Two sensitivity theorems in fuzzy integer programming *

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Abstract: We consider the problem of estimating optima of covering integer linear programs with 0-1 variables under the following conditions: we do not know exact values of elements in the constraint matrix A but we know what elements of A are zero and what are nonzero, and also know minimal and maximal values of nonzero elements. We find bounds for variation of the optima of such programs in the worst and average cases. We also find some conditions guaranteeing that the variation of the optimum of such programs in the average case is close to 1 as the number of variables tends to infinity. This means that the values of nonzero elements in A can vary without significantly affecting the value of the optimum of the integer program.

Key words: integer programming, sensitivity

1 Introduction

Linear and integer linear programming are known to capture well optimization problems relevant to high-speed networks [2, 9]. Often in such problems data are not known exactly. The reasons for such uncertainty can be different: for example, data may vary quickly (as in economics, traffics in

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networks, etc.) or a part of the data may even be unknown. What can one state about the solutions of such problems?

In [9] a model for linear programming with incomplete information (distributed among different agents) was presented. This model assumes, in particular, that we know what elements of the matrix of constraints are zero and what are nonzero. The following question was asked in [9] “What if the uncertainty about the coefficients is not complete, but they are known within some margin of error, or even have a known distribution?”.

We consider this question for a special class of covering integer linear programs with 0-1 variables. Note that different types of covering integer programs were intensively investigated [3, 5, 8, 10, 12].

Let $\mathbf{b} = (b_1, ..., b_m)^T > \mathbf{0}$ be a rational vector and M be a real number, $M \geq 1$. We consider integer programs of the form

$$\min \sum_{j=1}^{n} x_j \tag{1}$$

$$A\mathbf{x} \geq \mathbf{b} \tag{2}$$

$$0 \leq x_j \leq 1, \quad x_j - \text{ integer}, \quad j = 1, \dots, n. \tag{3}$$

where $\mathbf{x} = (x_1, \dots, x_n)^T$ and $A = (a_{ij})$ is a rational $m \times n$ "fuzzy" matrix: we do not know the elements of A exactly but we know

a) zero-nonzero pattern of A, i.e. what elements of A are zero and what are nonzero. This will be specified by a $(0, 1)$-matrix $G = (g_{ij})$ where $g_{ij} = 0$ if and only if $a_{ij} = 0$.

b) for every nonzero element we know that it is in the interval $[1, M]$, that is, $1 \leq a_{ij} \leq M$ holds for any $i,\ j$ such that $a_{ij} \neq 0$.

The value $\min \sum_{j=1}^{n} x_j$ is called the *integral optimum* of (1)-(3) and denoted by $Z(A, \mathbf{b})$. By a *feasible solution* of (1)-(3) we will mean any vector $\mathbf{x}$ satisfying (2)-(3).

The assumptions (a) and (b) take place, say, when combinatorial structure (zero-nonzero pattern) does not vary during long time but the values of nonzero elements vary during short time.

What can we do in this situation? Of course, we can solve each new integer program every time when we obtain new data. But there is a simple strategy to estimate optima variation: find maximum and minimum value of the optimum for such class of integer programs. Clearly the optimum of each program from this class will be between these two values. But how close are these values to each other?

In general case this problem seems difficult and is related to sensitivity of integer programs [4]. In this paper we consider the worst and average cases. We prove that even in the case when the variation of nonzero elements is small (i.e. M is close to 1) there are situations where the integral optimum can vary significantly. On the other hand we find some conditions guaranteeing that the variation of integral optimum in the average case (over all zero-nonzero patterns) is close to 1 as the number of variables tends to infinity. This means that the values of nonzero elements in A can vary without significantly affecting the value of the optimum of the integer program. The key step in our proof is the reduction of the problem to estimating the size of multiple covering of a $(0,1)$-matrix along with using combinatorial and probabilistic methods for the last problem.

