induced by the set of vertices of degree $m$ is acyclic. Assume that a subset $E_0 \subseteq E(G)$ is properly colored with colors $1, 2, ..., m$ such that exactly $n_i$ edges are colored $i$, for $i = 1, ..., m$. For each vertex $y$ let $C'(y)$ denote the set of colors of the edges incident with $y$ and
$\overline{C}(y) = \{1, 2, \ldots, m\} \setminus C(y)$. A generalization of Vizing’s theorem [17] was obtained in [6]. We use similar considerations for investigation of Problem 1.

**Definition** ([6]). Let $e = (x_0, y_0)$ be an uncolored edge of $G$ and $\alpha_1$ be a color such that $\alpha_1 \notin C(y_0)$ and $\alpha_1 \in C(x_0)$. We define a sequence $S(x_0, \alpha_1)$ of distinct edges $e_0, e_1, e_2, \ldots$ all incident with $x_0$, together with a function $f$ that associates to each edge $e_i$ of the sequence a color $\alpha_{i+1} = f(e_i)$, according to the following iterative procedure.

(I) Put $e_0 = e, f(e_0) = \alpha_1$.

(II) Suppose that the edges $e_0 = (x_0, y_0), \ldots, e_{i-1} = (x_0, y_{i-1})$ are already included in $S(x_0, \alpha_1)$ and $f(e_0) = \alpha_1, \ldots, f(e_{i-1}) = \alpha_i$ are already defined, $i \geq 1$.

a) If $\alpha_i \in C(x_0)$ and $\alpha_i \neq f(e_j)$ for all $j < i - 1$, consider the edge $e_i = (x_0, y_i)$ incident with $x_0$ that is colored with $\alpha_i$; let $\alpha_{i+1} = f(e_i)$ be a color satisfying the condition $\alpha_{i+1} \notin C(y_i)$.

b) If either $\overline{C}(y_{i-1}) = \emptyset$, or $\alpha_i \notin C(x_0)$, or $\alpha_i = f(e_j)$ for an index $j < i - 1$, then we stop, and the sequence $S(x_0, \alpha_1)$ is achieved, $S(x_0, \alpha_1) = (e_0, e_1, \ldots, e_{i-1})$.

**Proposition 3.1.** Let $G$ be a simple graph with $\Delta(G) = m$ where the subgraph induced by the set of vertices of degree $m$ is acyclic. Assume that a subset $E_0 \subseteq E(G)$ is colored with colors $1, 2, \ldots, m$ such that precisely $n_i$ edges are colored $i$, for $i = 1, \ldots, m$. Then for an uncolored edge $e$ the set $E_0 \cup \{e\}$ can be colored with colors $1, \ldots, m$ such that at least $n_i$ edges are colored $i$, for each $i = 1, \ldots, m$.

**Proof.** Without loss of generality we suppose that the only uncolored edge is $e$, that is, $E(G) = E_0 \cup \{e\}$. Let $e = (b_0, b_1)$. Consider the following algorithm. First we label vertices $b_0$ and $b_1$.

**Step** $r(r \geq 0)$. Suppose that the vertices $b_0, b_1, \ldots, b_{r+1}$ have been already labelled and $(b_r, b_{r+1})$ is the only uncolored edge of $G$. Choose a color $\alpha_1 \notin C(b_r)$. If $\alpha_1 \notin C(b_{r+1})$ then color the edge $(b_r, b_{r+1})$ with $\alpha_1$. If $\alpha_1 \in C(b_{r+1})$ then construct a sequence $S(b_{r+1}, \alpha_1)$. Let $(b_{r+1}, b_{r+2})$ be the last edge in $S(b_{r+1}, \alpha_1)$.

a) If $\overline{C}(b_{r+2}) \neq \emptyset$ then, by using the same considerations as in [6], the edges in $E_0 \cup \{e\}$ can be properly colored with colors $1, \ldots, m$. It is not difficult to check that the method of coloring described in [6] guarantees that at least $n_i$ edges are colored $i$, for $i = 1, \ldots, m$.

b) Suppose that $\overline{C}(b_{r+2}) = \emptyset$ and $S(b_{r+1}, \alpha_1) = (e_0, e_1, \ldots, e_t)$ where $e_0 = (b_r, b_{r+1}), e_t = (b_{r+1}, b_{r+2})$, $e_j$ is colored $\alpha_j$ and $f(e_j) = \alpha_{j+1}, j = 1, \ldots, t$. For each
$j = 1, \ldots, t$ remove the color $\alpha_j$ from $e_j$ and assign it instead to $e_{j-1}$. Now the only uncolored edge is $(b_{r+1}, b_{r+2})$. Label the vertex $b_{r+2}$ and go to Step $(r + 1)$.

