Pattern avoidance and Boolean elements in the Bruhat order on involutions

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Abstract We show that the principal order ideal of an element w in the Bruhat order on involutions in a symmetric group is a Boolean lattice if and only if w avoids the patterns 4321, 45312 and 456123. Similar criteria for signed permutations are also stated. Involutions with this property are enumerated with respect to natural statistics. In this context, a bijective correspondence with certain Motzkin paths is demonstrated.

1 Introduction

The Bruhat order on a Coxeter group is fundamental in a multitude of contexts. For example, the incidences among the closed cells in the Bruhat decomposition of a flag variety are governed by the Bruhat order on the corresponding Weyl group.

In spite of its importance, the Bruhat order is in many ways poorly understood. For example, much about the structure of intervals, or even principal order ideals, remains unclear. There are, however, several known connections between structural properties of principal order ideals in the Bruhat order and pattern avoidance properties of the corresponding group elements. Here are some examples:

– A Schubert variety is rationally smooth if and only if the corresponding Bruhat order ideal is rank-symmetric; see [3]. These properties have been

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characterized in terms of pattern avoidance by Lakshmibai and Sandhya [10] (type A) and Billey [1] (types B, C, D).
- Gasharov and Reiner [5] have shown that a Schubert variety is “defined by inclusions” precisely when the corresponding permutation avoids certain patterns. By work of Sjöstrand [12], these permutations are precisely those whose Bruhat order ideal is defined by the “right hull” of the permutation.
- Tenner [14] has demonstrated that the permutations whose Bruhat order ideals are Boolean lattices can be characterized in terms of pattern avoidance. By general theory, this characterizes the lattices among all principal order ideals in the Bruhat order.

An interesting subposet of the Bruhat order is induced by the involutions. Activity around this subposet was spawned by Richardson and Springer [11] who established connections with algebraic geometry that resemble (and, in some sense, generalize) the situation in the full Bruhat order. For example, the (dual of the) Bruhat order on the involutions in the symmetric group $\mathfrak{S}_{2n+1}$ encodes the incidences among the closed orbits under the action of a Borel subgroup on the symmetric variety $SL_{2n+1}(\mathbb{C})/SO_{2n+1}(\mathbb{C})$; cf. [11, Example 10.3].

Recently, it has been shown that the Bruhat order on involutions has many combinatorial and topological properties in common with the full Bruhat order [6, 9]. The purpose of this paper is to incorporate pattern avoidance into this picture. Specifically, we shall study analogues for involutions of the aforementioned results of Tenner.

Our main result is as follows:

Theorem 1.1 *The principal order ideal generated by an involution w in the Bruhat order on the involutions in a symmetric group is a Boolean lattice if and only if w avoids the patterns* 4321, 45312 *and* 456123.

The remainder of this paper is organised in the following way. In the next section, we recall standard definitions and agree on notation. That section also includes a brief review of some results on involutions in Coxeter groups. After that, we turn to the proof of Theorem 1.1 in Section 3. A corresponding result for signed permutations (the type B case) is also given. Section 4 is devoted to enumerative results; we count involutions with Boolean principal order ideals with respect to various natural statistics, and a bijective correspondence with certain Motzkin paths is constructed. Finally, we suggest a direction for further research in Section 5.

2 Preliminaries

2.1 Permutations and patterns

Let $\mathfrak{S}_n$ denote the symmetric group consisting of all permutations of $[n] = \{1, \dots, n\}$.

An *inversion* of $\pi \in \mathfrak{S}_n$ is a pair (i,j) such that $i < j$ and $\pi(i) > \pi(j)$. The number of inversions of π is denoted by $\mathrm{inv}(\pi)$.

The *excedances* and the *deficiencies* of $\pi \in \mathfrak{S}_n$ are the indices $i \in [n]$ such that $\pi(i) > i$ and $\pi(i) < i$, respectively. We use $\mathrm{exc}(\pi)$ to denote the number of excedances of π.

Given $\pi \in \mathfrak{S}_n$ and $p \in \mathfrak{S}_m$ (with $m \le n$), say that π *contains the pattern* p if there exist $1 \le i_1 < \cdots < i_m \le n$ such that for all $j,k \in [m]$, $\pi(i_j) < \pi(i_k)$ if and only if $p(j) < p(k)$. In this case, say that $\langle p \rangle = (\pi(i_1), \dots, \pi(i_m))$ is an *occurrence* of p in π. Furthermore, we write $\langle p(j) \rangle = \pi(i_j)$ for $j \in [m]$.

If π does not contain p, it *avoids* p.

Example 2.1 Consider $\pi = 84725631 \in \mathfrak{S}_8$. It has several occurrences of the pattern 4231; two of them are $(8,5,6,1)$ and $(8,4,5,3)$.

Recall that an *involution* is an element of order at most two. At times, we shall find it convenient to represent an involution $w \in \mathfrak{S}_n$ by the graph on vertex set $[n]$ in which two vertices are joined by an edge if they belong to the same 2-cycle in w. For an example, see Figure 3.2.

2.2 Coxeter groups

Here, we briefly review those facts from Coxeter group theory that we need in the sequel. For more details, see [2] or [8].

A *Coxeter group* is a group W generated by a finite set S of involutions where all relations among the generators are derived from equations of the form $(ss')^{m(s,s')} = e$ for some $m(s,s') = m(s',s) \ge 2$, where $s,s' \in S$ are disctinct generators. Here, $e \in W$ denotes the identity element. The pair (W,S) is referred to as a *Coxeter system*.

We may specify a Coxeter system using its *Coxeter graph*. This is an edge-labelled complete graph on vertex set S where the edge $\{s,s'\}$ has the label $m(s,s')$. For convenience, edges labelled 2 and edge labels that equal 3 are suppressed from the notation.

Let (W,S) be a Coxeter system. Given $w \in W$, suppose k is the smallest number such that $w = s_1 \cdots s_k$ for some $s_i \in S$. Then k is the *length* of w, denoted $\ell(w)$, and the word $s_1 \cdots s_k$ is called a *reduced expression* for w.

The set of *reflections* of W is $T = \{wsw^{-1} : w \in W, s \in S\}$. Define the *absolute length* $\ell'(w)$ to be the smallest k such that w is a product of k reflections.

Example 2.2 The symmetric group $\mathfrak{S}_n$ is a Coxeter group with the adjacent transpositions $\mathfrak{s}_i = (i, i+1)$, $i \in [n-1]$, as Coxeter generators. Its Coxeter graph is simply a path on $n-1$ vertices. In this setting, $\ell(w) = \mathrm{inv}(w)$.

In the $\mathfrak{S}_n$ case, T is the set of transpositions. It is well-known that the minimum number of transpositions required to express $w \in \mathfrak{S}_n$ as a product is $n - c(w)$, where $c(w)$ is the number of cycles in the disjoint cycle decomposition of w. In particular, if $w \in \mathfrak{S}_n$ is an involution, $\ell'(w)$ is the number of 2-cycles in w. In other words, $\ell'(w) = \mathrm{exc}(w)$.