2 Worst case analysis

Given (0,1)-matrix G and a number M, $M \geq 1$, let $K(G,M)$ denote the class of all matrices $A = (a_{ij})$ with zero-nonzero pattern G satisfying the condition $1 \leq a_{ij} \leq M$ for any pair $i,\ j$ such that $a_{ij} \neq 0$. Let

$$R(G,M,\mathbf{b}) = \max_{A_1,A_2 \in K(G,M)} \frac{Z(A_1,\mathbf{b})}{Z(A_2,\mathbf{b})}.$$

It is not difficult to see that

$$\max_{A \in K(G,M)} Z(A,\mathbf{b}) = Z(G,\mathbf{b}),$$

and

$$\min_{A \in K(G,M)} Z(A,\mathbf{b}) = Z(MG,\mathbf{b}),$$

where MG denotes the matrix where all 1's in G are replaced by M. Thus,

$$R(G,M,\mathbf{b}) = \frac{Z(G,\mathbf{b})}{Z(MG,\mathbf{b})}.$$

Moreover, it is clear that instead of the system

$$MG\mathbf{x} \geq \mathbf{b},$$

we can consider the equivalent system (by dividing each inequality by M)

$$G\mathbf{x} \geq \mathbf{b}/M.$$

This shows that

$$R(G, M, \mathbf{b}) = Z(G, \mathbf{b})/Z(MG, \mathbf{b}) = Z(G, \mathbf{b})/Z(G, \mathbf{b}/M).$$

In other words, we can estimate this ratio for integer programs with the same $(0, 1)$-matrix and different right-hand sides $\mathbf{b}$ and $\mathbf{b}/M$.

The most interesting question is, of course, when is this ratio close to 1? When this is the case, our upper and lower bounds for integer optima are close to each other and it is not necessary to solve every new integer program (1)-(3) with new data because we already have a good approximation of the optimum. In section 3 we find some conditions guaranteeing that this ratio is close to 1 for almost all G.

Note that for some G the value of $R(G, M, \mathbf{b})$ can be large even if M is close to 1. Consider the following example.

Let $G_0 = (g_{ij})$ be a (0,1)-pattern where $g_{i1} = 1$, $g_{ii} = 1$, for $i = 1, \dots, m$ and other elements of G_0 are zeroes. Furthermore, let δ be a small positive number, $M = 1 + \delta$ and let the vector $\mathbf{b}$ satisfy the condition: $b_1 = 1$ and $1 < b_i < 1 + \delta$ for $i = 2, ..., m$.

Then we have $R(G_0, M, \mathbf{b}) = Z(G_0, \mathbf{b})/Z(G, \mathbf{b}/M) = n$ because $Z(G_0, \mathbf{b}/M) = 1$ with the optimal solution $\mathbf{x} = (1, 0, ..., 0)^T$, and $Z(G_0, \mathbf{b}) = n$ with the optimal solution $\mathbf{x} = \mathbf{1} = (1, 1, ..., 1)^T$.

It seems difficult to find $R(G, M, \mathbf{b})$ for arbitrary G. For this reason we investigate the worst case of G. We consider only patterns G for which the inequality $A\mathbf{x} \geq \mathbf{b}$ has a nonnegative 0-1 solution.

Theorem 1.

$$1 + \lceil M - 1 \rceil m \leq \max_{G,n,\mathbf{b}} R(G, M, \mathbf{b}) \leq \lceil M \rceil m.$$

Proof. 1. *Lower bound.* Let G_0 be a matrix of size $m \times 1$ where all elements are equal to 1, and let $G_1, ..., G_{\lceil M-1 \rceil}$ be $m \times m$ diagonal $(0, 1)$-matrices.

Consider the program (1)-(3) where $\mathbf{b} = (M, \dots, M)^T$, $n = \lceil M - 1 \rceil m$ and the pattern G of A is the concatenation of matrices $G_0, G_1, ..., G_{\lceil M-1 \rceil}$.