It is not difficult to see that if the required coloring is not constructed on Step $r$ then $\overline{C(b_{r+2})} = \emptyset$, that is, the new labelled vertex $b_{r+2}$ has degree $\Delta(G)$. Since the subgraph of $G$ induced by the vertices of degree $\Delta(G)$ is acyclic, the vertices $b_2, b_3, \ldots$ constructed by the algorithm, are different. Therefore, on some step of the algorithm the required coloring of $E_0 \cup \{e\}$ will be constructed. ■

**Proposition 3.2.** Let $G$ be a simple graph with $q$ edges, and let $N = (n_1, \ldots, n_t)$ be a sequence in the set $D_q$ with $s(N) = (s(0), s(1), \ldots, s(l))$ such that $l \geq 2$ and $s(2) > \Delta_1(G)$. Then $N$ is color-feasible for $G$ if and only if there exist subsets $F_1, \ldots, F_l$ such that $F_1 \subset F_2 \subset \ldots \subset F_l$, the set $F_j$ is an $s(j)$-matching with $\sum_{i=1}^{s(j)} n_i$ edges, for $j = 1, \ldots, l$, and edges of $F_1$ can be properly colored with $s(1)$ colors.

**Proof.** The necessity is evident: if $G$ has a proper $t$-coloring corresponding to $N$ then the set of edges $F_i$ consisting of edges colored $1, 2, \ldots, s(i)$ is a $s(i)$-matching for each $i = 1, \ldots, l$, and $F_1 \subset F_2 \subset \ldots \subset F_l$.

Conversely, suppose that there exist subsets $F_1, \ldots, F_l$ satisfying the condition of the proposition. We will prove that the edges in $F_j$ can be properly colored with colors $1, \ldots, s(j)$ such that precisely $n_i$ edges are colored $i$, for $i = 1, \ldots, s(j)$. By the assumption, the edges in $F_1$ can properly colored with colors $1, \ldots, s(1)$. Therefore, by Corollary 1.2, there is a proper $s(1)$-coloring of $F_1$ corresponding to the sequence $(n_1, \ldots, n_{s(1)})$.

Suppose that the required coloring is already constructed for $F_j$, $1 \leq j < l$. Let $H_{j+1}$ denote the subgraph induced by the set $F_{j+1}$. Since $s(j+1) > \Delta_1(G)$, the subgraph of $H_{j+1}$ induced by the set of vertices of degree $s(j+1)$ in $H_{j+1}$, is acyclic. Then, by Proposition 3.1, the edges in $F_{j+1}$ can be colored with colors $1, 2, \ldots, s(j+1)$ such that at least $n_i$ edges in $F_{j+1}$ are colored $i$, for each $i = 1, 2, \ldots, s(j)$.

Suppose that precisely $n_i'$ edges are colored $i$, for $i = 1, \ldots, s(j+1)$. We may assume (possibly after permuting the colors) that $n_1' \geq n_2' \geq \cdots \geq n_{s(j+1)}'$ and $n_i' \geq n_i$, for each $i = 1, \ldots, s(j)$. Let $k(j+1)$ denote the maximal $i$ with $n_i' > 0$. Then $s(j) < k(j+1) \leq s(j+1)$ and $(n_1', \ldots, n_{k(j+1)}') \geq (n_1, \ldots, n_{s(j+1)})$ because $n_{1+s(j)} - n_{s(j+1)} \leq 1$.

If $(n_1', \ldots, n_{k(j+1)}) \neq (n_1, \ldots, n_{s(j+1)})$ then, by using the algorithm, suggested in [10] we can polynomially transform the coloring of $F_{j+1}$ corresponding to the
sequence \((n'_1, ..., n'_{k(j+1)})\), to a proper \(s(j+1)\)-coloring of the edges of \(F_{j+1}\) corresponding to the sequence \((n_1, n_2, ..., n_{s(j+1)})\). 