The *Bruhat order* is the partial order on W defined by $u \leq w$ if and only if $w = ut_1 \cdots t_m$ for some $t_i \in T$ such that $\ell(ut_1 \cdots t_i) < \ell(ut_1 \cdots t_{i+1})$ for all $i \in [m-1]$. Clearly, $e \in W$ is the minimum element under the Bruhat order.

2.3 Involutions in Coxeter groups

As before, let (W, S) be a Coxeter system. Denote by $\mathcal{I}(W) \subseteq W$ the set of involutions in W. We now review some results on the combinatorics of $\mathcal{I}(W)$. They can all be found in [6] or [7]. The reader who is acquainted with the subject will notice that all these properties are completely analogous to standard statements about the full group W.

Introduce a set of symbols $\underline{S} = \{\underline{s} : s \in S\}$. By abuse of notation, we will denote the symbols corresponding to the generators $\mathfrak{s}_i$ of the symmetric group $\mathfrak{S}_n$ by $\underline{\mathfrak{s}}_i$ instead of $\underline{\mathfrak{s}_i}$ for better readability. Define an action of the free monoid $\underline{S}^*$ from the right on (the set) W by

$$w\underline{s} = \begin{cases} ws & \text{if } sws = w, \\ sws & \text{otherwise,} \end{cases}$$

and $w\underline{s}_1 \cdots \underline{s}_k = (\cdots(w\underline{s}_1)\underline{s}_2 \cdots)\underline{s}_k$ for $w \in W$, $\underline{s}_i \in \underline{S}$. By abuse of notation, we write $\underline{s}_1 \cdots \underline{s}_k$ instead of $e\underline{s}_1 \cdots \underline{s}_k$. The elements of this kind are precisely the involutions in W:

Proposition 2.3 *The orbit of e under the $\underline{S}^*$-action is $\mathcal{I}(W)$.* □

When $w \in \mathcal{I}(W)$, the condition $sws = w$ which appears in the definition of the $\underline{S}^*$-action is equivalent to $\ell(sws) = \ell(w)$.

If $w = \underline{s}_1 \cdots \underline{s}_k$ for some $\underline{s}_i \in \underline{S}$, then the sequence $\underline{s}_1 \cdots \underline{s}_k$ is called an *$\underline{S}$-expression* for w. This expression is *reduced* if k is minimal among all such expressions. In this case, k is called the *rank* and denoted $\rho(w)$.

Proposition 2.4 (Deletion property) *Suppose $\underline{s}_1 \cdots \underline{s}_k$ is an $\underline{S}$-expression for w which is not reduced. Then, $w = \underline{s}_1 \cdots \widehat{\underline{s}_i} \cdots \widehat{\underline{s}_j} \cdots \underline{s}_k$ for some $1 \leq i < j \leq k$, where a hat means omission of that element.* □

Let $\mathrm{Br}(\mathcal{I}(W))$ denote the subposet of the Bruhat order on W induced by $\mathcal{I}(W)$. Next, we recall a convenient characterization of its order relation.

Proposition 2.5 (Subword property) *Suppose that $\underline{s}_1 \cdots \underline{s}_k$ is a reduced $\underline{S}$-expression for $w \in \mathcal{I}(W)$. For $u \in \mathcal{I}(W)$, we have $u \leq w$ if and only if $u = \underline{s}_{i_1} \cdots \underline{s}_{i_m}$ for some $1 \leq i_1 < \cdots < i_m \leq k$.* □

The poset $\mathrm{Br}(\mathcal{I}(W))$ is graded with rank function ρ. Furthermore, $\rho(w) = (\ell(w) + \ell'(w))/2$ for all $w \in \mathcal{I}(W)$. In fact, given a reduced $\underline{S}$-expression $\underline{s}_1 \cdots \underline{s}_k$ for $w \in \mathcal{I}(W)$, one has

$$\ell'(w) = |\{i \in [k] : \underline{s}_1 \cdots \underline{s}_i = \underline{s}_1 \cdots \underline{s}_{i-1} s_i\}|$$

and, consequently,

$$\ell(w) = \ell'(w) + 2 \cdot |\{i \in [k] : \underline{s}_1 \cdots \underline{s}_i \neq \underline{s}_1 \cdots \underline{s}_{i-1} s_i\}|.$$

3 Boolean involutions and pattern avoidance

As before, let (W, S) be a Coxeter system. For $w \in \mathcal{I}(W)$, denote by $B(w)$ the principal order ideal below w in the Bruhat order on involutions. In other words, $B(w)$ is the subposet of $\mathrm{Br}(\mathcal{I}(W))$ induced by $\{u \in \mathcal{I}(W) : u \leq w\}$.

We call an involution $w \in \mathcal{I}(W)$ *Boolean* if $B(w)$ is isomorphic to a Boolean lattice. In this section we shall prove the characterization of Boolean involutions in $\mathcal{I}(\mathfrak{S}_n)$ which was stated as Theorem 1.1.

First, we observe a useful characterization of Boolean involutions which is valid in any Coxeter group. See [14, Theorem 4.3] for an analogous statement about the full Bruhat order.

Proposition 3.1 *Let $w \in \mathcal{I}(W)$. Then w is Boolean if and only if no reduced $\underline{S}$-expression for w has repeated letters. This is the case if and only if there is an $\underline{S}$-expression for w without repeated letters.*

Proof Observe that, by the subword property, every reduced $\underline{S}$-expression of $w \in \mathcal{I}(W)$ contains the same set of letters, namely $\{\underline{s} \in \underline{S} : \underline{s} \leq w\}$. If $\underline{s}_1 \cdots \underline{s}_{k-1}$ is a reduced $\underline{S}$-expression for $w \in \mathcal{I}(W)$ and all $\underline{s}_i$, $i \in [k]$, are distinct, then $\underline{s}_1 \cdots \underline{s}_k$ is reduced, too; otherwise the deletion property would imply that $w = \underline{s}_1 \cdots \underline{s}_k \underline{s}_k$ has a reduced expression containing the letter $\underline{s}_k$, contradicting the above assertion. We conclude that every $\underline{S}$-expression containing only distinct letters is reduced. The "if" direction (of both assertions) therefore follows directly from the subword property.