Clearly, $Z(G, \mathbf{b}/M) = Z(G, \mathbf{1}) = 1$ with the optimal solution $\mathbf{x} = (1, 0, ..., 0)^T$. On the other hand, $Z(G, \mathbf{b}) = 1 + \lceil M - 1 \rceil m$ with the optimal solution $\mathbf{x} = \mathbf{1} = (1, 1, ..., 1)^T$. We have (see Section 2) that

$$R(G, M, \mathbf{b}) = Z(G, \mathbf{b})/Z(G, \mathbf{b}/M).$$

Hence,

$$R(G, M, \mathbf{b}) \geq 1 + \lceil M - 1 \rceil m,$$

where $n = \lceil M - 1 \rceil m$, $\mathbf{b} = (M, ..., M)^T$ and G is the $(0, 1)$-matrix defined above. Therefore

$$\max_{G,n,\mathbf{b}} R(G, M, \mathbf{b}) \geq 1 + \lceil M - 1 \rceil m.$$

2. *Upper bound.* Consider an arbitrary program (1)-(3) with an arbitrary pattern G. We have

$$R(G, M, \mathbf{b}) = Z(G, \mathbf{b})/Z(G, \mathbf{b}/M).$$

Clearly, the number of 1's in any integral (0,1)-solution of the system $G\mathbf{x} \geq \mathbf{b}/M$ must be at least $\lceil \frac{b_{\max}}{M} \rceil$. Therefore,

$$Z(G, \mathbf{b}/M) \geq \lceil \frac{b_{\max}}{M} \rceil.$$

On the other hand, the number of 1's in any integral (0,1)-solution of the system $G\mathbf{x} \geq \mathbf{b}$ is at most $m\lceil b_{\max} \rceil$. Therefore,

$$Z(G, \mathbf{b}) \leq \lceil b_{max} \rceil m = \lceil \frac{b_{max}}{M} M \rceil m \leq \lceil M \rceil \lceil \frac{b_{max}}{M} \rceil m.$$

Thus,

$$R(G, M, \mathbf{b}) = Z(G, \mathbf{b})/Z(G, \mathbf{b}/M) \leq \lceil M \rceil m$$

for arbitrary G, n and $\mathbf{b}$.

3 Average case analysis

In this section we investigate the variation of the integral optimum in the average case, over all zero-nonzero patterns. To do this we can introduce the probability distribution on G assuming that $G = (g_{ij})$ is a random (0,1)-matrix where each g_{ij} independently takes value 1 with probability 1/2, and 0 with probability 1/2. Then the expectation of $R(G, M, \mathbf{b})$ is equal to its average value

$$\sum_{G \in W} \frac{1}{2^{mn}} R(G, M, \mathbf{b})$$

where W is the set of all 0-1 matrices of size $m \times n$, that is, all possible patterns $G = (g_{ij})$.

But we will consider here a more general probabilistic model in which $G = (g_{ij})$ is a random (0,1)-matrix such that $\mathbf{P}\{g_{ij} = 1\} = p$ and $\mathbf{P}\{g_{ij} = 0\} = 1 - p$ independently for all i, j. Then, for any M and $\mathbf{b}$ the value $R(G, M, \mathbf{b})$ is a random variable.

Our main result is the following theorem.

Theorem 2. Consider the problem (1)-(3) with a random G defined above. Let the probability p be such that $p \leq c < 1$ for some constant c and let $\max_i b_i = o(\ln mp)$, and

$$\frac{\ln \ln n}{\ln mp} \to 0 \text{ as } n \to \infty, \tag{4}$$

$$\frac{\ln m}{np} \to 0 \text{ as } n \to \infty. \tag{5}$$

Then for any fixed $\varepsilon > 0$ $\mathbf{P}\{R(G, M, \mathbf{b}) \leq 1 + \varepsilon\} \to 1$ as $n \to \infty$.

This means that for most zero-nonzero patterns under the conditions (1)-(2) the variation of values of nonzero elements does not significantly affect the value of the optimum of (1)-(3), that is a very desirable situation.