Note that the proof of Proposition 3.2 provides a polynomial algorithm for constructing a coloring corresponding to the sequence \(N\), if the required sets \(F_1, ..., F_l\) are given.

**Proposition 3.3.** Let \(G\) be a simple graph. Then all 1- and 2-step sequences with threshold \(\Delta_1(G)\) in the set \(D_q(G)\) are color-feasible for \(G\).

**Proof.** Proposition 1.7 and Corollary 1.2 imply that all 1-step sequences with threshold \(\Delta_1(G)\) in \(D_q(G)\) are color-feasible for \(G\). Now consider a 2-step sequence \(N = (n_1, ..., n_l)\) in \(D_q(G)\) with \(s(N) = (s(0), s(1), s(2))\) and \(s(1) > \Delta_1(G)\). Construct a maximum \(s(1)\)-matching \(F\) of \(G\). Clearly, \(\sum_{i=1}^{s(1)} n_i \leq |F|\) since \(N \in D_q(G)\). Choose in \(F\) a subset \(F_1\) of \(\sum_{i=1}^{s(1)} n_i\) edges. Let \(H\) denote the subgraph induced by \(F_1\). Since \(s(1) > \Delta_1(G)\), the subgraph of \(H\) induced by vertices of degree \(s(1)\) is acyclic. By Proposition 1.7, edges of \(F_1\) can be properly colored by \(s(1)\) colors. Put \(F_2 = E(G)\). Then \(F_1\) and \(F_2\) satisfy the condition of Proposition 3.2. Therefore, \(N\) is color-feasible for \(G\).

**Theorem 3.4.** Let \(G\) be a simple graph. Then all sequences with threshold \(\Delta_2(G)\) in the set \(D_q(G)\) are color-feasible for \(G\).

**Proof.** Let \(N = (n_1, ..., n_l) \in D_q(G), s(N) = (s(0), s(1), ..., s(l))\) and \(s(1) > \Delta_2(G)\). Then \(s(1) > \Delta_1(G)\) since \(\Delta_2(G) \geq \Delta_1(G)\). If \(l \leq 2\) then, by Proposition 3.3, \(N\) is color-feasible for \(G\). Now suppose that \(l \geq 3\).

Let \(X_1\) be a maximum \(s(1)\)-matching of \(G\). We shall construct edge subsets \(X_2, ..., X_l\) in the following way: suppose that \(X_1, ..., X_{i-1}\) are already constructed \((i \leq l)\). If \(s(i) \geq \Delta(G)\), put \(X_i = E(G)\). Otherwise, at each vertex \(u\) with \(d_G(u) > s(i)\) delete precisely \(d_G(u) - s(i)\) edges from \(E(G)\). The remaining edges of \(E(G)\) together with \(X_{i-1}\) form the next edge subset \(X_i\). It is not difficult to check that every alternating path \(P = a_0a_1 ... a_{2r+1}\) relative to \(X_i\) with end edges in \(E(G)\) has the property that if \(a_0\) is incident with less than \(s(i)\) edges of \(X_i\) then each of the vertices \(a_1, a_2, a_3, ..., a_{2r+1}\) is incident with exactly \(s(i)\) edges of \(X_i\), and each of the vertices \(a_1, a_2, a_3, ..., a_{2r}\) is incident with less than \(s(i)\) edges of \(X_i\). This property and Lemma 2.3 imply that \(X_i\) is a maximum \(s(i)\)-matching of \(G\). Clearly \(X_1 \subseteq X_2 \subseteq ... \subseteq X_l\). From \(X_1\) choose a subset of edges \(F_1\) of size \(\sum_{i=1}^{s(1)} n_i\), and from \(X_j\) choose a subset \(F_j\) of size \(\sum_{i=1}^{s(j)} n_i\) containing \(F_{j-1}\), for each
Let $G$ be a simple graph with $|E(G)| = q$ and $\Delta(G) = \Delta$, and let $H(G) = (h_1, ..., h_{\Delta})$ be a sequence which was defined in Section 2. It is known that $H(G) \in D_q$ if $G$ is bipartite and it may not be true if $G$ is non-bipartite [18]. The next result describes a class of graphs where $H(G) \in D_q$ and, moreover, $H(G)$ is color-feasible for $G$.