Since ρ is the rank function of $\mathrm{Br}(\mathcal{I}(W))$, the elements of rank one in $[e, w]$ are the $\underline{s}_i \leq w$. Thus, if w has a reduced $\underline{S}$-expression containing repeated letters, $[e, w]$ will have fewer elements of rank one than the Boolean lattice of rank $\rho(w)$, so that w cannot be Boolean. This shows the "only if" part of the assertions. □

Remark 3.2 As a consequence of [6, Theorem 4.5], the principal order ideals in $\mathrm{Br}(\mathcal{I}(W))$ are compressible Eulerian posets in the sense of du Cloux [4]. It then follows from [4, Corollary 5.4.1], that such an ideal is a lattice if and only if it is a Boolean lattice. Thus, the Boolean involutions are precisely the involutions whose principal order ideals are lattices.

Remark 3.3 The map $w \mapsto w^{-1}$ is an automorphism of the Bruhat order on the full group W. The fixed point poset is $\mathrm{Br}(\mathcal{I}(W))$. It is easy to see that the fixed point poset of any automorphism of a Boolean lattice is itself a Boolean lattice. Therefore, an involution w is Boolean if its principal order ideal in the full Bruhat order on W is Boolean. The converse, however, does not hold. For example, 321 and 3412 are the smallest non-Boolean permutations in the full Bruhat order, but they are Boolean in the Bruhat order on involutions.

3.1 Proof of Theorem 1.1

We now proceed to prove Theorem 1.1. First, however, let us give a short outline of the idea of the proof. We shall introduce the notions of connected components and long-crossing pairs for purely technical purposes. Then, Propositions 3.8 and 3.9 establish the fact that being Boolean is equivalent to the non-existence of a long-crossing pair. Finally, we show in Proposition 3.12 that $w \in \mathcal{I}(\mathfrak{S}_n)$ has a long-crossing pair if and only if it contains one or more of the patterns 4321, 45312 and 456123.

An *orbit* of $w \in \mathfrak{S}_n$ is a set of the form $\{i, w(i), w^2(i), \ldots\} \subseteq [n]$.

Definition 3.4 Let $w \in \mathfrak{S}_n$. An interval $C \subseteq [n]$ is called a *connected component* of w if it is the union of some orbits of w and it cannot be partitioned into two intervals that also are unions of orbits of w. The permutation $w \in \mathfrak{S}_n$ is called *connected* if $[n]$ is the unique connected component of w.

For $w \in \mathfrak{S}_n$ and $D \subseteq [n]$ being any union of orbits of w, we define the *restriction* w_D of w to D by

$$w_D(i) = \begin{cases} w(i) & \text{if } i \in D, \\ i & \text{otherwise.} \end{cases}$$

We see that if $w \in \mathcal{I}(\mathfrak{S}_n)$ is an involution, then w_D is also an involution.

Example 3.5 Consider the involution $w = 532614798 \in \mathcal{I}(\mathfrak{S}_9)$. Its cycle decomposition is given by $(15)(23)(46)(89)$ and it has connected components $[1,6], \{7\}$ and $[8,9]$. The restriction of w to $D = \{1,5,8,9\}$ is the involution $w_D = 523416798 \in \mathcal{I}(\mathfrak{S}_9)$ which has connected components $[1,5], \{6\}, \{7\}$ and $[8,9]$.

Recall that, as a Coxeter group, $\mathfrak{S}_n$ is generated by the adjacent transpositions $\mathfrak{s}_i = (i, i+1)$, $i \in [n-1]$.

Let $w \in \mathcal{I}(\mathfrak{S}_n)$ have connected components $C_1, \ldots, C_k$. Then w_{C_i} belongs to the *standard parabolic subgroup* of $\mathfrak{S}_n$ generated by $\mathfrak{s}_{a_i}, \mathfrak{s}_{a_i+1}, \ldots, \mathfrak{s}_{b_i}$ where $C_i = [a_i, b_i + 1]$. In particular, those subgroups have pairwise trivial intersections and generators of different subgroups commute. This implies that the concatenation of reduced $\underline{S}$-expressions for w_{C_i} and w_{C_j} is a reduced $\underline{S}$-expression for $w_{C_i \cup C_j}$ for all $i, j \in [k]$ with $i \neq j$. The following lemma is now immediate.

Lemma 3.6 *Let $w \in \mathcal{I}(\mathfrak{S}_n)$ have connected components $C_1, \ldots, C_k$. Then the following holds:*

1. *If w_i is a reduced $\underline{S}$-expression for w_{C_i} for all $i \in [k]$, then the concatenation $w_{\pi(1)} w_{\pi(2)} \cdots w_{\pi(k)}$ is a reduced $\underline{S}$-expression for w for any $\pi \in S_k$.*
2. *$B(w) \cong B(w_{C_1}) \times \cdots \times B(w_{C_k})$.*
3. *The involution w is Boolean if and only if w_{C_i} is Boolean for all $i \in [k]$.* □

Definition 3.7 Let $w \in \mathcal{I}(\mathfrak{S}_n)$ and $i, j \in [n]$. The pair (i, j) is *long-crossing* in w if $i < j < w(j)$ and $w(i) > j + 1$.

We note that the elements i and j of a long-crossing pair (i, j) in some $w \in \mathcal{I}(\mathfrak{S}_n)$ are in the same connected component.

Proposition 3.8 (A sufficiency criterion) *Let $w \in \mathcal{I}(\mathfrak{S}_n)$. If there is no long-crossing pair (i, j) in w, then w is Boolean.*

Proof Assume that $w \in \mathcal{I}(\mathfrak{S}_n)$ has no long-crossing pair and that $n \geq 3$. Using Lemma 3.6 we can assume that w is connected (otherwise consider each connected component separately). Assume that the set of 2-cycles of w is given by $\{(i_l, w(i_l)) : l \in [k]\}$ with $i_l < w(i_l)$ for all $l \in [k]$ and $1 = i_1 < i_2 < \ldots < i_k$. Because (i_l, i_{l+1}) is not a long-crossing pair for $l \in [k-1]$, it holds that $w(i_l) \leq i_{l+1} + 1 \leq w(i_{l+1})$ and in particular $w(i_k) = n$. If now $w(i_l) < i_{l+1}$ for some $l \in [k-1]$, then $[1, w(i_l)]$ and $[w(i_l) + 1, n]$ are unions of orbits of w and w is not connected. Thus, $w(i_l) = i_{l+1} + 1$ for all $l \in [k-1]$.