Corollary 1. Let $m = cn$, where c is some constant and $p = n^{-(1-\delta)}$, for some constant $\delta > 0$. Consider the problem (1)-(3) with a random G defined above. Let $\max_i b_i = o(\ln n)$ as $n \to \infty$. Then, for any fixed $\varepsilon > 0$ $\mathbf{P}\{R(G, M, \mathbf{b}) \leq 1 + \varepsilon\} \to 1$ as $n \to \infty$.

Corollary 1 follows from Theorem 2 because the conditions (4) and (5) of Theorem 2 obviously hold when $m = cn$, $p = n^{-(1-\delta)}$.

The key roles in the proof of Theorem 2 are played by the concept of a cover and r-cover of a 0-1 matrix.

Definition. A *cover* of (0,1)-matrix G is a subset of its columns such that each row contains at least one 1 in these columns. The number of columns in a cover is called the *size* of the cover. An r-*cover* is a subset of columns such that each row contains at least r 1's in these columns.

We will also use the following two lemmas.

Lemma 1 [1]. Let Y be a sum of n independent random variables each taking the value 1 with probability p and 0 with probability $1 - p$. Then, for any $\delta > 0$

$$\mathbf{P}\{|Y - np| > \delta np\} \leq 2 \exp\{-(\delta^2/3) np\}.$$

Lemma 2. Let g_{ij} be independent random variables such that $\mathbf{P}\{g_{ij} = 1\} = p$ and $\mathbf{P}\{g_{ij} = 0\} = 1 - p$, for each pair $i,\ j, 1 \le i \le m, 1 \le j \le n$. If the condition (5) holds then, for any $\delta > 0$ with probability tending to 1 as n tends to infinity, every row of the random matrix $G = (g_{ij})$ contains at least $(1-\delta)pn$ and at most $(1+\delta)pn$ 1's.

Proof. Let $Y_i = \sum_{i=1}^{n} g_{ij}$ and A_i be the event that $|Y_i - pn| > \delta pn$, $i = 1, ..., m$. Lemma 1 implies that

$$\mathbf{P}\{A_i\} \le 2\exp\{-(\delta^2/3)np\}.$$

This and condition (5) imply that

$$\mathbf{P}\{\bigcup_{i=1}^{m} A_i\} \le \sum_{i=1}^{m} \mathbf{P}\{A_i\} \le 2m\exp\{-(\delta^2/3)np\} =$$

$$= 2\exp\{\ln m - O(n)\} \to 0$$

as $n \to \infty$. Clearly, $\overline{\bigcup_{i=1}^{m} A_i}$ is the event: each row of G contains at least $(1-\delta)pn$ and at most $(1+\delta)pn$ 1's. We have

$$\mathbf{P}\{\overline{\bigcup_{i=1}^{m} A_i}\} = 1 - \mathbf{P}\{\bigcup_{i=1}^{m} A_i\} \to 1 \text{ as } n \to \infty.$$

Proof of Theorem 2. We have (see Section 2) that

$$R(G, M, \mathbf{b}) = Z(G, \mathbf{b})/Z(G, \mathbf{b}/M).$$

To give an upper bound of $R(G, M, \mathbf{b})$ we will give a lower bound of $Z(G, \mathbf{b}/M)$ and an upper bound of $Z(G, \mathbf{b})$.

1. *Lower bound of* $Z(G, \mathbf{b}/M)$. Every feasible solution $\mathbf{x} = (x_1, ..., x_n)$ of the covering integer program (1)-(3) in the case when $\mathbf{b} = \mathbf{1}$, represents a cover of the (0,1)-matrix (pattern) G corresponding to A: if

$$S = \{i : x_i = 1,\ 1 \le i \le n\}$$

then the set of columns $\{C_i : i \in S\}$ in G is a cover of G. Therefore, $Z(G, \mathbf{1})$ is the size of a minimum cover of G.