**Theorem 3.6.** Let $G$ be a connected simple graph with $q$ edges where $\Delta(G) \geq 3$ and $\Delta_2(G) = 2$, that is, every pair of vertices of degree at least 3 are non-adjacent. Then $H(G)$ is color-feasible for $G$ and $D_q(G)$ is the set of all color-feasible sequences for $G$.

**Proof.** Let $H(G) = (h_1, ..., h_{\Delta})$ where $\Delta = \Delta(G)$. We will show that $H(G)$ is color-feasible for $G$. Let $F_1$ be a maximum matching of $G$. We will sequentially construct edge subsets $F_2, ..., F_{\Delta}$.

At each vertex $x$ with $d_G(x) > 2$ delete precisely $d_G(x) - 2$ edges from $E(G) \setminus F_1$. The remaining set of edges we denote by $F_2$. It is clear that $F_2$ is a maximum 2-matching of $G$.

Suppose that the set $F_2$ induces a non-bipartite graph. Consider in this graph a cycle $C$ of odd length. Since $G$ is connected and $\Delta \geq 3$, there is an edge $(x, y)$ in $C$ and a vertex $z \notin C$ such that $d_G(x) \geq 3, d_G(y) = 2$ and $(x, z) \in E(G)$. Clearly, $d_G(z) \leq 2$. Now we delete the edge $(x, y)$ from $F_2$ and introduce $(x, z)$, that is, $F_2 := (F_2 \setminus \{(x, y)\}) \cup \{(x, z)\}$.

Then the number of odd cycles in the subgraph induced by $F_2$ decreases by 1. We repeat this procedure until $F_2$ induces a bipartite graph.

Suppose that we have already constructed subsets $F_1, ..., F_{i-1}$ where $2 < i \leq \Delta(G)$ and $F_1 \subset ... \subset F_{i-1}$. At each vertex $x$ with $d_G(x) > i$ delete precisely $d_G(x) - i$ edges from $E(G) \setminus F_{i-1}$. The remaining set of edges we denote by $F_i$. It is not difficult to check that every alternating path $P = a_0a_1 \ldots a_{2r+1}$ relative to $F_i$ with end edges in $E(G) \setminus F_i$ has the property that if $a_0$ is incident with less than $i$ edges of $F_i$ then each of the vertices $a_1, a_3, \ldots, a_{2r+1}$ is incident with exactly $i$
edges of \( F_i \), and each of the vertices \( a_2, a_4, \ldots, a_{2r} \) is incident with less than \( i \) edges of \( F_i \). This property and Lemma 2.3 imply that \( F_i \) is a maximum \( i \)-matching of \( G \).

By repeating this process we obtain the sets \( F_1, \ldots, F_\Delta \) such that \( F_i \) is a maximum \( i \)-matching of \( G \), for \( i = 1, \ldots, \Delta \), and \( F_1 \subset F_2 \subset \ldots \subset F_\Delta \).

Let \( H_i \) be the subgraph induced by the set \( F_i, i = 1, \ldots, \Delta \). Since \( H_2 \) is a bipartite graph with \( \Delta(H_2) = 2 \) and \( F_1 \) is a maximum matching, \( q_1(G) \geq q_2(G) - q_1(G), \) that is, \( h_1 \geq h_2 \). Moreover, it is not difficult to see that the edges of \( H_2 \) can be colored with colors 1 and 2 such that \( h_1 \) edges colored 1 and \( h_2 \) edges colored 2.

Suppose that we have already properly colored edges in \( F_i \) with \( i \geq 2 \) colors 1, ..., \( i \) such that precisely \( h_j \) edges colored \( j \), for \( j = 1, \ldots, i \). If \( i < \Delta \) then, by Proposition 3.1, edges in \( F_{i+1} \) can properly colored with \( i+1 \) colors such that at least \( h_j \) edges are colored \( j \), for \( j = 1, 2, \ldots, i \). The condition \( \sum_{j=1}^{i} h_j = q_j(G) \) implies that under this coloring precisely \( h_j \) edges receive color \( j \), for each \( j = 1, 2, \ldots, i+1 \).

By repeating this process we obtain a proper \( \Delta \)-coloring corresponding to the sequence \( H(G) \). Therefore, by Remark 2.1, \( D_q(G) \) is the set of all color-feasible sequences for \( G \). \( \blacksquare \)

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