Consider the involution

$$v = (1, i_2)(i_2 + 1, i_3)(i_3 + 1, i_4) \cdots (i_{k-1} + 1, i_k)(i_k + 1, n) \in \mathcal{I}(\mathfrak{S}_n).$$

An $\underline{S}$-expression for v is given by

$$\underline{\mathfrak{s}}_1 \underline{\mathfrak{s}}_2 \cdots \underline{\mathfrak{s}}_{i_2-1} \underline{\mathfrak{s}}_{i_2+1} \cdots \underline{\mathfrak{s}}_{i_3-1} \underline{\mathfrak{s}}_{i_3+1} \cdots \underline{\mathfrak{s}}_{i_k-1} \underline{\mathfrak{s}}_{i_k+1} \cdots \underline{\mathfrak{s}}_{n-1}.$$

But w is obtained by letting $\underline{\mathfrak{s}}_{i_2} \underline{\mathfrak{s}}_{i_3} \cdots \underline{\mathfrak{s}}_{i_k}$ act on v from the right, that is

$$\underline{\mathfrak{s}}_1 \underline{\mathfrak{s}}_2 \cdots \underline{\mathfrak{s}}_{i_2-1} \underline{\mathfrak{s}}_{i_2+1} \cdots \underline{\mathfrak{s}}_{i_3-1} \underline{\mathfrak{s}}_{i_3+1} \cdots \underline{\mathfrak{s}}_{i_k-1} \underline{\mathfrak{s}}_{i_k+1} \cdots \underline{\mathfrak{s}}_{n-1} \underline{\mathfrak{s}}_{i_2} \underline{\mathfrak{s}}_{i_3} \cdots \underline{\mathfrak{s}}_{i_k}$$

is an $\underline{S}$-expression for w without repeated letters, and thus w is Boolean by Corollary 3.1. □

Proposition 3.9 (A necessity criterion) *Let $w \in \mathcal{I}(\mathfrak{S}_n)$. If there is a long-crossing pair (i, j) in w, then w is not Boolean.*

Proof Fix $i, j \in [n]$ such that (i, j) is a long-crossing pair in w. Choose any cycle $(k, w(k))$ with $k \neq i, j$. By multiplication of w with $(k, w(k))$ from the right, we get an involution $w' \leq w$ with the same entries as w except that k and $w(k)$ are now fixed points. Repeat this for all cycles except $(i, w(i))$ and $(j, w(j))$, and call the resulting involution w'. Thus, the only non-fixed points of w' are i, $w(i)$, j and $w(j)$, and we have $w' \leq w$.

By conjugating w' first with $(j + 1, w(j))$, then with $(i, j - 1)$ and finally with $(j + 2, w(i))$, we get an involution $w'' \leq w'$ having the cycles $(j - 1, j + 2)$ and $(j, j + 1)$ and fixed points in all other positions. (Here, (k, k) for any $k \in [n - 1]$ should be interpreted as the identity permutation.)

A reduced $\underline{S}$-expression for w'' is given by $\underline{\mathfrak{s}}_{j-1} \underline{\mathfrak{s}}_j \underline{\mathfrak{s}}_{j+1} \underline{\mathfrak{s}}_j$, and thus w'' is not Boolean. But we have $w'' \leq w' \leq w$, and therefore w is not Boolean either. □

Example 3.10 In Figure 3.1 the steps of the proof of Proposition 3.9 are demonstrated for $w = 5764132$ and the long-crossing pair $(1, 2)$.

In fact, we have shown that $w \in \mathcal{I}(\mathfrak{S}_n)$ is Boolean if and only if $B(w)$ contains no element of the form $\underline{\mathfrak{s}}_{j-1} \underline{\mathfrak{s}}_j \underline{\mathfrak{s}}_{j+1} \underline{\mathfrak{s}}_j$. Using similar terminology as in [14], such an element may be called a *shift* of $\underline{\mathfrak{s}}_1 \underline{\mathfrak{s}}_2 \underline{\mathfrak{s}}_3 \underline{\mathfrak{s}}_2 = 4321 \in \mathcal{I}(\mathfrak{S}_4)$. Thus, 4321 in some sense is the unique minimal non-Boolean involution.

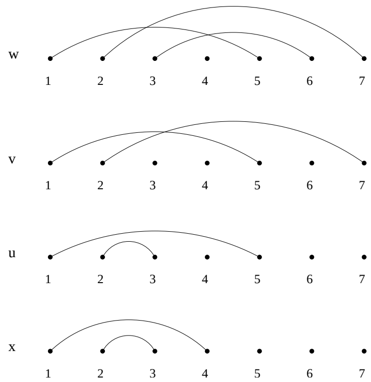


Fig. 3.1 Illustration for the proof of Proposition 3.9.

Definition 3.11 Suppose $\pi \in \mathfrak{S}_n$, $p \in \mathfrak{S}_m$ and that $\langle p \rangle$ is an occurrence of p in π. We say that this occurrence is *induced* if $(\langle p_1 \rangle, \langle p_2 \rangle, \ldots, \langle p_k \rangle)$ is a cycle of π whenever $(p_1, p_2, \ldots, p_k)$ is a cycle of p.

Recall the involution $\pi = 84725631 \in \mathfrak{S}_8$ from Example 2.1 with its cycle decomposition $(1,8)(2,4)(3,7)(5)(6)$ Two occurrences of the pattern 4231 in π are $(8,5,6,1)$ and $(8,4,5,3)$ whereof the first one is induced but the second one is not.

Proposition 3.12 (A pattern criterion) *Let $w \in \mathcal{I}(\mathfrak{S}_n)$. The following are equivalent:*

1. *There is a long-crossing pair (i,j) in w.*
2. *The involution w contains at least one of the patterns* 4321*,* 45312 *and* 456123 *as an induced pattern.*
3. *The involution w contains at least one of the patterns* 4321*,* 45312 *and* 456123*.*

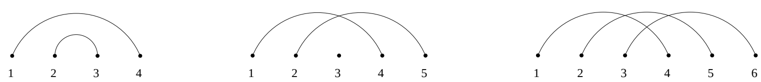
Proof "1. $\Rightarrow$ 2." Let (i,j) be a long-crossing pair in w. Assume that w does not contain an induced occurence of 4321. In particular, this implies that $w(i) < w(j)$. If $j+1$ is a fixed point then w contains the induced pattern 45312. Otherwise, we have $w(j+1) < i$ or $w(j+1) > w(j)$ because we assumed w to avoid the induced pattern 4321. But then w contains 456123 as an induced pattern.

"2. $\Rightarrow$ 3." is obvious.

"3. $\Rightarrow$ 1." We distinguish three cases. First, assume that w contains 4321 and that $\langle 4321 \rangle$ is an occurrence. Then, at least one of $\langle 3 \rangle$ or $\langle 2 \rangle$ is not a fixed point of w; denote that value by k. Recall that $\langle x \rangle$ refers to the value corresponding to x in the occurrence and that $w(\langle x \rangle) = w^{-1}(\langle x \rangle)$ denotes the position of that value in w. If $w(k) > k$, then $w(\langle 1 \rangle) > w(k) > k > \langle 1 \rangle$ and $(\langle 1 \rangle, k)$ is a long-crossing pair in w. Otherwise, it follows that $w(\langle 4 \rangle) < w(k) < k < \langle 4 \rangle$ and $(w(\langle 4 \rangle), w(k))$ is such a pair.