Furthermore, it is clear that

$$Z(G, \mathbf{b}/M) \ge Z(G, \mathbf{1}),$$

because $\mathbf{b}$ is a positive vector, $\mathbf{x}$ is a (0,1)-vector and the elements of the matrix G are zeroes and ones. Now we derive a lower bound of $Z(G, \mathbf{1})$, the size of a minimum cover of G.

Let X be the random variable equal to the number of covers in G of size $l_0 = -\lceil (1-\delta)\ln mp/\ln(1-p)\rceil$. Clearly,

$$\mathbf{E}X = \binom{n}{l_0} P(l_0),$$

where $P(l_0)$ is the probability that fixed l_0 columns form a cover in G. It is not difficult to see that

$$P(l_0) = \Big(1-(1-p)^{l_0}\Big)^m \le \exp\{-m(1-p)^{l_0}\}.$$

Thus, using the inequality $\binom{n}{k} \le n^k$, we have

$$\ln \mathbf{E}X \le l_0 \ln n - m(1-p)^{l_0} \le (-\ln mp/\ln(1-p))\ln n -$$

$$-m\exp\{-(1-\delta)\ln(1-p)\frac{\ln mp}{\ln(1-p)}\} =$$

$$-(\ln mp/\ln(1-p))\ln n - m(mp)^{-1+\delta} =$$

$$-(\ln mp/\ln(1-p))\ln n - p^{-1}(mp)^{\delta}.$$

It is not difficult to see that for any fixed $0 < \delta < 1$ under the condition (4) the last expression tends to $-\infty$ as n tends to infinity. This and Markov's inequality $\mathbf{P}\{X \ge 1\} \le \mathbf{E}X$ imply that $\mathbf{P}\{X \ge 1\} \to 0$ as $n \to \infty$.

Thus, the probability that there is no cover of size l_0 in a random $(0,1)$-matrix G tends to 1. Clearly, if there is no cover of size l_0 in G then there is also no cover of size smaller than l_0. Therefore, $\mathbf{P}\{Z(G,\mathbf{1}) > l_0\} \to 1$ as $n \to \infty$. Since $Z(G, \mathbf{b}/M) \ge Z(G, \mathbf{1})$, we have

$$\mathbf{P}\{Z(G, \mathbf{b}/M) > l_0\} \to 1$$

as $n \to \infty$.

2. *Upper bound of Z(G,*$\mathbf{b}$*).* Let $B_1 = \max_i b_i$ and $\mathbf{B_1} = (B_1, \ldots, B_1)$. Furthermore, let $B = 1 + B_1$ and $\mathbf{B} = (B, \ldots, B)$. It is clear that $B > 1$ and $Z(G, \mathbf{b}) \le Z(G, \mathbf{B_1}) \le Z(G, \mathbf{B})$. On the other hand, it is not difficult to see that $Z(G, \mathbf{B})$ is exactly the minimum size of a B-cover of the $(0,1)$-matrix G.

Let k be a positive integer such that each row of G contains at least k 1's. Then we use the estimate of the size of a B-cover ([7]) obtained by averaging over all l-subsets of columns of G:

$$Z(G, \mathbf{B}) \leq \min_l \{l + \frac{m}{\binom{n}{l}} \sum_{i=0}^{B-1} \binom{k}{i} \binom{n-k}{l-i}\}.$$

The following simple proposition can be proved by induction by considering the ratio of two consecutive members of such a sequence.

Proposition 1. If $l \geq (n/k)B$ then

$$\max_i \binom{k}{i} \binom{n-k}{l-i} = \binom{k}{B-1} \binom{n-k}{l-B+1}.$$

Thus, by Proposition 1,

$$Z(G, \mathbf{B}) \leq \min_{l \geq (n/k)B} \{l + \frac{Bm}{\binom{n}{l}} \binom{k}{B-1} \binom{n-k}{l-B+1}\}.$$

By Lemma 2, with probability tending to 1,

$$k \geq (1-\delta)pn \tag{6}$$

for any $\delta > 0$. In Section 5 we prove the following proposition.