Next, assume that w avoids 4321 but contains 45312. Let $\langle 45312\rangle$ be an occurrence. Then $\langle 3\rangle$ is a fixed point, because otherwise w will contain 4321 by similar arguments as in the first case. This implies that $(\langle 1\rangle, \langle 2\rangle)$ is a long-crossing pair.

Finally, assume that w avoids 4321 and 45312 and let $\langle 456123\rangle$ be an occurrence of 456123 in w. The fact that w avoids 45312 implies that none of $\langle 1\rangle, \langle 2\rangle, \dots, \langle 6\rangle$ is a fixed point. Furthermore, if $\langle 1\rangle$, $\langle 2\rangle$ or $\langle 3\rangle$ is a deficiency, denote that value by k. Then $w(\langle 4\rangle) < w(k) < k < \langle 4\rangle$ and w contains 4321 in contradiction to our assumption. Thus, $\langle 1\rangle$, $\langle 2\rangle$ and $\langle 3\rangle$ are excedances. Analogously, $\langle 4\rangle, \langle 5\rangle$ and $\langle 6\rangle$ are deficiencies. Now, $w(\langle 1\rangle) \neq \langle x\rangle$ for at least one of $x = 4$ or $x = 5$, and for such x $(w(\langle x\rangle), \langle 1\rangle)$ or $(\langle 1\rangle, w(\langle x\rangle))$ is a long-crossing pair because $w(\langle x\rangle) < \langle 1\rangle < \langle 2\rangle < \langle x\rangle$ or $\langle 1\rangle < w(\langle x\rangle) < w(\langle 6\rangle) < w(\langle 1\rangle)$, respectively. □



(a) 4321 (b) 45312 (c) 456123

Fig. 3.2 Non-Boolean patterns for $\mathcal{I}(\mathfrak{S}_n)$.

3.2 Other Coxeter groups

The knowledge we gained in Section 3.1 about Boolean involutions in $\mathcal{I}(\mathfrak{S}_n)$ can be used to classify Boolean involutions in $\mathcal{I}(W)$ for some other W. Here, we shall develop results for the case that W is the group of signed permutations $\mathfrak{S}_n^B$. This is the group of permutations π of the set $[\pm n] = \{-n, \dots, -1\} \cup [n]$ such that $\pi(i) = -\pi(-i)$ for all $i \in [n]$.

Let $\mathfrak{s}'_i = (-i, -i-1)$, $i > 0$, and $\mathfrak{s}_0 = (1, -1)$. Define $\mathfrak{s}_i^B = \mathfrak{s}_i\mathfrak{s}'_i$, $i > 0$, and $\mathfrak{s}_0^B = \mathfrak{s}_0$. Then, $\mathfrak{S}_n^B$ is generated as a Coxeter group by $\{\mathfrak{s}_0^B, \dots, \mathfrak{s}_{n-1}^B\}$, whereas the symmetric group $\mathfrak{S}([\pm n])$ is generated by $\{\mathfrak{s}'_{n-1}, \dots, \mathfrak{s}'_1, \mathfrak{s}_0, \dots, \mathfrak{s}_{n-1}\}$.

We have an obvious inclusion $\mathcal{I}(\mathfrak{S}_n^B) \subseteq \mathcal{I}(\mathfrak{S}([\pm n]))$; let ϕ denote the inclusion map.

Lemma 3.13 *Let $w \in \mathcal{I}(\mathfrak{S}_n^B)$. Then, $\phi(w\underline{\mathfrak{s}}_0^B) = \phi(w)\underline{\mathfrak{s}}_0$. Furthermore, for $i \in [n-1]$,*

$$\phi(w\underline{\mathfrak{s}}_i^B) = \begin{cases} \phi(w)\underline{\mathfrak{s}}_i & \text{if } \mathfrak{s}_i w \mathfrak{s}_i = \mathfrak{s}'_i w \mathfrak{s}'_i \neq w, \\ \phi(w)\underline{\mathfrak{s}}_i\underline{\mathfrak{s}}'_i & \text{otherwise.} \end{cases}$$

Proof Let $w \in \mathcal{I}(\mathfrak{S}_n^B)$. Assume first that $w = \mathfrak{s}_i w \mathfrak{s}_i$. This implies $w = \mathfrak{s}'_i w \mathfrak{s}'_i$ as well as $w\mathfrak{s}_i^B = \mathfrak{s}_i^B w$ and thus $\phi(w\underline{\mathfrak{s}}_i^B) = \phi(w\mathfrak{s}_i^B) = \phi(w)\mathfrak{s}_i\mathfrak{s}'_i = \phi(w)\underline{\mathfrak{s}}_i\underline{\mathfrak{s}}'_i$. If, on the other hand, $\mathfrak{s}_i w \mathfrak{s}_i \neq \mathfrak{s}'_i w \mathfrak{s}'_i$ it follows that $\mathfrak{s}_i w \mathfrak{s}_i \neq w \neq \mathfrak{s}'_i w \mathfrak{s}'_i$ and thus $\phi(w\underline{\mathfrak{s}}_i^B) = \phi(\mathfrak{s}_i^B w \mathfrak{s}_i^B) = \mathfrak{s}_i\mathfrak{s}'_i\phi(w)\mathfrak{s}_i\mathfrak{s}'_i = \phi(w)\underline{\mathfrak{s}}_i\underline{\mathfrak{s}}'_i$. Finally, assume that $w \neq \mathfrak{s}_i w \mathfrak{s}_i = \mathfrak{s}'_i w \mathfrak{s}'_i$. By the remark after Proposition 2.3, $\ell(\mathfrak{s}_i \phi(w) \mathfrak{s}_i) = \ell(\phi(w)) \pm 2$. Assume the plus sign holds; otherwise a completely analogous argument applies. We claim that $\phi(w)\mathfrak{s}_i = \mathfrak{s}'_i \phi(w)$ and $\mathfrak{s}_i \phi(w) = \phi(w)\mathfrak{s}'_i$. To see this, consider the open interval $I = (\phi(w), \mathfrak{s}_i \phi(w) \mathfrak{s}_i)$ in the Bruhat order on $\mathfrak{S}([\pm n])$. Known facts about the Bruhat order (see e.g. [2, Lemma 2.7.3]) imply that I consists of exactly two elements. Thus, $I = \{\phi(w)\mathfrak{s}'_i, \mathfrak{s}'_i \phi(w)\} = \{\phi(w)\mathfrak{s}_i, \mathfrak{s}_i \phi(w)\}$, proving the claim. We conclude

$$\phi(w \underline{\mathfrak{s}}_i^B) = \phi(w \mathfrak{s}_i^B) = \phi(w)\mathfrak{s}_i \mathfrak{s}'_i = \mathfrak{s}_i \phi(w) \mathfrak{s}_i = \phi(w) \underline{\mathfrak{s}}_i$$

and the lemma is proved. □

In conjunction with Proposition 3.1, this in particular implies the following.

Corollary 3.14 *An involution $w \in \mathcal{I}(\mathfrak{S}_n^B)$ is Boolean if and only if $\phi(w) \in \mathcal{I}(\mathfrak{S}([\pm n]))$ is Boolean.* □

There are several possible ways to extend the notion of pattern avoidance from $\mathfrak{S}_n$ to $\mathfrak{S}_n^B$. We now describe the version which we shall use.

Given $\pi \in \mathfrak{S}_n^B$ and $p \in \mathfrak{S}_m^B$ (with $m \leq n$), we say that π *contains the signed pattern* p if there exist $1 \leq i_1 < \cdots < i_m \leq n$ such that $(|\pi(i_1)|, \ldots, |\pi(i_m)|)$ is an occurrence of the (unsigned) pattern $|p(1)| \cdots |p(m)|$ in the ordinary sense, and $\mathrm{sgn}(\pi(i_j)) = \mathrm{sgn}(p(j))$ for all $j \in [m]$.

We have a characterization of the Boolean elements of $\mathcal{I}(\mathfrak{S}([\pm n]))$ in terms of patterns. This can be translated into signed pattern avoidance in $\mathcal{I}(\mathfrak{S}_n^B)$.

Below, we use window notation for signed permutations. Thus, $\pi \in \mathfrak{S}_n^B$ is represented by the sequence $\pi(1) \cdots \pi(n)$. For compactness, we write $\underline{i}$ instead of $-i$. As an example, $\underline{2}31$ denotes the signed permutation defined by $\pm 1 \mapsto \mp 2$, $\pm 2 \mapsto \pm 3$ and $\pm 3 \mapsto \pm 1$.

Proposition 3.15 *Let $w \in \mathcal{I}(\mathfrak{S}_n^B)$. Then w is Boolean if and only if it avoids all of the following signed patterns.*

$$\begin{array}{lll}
4321 & 45312 & 456123 \\
\underline{1}\underline{2} & 1\underline{3}\underline{2} & \underline{3}\underline{2}\underline{1} \\
21\underline{3} & 42\underline{3}1 & 4\underline{3}\underline{2}1 \\
3\underline{4}1\underline{2} & \underline{4}531\underline{1}2 & 45\underline{3}12 \\
\underline{4}32\underline{1} & 5\underline{4}3\underline{2}1 & \underline{4}56\underline{1}23 \\
 & & 5\underline{4}6\underline{2}13
\end{array}$$

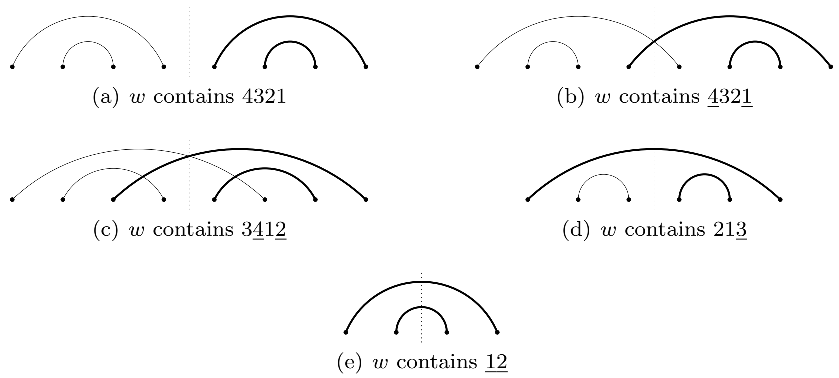
Proof Recalling Theorem 1.1 and Corollary 3.14, we need only show that w contains one of the patterns from the above list if and only if $\phi(w)$ contains 4321, 45312 or 456123.

"$\Rightarrow$". It is straightforward to check, that if w contains any of the signed patterns listed in the lemma, then $\phi(w)$ contains 4321, 45312 or 456123. For example, assume that w contains $21\underline{3}$. Then the definition of ϕ implies that $\phi(w)$ contains $3\underline{1}\underline{2}21\underline{3}$, which in turn contains 4321.

"$\Leftarrow$". Recall from Proposition 3.12 that $\phi(w)$ contains 4321, 45312 or 456123 if and only if it has an induced occurrence of one of those three patterns.

We will show for 4321 that such an induced occurrence in $\phi(w)$ implies that w contains one of the signed patterns listed in the lemma. Similar arguments apply in the other cases.

Assume that $\phi(w)$ contains an induced 4321-pattern. The graph representation of $\phi(w)$ is symmetric with respect to the vertical axis bisecting the segment between 1 and -1, because $\phi(w)$ is the image of a signed permutation. In Figure 3.3 we have indicated with thick edges all possibilities of how the induced occurrence of 4321 can be placed in the graph representation and completed to a symmetric pattern. This leads to the list of signed patterns in the first column of the proposition. □



(a) w contains 4321

(b) w contains $\underline{4}32\underline{1}$

(c) w contains $3\underline{4}1\underline{2}$

(d) w contains $21\underline{3}$

(e) w contains $\underline{12}$

Fig. 3.3 Graph representations for 4321-containing $\phi(w)$.

The subgroup $\mathfrak{S}_n^D$ of $\mathfrak{S}_n^B$ consists of the permutations with an even number of negative elements in the window notation. It is a Coxeter group in its own right. The interested reader is referred to [15, Corollary 5.25] for a list of forbidden patterns that characterize Boolean involutions in $\mathfrak{S}_n^D$. However, the proofs get even more technical because the obvious analogue of Corollary 3.14 does not hold. For example, the signed involution $w = \underline{12}3$ is Boolean in $\mathcal{I}(\mathfrak{S}_n^D)$ but its image $\phi(w) = \underline{3}21\underline{12}3$ is not Boolean in $\mathcal{I}(\mathfrak{S}([\pm n]))$ as it contains the pattern 4321.

4 Enumeration

In this section we shall deduce some enumerative facts about Boolean involutions. The key is a simple linear recurrence formula valid for a class of Coxeter groups which we now specify.

Let W be a Coxeter group with Coxeter generator set $S = \{s_1, \dots, s_n\}$, $n \geq 3$, such that s_n commutes with all s_i for $i \leq n-2$. Further, assume s_{n-1} commutes with all s_i for $i \leq n-3$. Finally, suppose $s_n s_{n-1} s_n = s_{n-1} s_n s_{n-1}$ and $s_{n-1} s_{n-2} s_{n-1} = s_{n-2} s_{n-1} s_{n-2}$. This means that the Coxeter graph of

(W,S) is of the form displayed in Figure 4.1. Examples of such W include $\mathfrak{S}_n$, $n \geq 3$, as well as $\mathfrak{S}_n^B$, $n \geq 4$, and $\mathfrak{S}_n^D$, $n \geq 5$.