Proposition 2. Under the conditions of Theorem 2

$$\min_{l \geq (n/k)B} \{l + \frac{Bm}{\binom{n}{l}} \binom{k}{B-1} \binom{n-k}{l-B+1}\} \leq l_1 + n/k$$

where

$$l_1 = \lceil (1+\delta) \frac{\ln \frac{mk}{n}}{\ln \frac{n}{n-k}} \rceil.$$

Note that $l_1 \geq (n/k)B$ since the condition $B = o(\ln mp)$ and inequality (6) imply that $(n/k)B = o(l_1)$. Thus, Proposition 2 implies that $Z(G, \mathbf{B}) \leq l_1 + n/k$. Let C be the event that

$$Z(G, \mathbf{B}) \leq (1+\delta) \frac{\ln mp}{\ln \frac{1}{1-p(1-\delta)}} + ((1+\delta)p)^{-1}.$$

Inequality (6) and the inequality $Z(G, \mathbf{b}) \le l_1 + n/k$ imply that $\mathbf{P}\{C\} \to 1$ as $n \to \infty$. On the other hand (see the proof of the lower bound), the probability of the event D:

$$Z(G, \mathbf{b}/M) > l_0$$

tends to 1 as $n \to \infty$. Clearly, $\mathbf{P}\{C \cap D\} \to 1$ as $n \to \infty$. Combining the upper bound of $Z(G, \mathbf{b})$ and the lower bound of $Z(G, \mathbf{b}/M)$ we obtain that for any fixed $\varepsilon > 0$ the probability of the event

$$\frac{Z(G, \mathbf{b})}{Z(G, \mathbf{b}/M)} \le 1 + \varepsilon.$$

tends to 1 as $n \to \infty$. This implies that

$$\mathbf{P}\{R(G, M, \mathbf{b}) \le 1 + \varepsilon\} \to 1$$

as $n \to \infty$ because $R(G, M, \mathbf{b}) = Z(G, \mathbf{b})/Z(G, \mathbf{b}/M)$. The proof of Theorem 2 is complete.

4 Proof of Proposition 2

We will use the inequalities (see [6], p. 243)

$$G(n, \lambda) > \binom{n}{\lambda n} > \frac{\sqrt{\pi}}{2} G(n, \lambda), \tag{7}$$

where $0 < \lambda < 1,\ n \ge 2$ and

$$G(n, \lambda) = \frac{\lambda^{-\lambda n}(1-\lambda)^{-(1-\lambda)n}}{\sqrt{2\pi\lambda(1-\lambda)n}}.$$

To prove Proposition 2 it is enough to show that

$$\frac{B}{\binom{n}{l_1}} \binom{k}{B-1} \binom{n-k}{l_1 - B + 1} \le \frac{n}{mk}$$

where

$$l_1 = \lceil (1+\delta) \frac{\ln \frac{mk}{n}}{\ln \frac{n}{n-k}} \rceil.$$

We will prove the above inequality in the following equivalent form:

$$\frac{\binom{n}{l_1}}{B}\binom{k}{B-1}^{-1}\binom{n-k}{l_1-B+1}^{-1} \geq \frac{mk}{n}.$$

Denote the left-hand side of this inequality by L. Applying the inequality (7) we have

$$L \geq (B\binom{k}{B-1})^{-1}\frac{\sqrt{\pi}}{2}G(n,\lambda_1)/G(n-k,\lambda_2), \tag{8}$$

where $\lambda_1 = l_1/n$, and $\lambda_2 = (l_1 - B + 1)/(n-k)$.