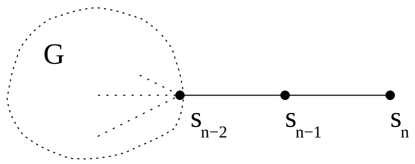


Fig. 4.1 The Coxeter graph of $W = W_n$.

For brevity, denote by W_i, $i \in [n]$, the standard parabolic subgroup of W generated by $\{s_1, \ldots, s_i\}$. Let $f(W_i, l, a)$ be the number of Boolean involutions in W_i of Coxeter length l and absolute length a. In other words,

$$f(W_i, l, a) = |\{w \in \mathcal{I}(W_i) : w \text{ is Boolean}, \ell(w) = l \text{ and } \ell'(w) = a\}|.$$

Theorem 4.1 *Let (W,S) be as above. Then, for $n, l \geq 3$ and $a \geq 1$,*

$$\begin{aligned} f(W_n, l, a) = {} & f(W_{n-1}, l, a) + f(W_{n-1}, l-2, a) \\ & + f(W_{n-2}, l-1, a-1) - f(W_{n-2}, l-2, a) \\ & + f(W_{n-2}, l-3, a-1) - f(W_{n-3}, l-3, a-1). \end{aligned} \tag{4.1}$$

Proof Suppose $w \in \mathcal{I}(W_n)$ is Boolean with $\ell(w) = l$ and $\ell'(w) = a$. If $s_n \not\leq w$ then w is a Boolean involution in W_{n-1}. There are exactly $f(W_{n-1}, l, a)$ such w. Otherwise, consider the lexicographically first (with respect to the indices of the generators) reduced $\underline{S}$-expression for w; call this expression E. We have two cases, depending on whether E ends with $\underline{s}_n$. If it does not, then it necessarily ends with $\underline{s}_n\underline{s}_{n-1}$. Because w is Boolean, exactly one $\underline{s}_n$ appears in E.

Case 1, E ends with $\underline{s}_n$. This is the case if and only if $w\underline{s}_n \in \mathcal{I}(W_{n-1})$. If w commutes with s_n, we have $\ell(w) = \ell(w\underline{s}_n) + 1$ and $\ell'(w) = \ell'(w\underline{s}_n) + 1$. If not, $\ell(w) = \ell(w\underline{s}_n) + 2$ and $\ell'(w) = \ell'(w\underline{s}_n)$.

Now, w commutes with s_n if and only if $\underline{s}_{n-1}$ does not occur in E, that is if and only if $w\underline{s}_n \in \mathcal{I}(W_{n-2})$. Hence, the number of w that fall into Case 1 is $f(W_{n-2}, l-1, a-1) + f(W_{n-1}, l-2, a) - f(W_{n-2}, l-2, a)$.

Case 2, E ends with $\underline{s}_n\underline{s}_{n-1}$. Let $u = w\underline{s}_{n-1}\underline{s}_n$. Then $w = u\underline{s}_n\underline{s}_{n-1} = u\underline{s}_{n-1}\underline{s}_n$, that is $\underline{s}_{n-1}$ and $\underline{s}_n$ commute, if and only if no reduced $\underline{S}$-expression for u contains $\underline{s}_{n-2}$. Thus, we are in Case 2 if and only if $u \in \mathcal{I}(W_{n-2}) \setminus \mathcal{I}(W_{n-3})$. Then, u commutes with s_n whereas $u\underline{s}_n$ does not commute with s_{n-1}. Hence, $\ell(w) = \ell(u) + 3$ and $\ell'(w) = \ell'(u) + 1$. Consequently, there are $f(W_{n-2}, l-3, a-1) - f(W_{n-3}, l-3, a-1)$ elements w that belong to Case 2. $\square$

Corollary 4.2 *Keeping the above assumptions on (W,S), let $g(W_i, k)$ denote the number of Boolean involutions $w \in \mathcal{I}(W_i)$ with rank $\rho(w) = k$. Also, define*

$h(W_i)$ to be the number of Boolean involutions in $\mathcal{I}(W_i)$. Then, for $n \geq 3$ and $k \geq 2$,

$$g(W_n, k) = g(W_{n-1}, k) + g(W_{n-1}, k-1) + g(W_{n-2}, k-2) - g(W_{n-3}, k-2)$$

and

$$h(W_n) = 2h(W_{n-1}) + h(W_{n-2}) - h(W_{n-3}).$$

Proof Once we recall that $\rho(w) = (\ell(w) + \ell'(w))/2$, the identities follow by summing equation (4.1) over appropriate l and a. □

From now on, let us stick to the case of symmetric groups. With $W = \mathfrak{S}_{n+1}$, we have $W_j = \mathfrak{S}_{j+1}$ and $f(\mathfrak{S}_j, i, e)$ is the number of Boolean involutions in $\mathfrak{S}_j$ with i inversions and e excedances.

Proposition 4.3 *Consider the generating function for the number of Boolean involutions in $\mathfrak{S}_n$ with respect to inversion number and excedance number. That is, define*

$$F(x, y, z) = \sum_{n \geq 1,\, i \geq 0,\, e \geq 0} f(\mathfrak{S}_n, i, e) x^n y^i z^e.$$

Then,

$$F(x, y, z) = \frac{x^2yz + x - x^2y^2 - x^3y^3z}{1 - x - x^2yz - xy^2 + x^2y^2 - x^2y^3z + x^3y^3z}.$$

Proof This follows from equation (4.1) via standard techniques once one has computed $f(\mathfrak{S}_n, l, a)$ for $n \leq 3$ or $i \leq 2$ or $e = 0$. These numbers vanish except in the following cases: $f(\mathfrak{S}_n, 0, 0) = 1$ $(n \geq 1)$, $f(\mathfrak{S}_n, 1, 1) = n - 1$ $(n \geq 2)$, $f(\mathfrak{S}_n, 2, 2) = (n^2 - 5n + 6)/2$ $(n \geq 4)$ and $f(\mathfrak{S}_3, 3, 1) = 1$. □

Plugging in $y = z = t^{1/2}$ and $y = z = 1$, one obtains the generating functions for $g(\mathfrak{S}_n, k)$ and $h(\mathfrak{S}_n)$, respectively.

Corollary 4.4 *We have*

$$\sum_{n \geq 1,\, k \geq 0} g(\mathfrak{S}_n, k) x^n t^k = \frac{x(1 - x^2t^2)}{(1 - x^2t^2)(1 - x) - xt}$$

and

$$\sum_{n \geq 1} h(\mathfrak{S}_n) x^n = \frac{x(1 - x^2)}{1 - 2x - x^2 + x^3}.$$

□

Recall that a *Motzkin path of length* n is a lattice path from $(0,0)$ to $(n,0)$ which never goes below the x-axis and whose steps are either $(1,1)$, $(1,0)$ or $(1,-1)$. These steps are called *upsteps*, *flatsteps* and *downsteps*, respectively. We denote by M_n the set of Motzkin paths of length n.