We have

$$\frac{G(n,\lambda_1)}{G(n-k,\lambda_2)} = F\exp\{l_1 \ln\frac{n}{l_1} - (n-l_1)\ln(1-\frac{l_1}{n}) -$$

$$(l_1 - B + 1)\ln\frac{n-k}{l_1-B+1} - (n-k-l_1+B-1)\ln\frac{n-k}{n-k-l_1+B-1}\},$$

where

$$F = \sqrt{\frac{\lambda_2(1-\lambda_2)(n-k)}{\lambda_1(1-\lambda_1)n}} \geq \sqrt{(1-\frac{B-1}{l_1})(1-\frac{l_1-B+1}{n-k})}.$$

The conditions of Theorem 2 ($B = o(\ln mp)$ and $\ln m = o(np)$) and Lemma 2 ($pn(1+\delta) \geq k \geq (1-\delta)pn$) imply that $F = 1 - o(1)$.

Using the inequalities

$$-x \leq -\ln(1+x), \qquad \ln(1-x) \leq -x, \qquad 0 < x < 1,$$

we have

$$\ln(\frac{G(n,\lambda_1)}{G(n-k,\lambda_2)}) \geq \ln F + l_1 - \frac{l_1^2}{n} + l_1\ln\frac{n}{l_1} - (l_1-B+1)\ln\frac{n-k}{l_1-B+1} -$$

$$-(n-k-l_1+B-1)\frac{l_1-B+1}{n-k-l_1+B-1}.$$

Combining this inequality with inequality (8) and taking into account that $B > 1$ and

$$\binom{k}{B-1} \leq \left(\frac{ek}{B-1}\right)^{B-1},$$

we obtain

$$\ln L \geq O(1) - \ln B - (B-1)\ln\frac{ek}{B-1} + (B-1)\ln\frac{n-k}{l_1-B+1} -$$

$$-\frac{l_1^2}{n} + B - 1 + l_1\ln\frac{n(l_1-B+1)}{l_1(n-k)}. \tag{9}$$

The conditions $pn(1+\delta) \geq k \geq (1-\delta)pn$, $B = o(\ln mp)$ and $\ln m = o(np)$ imply that $l_1 = o(n)$, $B = o(pl_1)$, and

$$-(B-1)\ln\frac{ek}{B-1} + (B-1)\ln\frac{n-k}{l_1-B+1} =$$

$$= (B-1)\ln\frac{(B-1)(n-k)}{(l_1-B+1)ek} = O((B-1)\ln\frac{B}{pl_1}).$$

Let $pl_1/B = \varphi$. We have $\varphi \to \infty$ and

$$-(B-1)\ln\frac{pl_1}{B} \geq -B\ln\frac{pl_1}{B} = -\frac{pl_1}{\varphi}\ln\varphi = o(l_1).$$

Hence, the right-hand side of the inequality (9) is

$$(1-o(1))l_1\ln\frac{n}{n-k} \geq (1-o(1))(1+\delta)\ln(mk/n),$$

and the inequality $\ln L \geq \ln(mk/n)$ holds for sufficiently large n.

5 Concluding remarks

In this paper we considered covering programs with 0-1 variables and cost function of the form $\sum_j x_j$ under the assumption that we know a pattern of coefficients in constraints which are nonzero and only those coefficients can vary in some interval $[1, M]$ (zero elements do not change their value). In our main result we found some sufficient conditions guaranteeing that the variation of integral optimum in the average case (over all zero-nonzero patterns) is close to 1 as the numbers of variables tends to infinity. This means that for typical patterns the values of nonzero elements in A can vary without affecting significantly the value of the optimum of the integer program (that is the optimum value depends mostly on pattern but not on values of nonzero elements).

At the same time the typical patterns depends heavily on the probability measure. For example, if the value of the probability p is constant, the matrix A is in fact dense. If, however, the value of p tends to zero when the number of variables increases, the matrix A is sparse. All the restrictions in Theorem 2 also depend on the probability p. That is why it is difficult to describe explicitly the class of programs given by Theorem 2.

However it seems interesting to find similar conditions for more general classes of integer programs where negative elements are allowed.

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