The sequence $h(\mathfrak{S}_n)$, $n \geq 1$, can be found in [13, A052534] where it is referred to as the number of Motzkin paths with certain properties. Let $M_n^{\mathrm{r}} \subseteq M_n$ denote the set of Motzkin paths of length n that never go higher than level 2 and whose flatsteps all occur on level at most 1. We call a path in M_n^{r} a *restricted Motzkin path* of length n.

Proposition 4.5 *Let $\psi : \mathcal{I}(\mathfrak{S}_n) \to M_n$ be the mapping which sends an involution w to the Motzkin path $\psi(w)$ with a flatstep, upstep or downstep as k-th step if $w(k)$ is a fixed point, an excedance or a deficiency, respectively. Then ψ induces a bijection between the Boolean involutions in $\mathcal{I}(\mathfrak{S}_n)$ and the restricted Motzkin paths of length n.*

An example is shown in Figure 4.2.

Proof For every $w \in \mathcal{I}(\mathfrak{S}_n)$, $\psi(w)$ is a lattice path by definition. It goes from $(0,0)$ to $(n,0)$, because w has the same number of excedances and deficiencies, and it obviously does not go below the x-axis. Thus, $\psi(w)$ is a Motzkin path for all $w \in \mathcal{I}(\mathfrak{S}_n)$ and ψ is well-defined.

Assume that the k-th step of $\psi(w)$ is a flatstep on level p (that is it goes from $(k-1,p)$ to (k,p)). Then there are exactly p elements $l \in [k-1]$ such that $w(l) > k$. If $p > 1$ there are $l_1, l_2 \in [k-1]$ such that $w(l_1) > k$ and $w(l_2) > k$. Assuming $l_1 < l_2$, (l_1, l_2) is a long-crossing pair. Thus, if $\psi(w)$ is a path with a flatstep on level 2 or higher, then w is not Boolean. Similarly, it follows that if $\psi(w)$ goes to a level > 2, then w is not Boolean. Therefore every Boolean involution is mapped to a restricted Motzkin path and

$$\psi(\{w \in \mathcal{I}(\mathfrak{S}_n) : w \text{ is Boolean}\}) \subseteq M_n^{\mathrm{r}}.$$

In order to show the reverse inclusion, fix a restricted Motzkin path Γ. We construct an involution $w \in \mathcal{I}(\mathfrak{S}_n)$ such that $\psi(w) = \Gamma$. For $k \in [n]$ define $w(k) = k$ if the k-th step of Γ is a flatstep. If the k-th step is an upstep or a downstep, and it is the m-th upstep or downstep, respectively, then define $w(k) = p$ where p is such that the p-th step in Γ is the m-th downstep or upstep, respectively. This obviously defines a unique involution in $\mathcal{I}(\mathfrak{S}_n)$. Observe that the given restrictions on the Motzkin path ensure that long-crossing pairs never occur. Hence, the constructed involution is Boolean. This proves $\psi(\{w \in \mathcal{I}(\mathfrak{S}_n) : w \text{ is Boolean}\}) = M_n^{\mathrm{r}}$.

Note that the proof of Proposition 3.8 implies that a Boolean involution is uniquely determined by its sets of excedances and deficiencies. Thus, ψ yields a bijection between the Boolean elements of $\mathcal{I}(\mathfrak{S}_n)$ and M_n^{r}. □

We conclude this section by pointing out what happens to our favourite statistics under the bijection ψ.

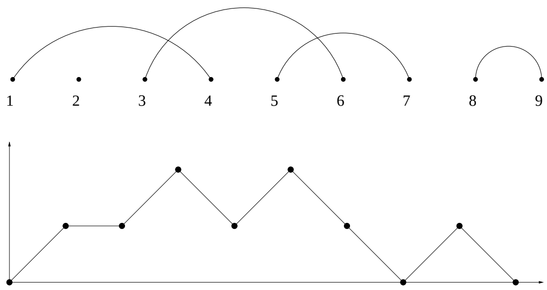


Fig. 4.2 A Boolean involution and the corresponding restricted Motzkin path.

Proposition 4.6 *Suppose* $w \in \mathcal{I}(\mathfrak{S}_n)$ *is Boolean. Let* $\alpha(w)$ *be the number of indices* $i \in [n]$ *such that* $\psi(w)$ *contains the point* $(i, 0)$. *Then,* $\rho(w) = n - \alpha(w)$.

Proof Because w is Boolean, $\rho(w)$ is the number of distinct generators $\mathfrak{s}_i$, $i \in [n-1]$ that appear in reduced $\underline{S}$-expressions for w, that is that are below w in the Bruhat order. On the other hand, for $i \in [n-1]$, $(i, 0)$ belongs to $\psi(w)$ if and only if $w(j) \leq i$ for all $j \leq i$. This holds if and only if $\mathfrak{s}_i \not\leq w$. □

By construction, the number of excedances (or deficiencies) of w is precisely the number of upsteps (or downsteps) of $\psi(w)$. Since $2\rho = \text{exc} + \text{inv}$, Proposition 4.6 also provides an interpretation for the inversion number of w in terms of the corresponding Motzkin path.

As an example, the path in Figure 4.2 touches the x-axis in two points (excluding the origin). Thus, the rank of the corresponding involution w is $9 - 2 = 7$. There are four upsteps, so $\text{exc}(w) = 4$. Hence, $\text{inv}(w) = 10$.

5 Twisted involutions

As was mentioned in the introduction, a good reason to study $\text{Br}(\mathcal{I}(W))$ is the connection with orbit decompositions of symmetric varieties which is explained in [11]. In this context, the more general setting of *twisted involutions* with respect to an involutive automorphism θ of (W, S) is important. These are the elements $w \in W$ such that $\theta(w) = w^{-1}$. Thus, $\mathcal{I}(W)$ corresponds to the $\theta = \text{id}$ case.

In the context of a symmetric group, there is only one non-trivial θ; it is given by $w \mapsto w_0 w w_0$, where $w_0 \in \mathfrak{S}_n$ is the longest element (the reverse permutation).

Problem 5.1 Find an analogue of Theorem 1.1 valid for $\theta \neq \text{id}$.

In order to attack this problem, [15, Proposition 5.1] is likely to be useful. It provides a generalization to arbitrary θ of Proposition 3.1. Also, the tools mentioned in Subsection 2.3 have direct counterparts in this more general setting; see [6, 7].

We remark that whenever θ is given by $w \mapsto w_0 w w_0$, the Bruhat order on twisted involutions is isomorphic to the dual of $\mathrm{Br}(\mathcal{I}(W))$. Thus, Problem 5.1 is equivalent to the problem of characterizing Boolean principal order filters in $\mathrm{Br}(\mathcal{I}(\mathfrak{S}_n))$.

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