CRITERIA FOR RATIONAL SMOOTHNESS OF SOME SYMMETRIC ORBIT CLOSURES

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Abstract. Let G be a connected reductive linear algebraic group over $\mathbb{C}$ with an involution θ. Denote by K the subgroup of fixed points. In certain cases, the K-orbits in the flag variety G/B are indexed by the twisted identities $\iota(\theta) = \{\theta(w^{-1})w \mid w \in W\}$ in the Weyl group W. Under this assumption, we establish a criterion for rational smoothness of orbit closures which generalises classical results of Carrell and Peterson for Schubert varieties. That is, whether an orbit closure is rationally smooth at a given point can be determined by examining the degrees in a "Bruhat graph" whose vertices form a subset of $\iota(\theta)$. Moreover, an orbit closure is rationally smooth everywhere if and only if its corresponding interval in the Bruhat order on $\iota(\theta)$ is rank symmetric.

In the special case $K = \mathrm{Sp}_{2n}(\mathbb{C})$, $G = \mathrm{SL}_{2n}(\mathbb{C})$, we strengthen our criterion by showing that only the degree of a single vertex, the "bottom one", needs to be examined. This generalises a result of Deodhar for type A Schubert varieties.

1. Introduction

Let G be a connected reductive complex linear algebraic group equipped with an automorphism θ of order 2. There is a θ-stable Borel subgroup B which contains a θ-stable maximal torus T [20, §7] with normaliser N. Let $K = G^\theta$ be the fixed point subgroup. We may always assume θ to be the complexification of the Cartan involution of some real form $G_\mathbb{R}$ of G [15].

The flag variety $X = G/B$ decomposes into finitely many orbits under the action of the symmetric subgroup K by left translations. A natural "Bruhat-like" partial order on the set of orbits $K\backslash X$ is defined by inclusion of their closures. Let V denote this poset. Richardson and Springer [15, 16] defined a poset map $\varphi : V \to \mathrm{Br}(W)$, where $\mathrm{Br}(W)$ is the Bruhat order on the Weyl group $W = N/T$. The image of φ is contained in the set of *twisted involutions* $\mathcal{I}(\theta) = \{w \in W \mid \theta(w) = w^{-1}\}$. In general, φ is neither injective nor surjective. For certain choices of G and θ, however, φ produces a poset isomorphism $V \cong \mathrm{Br}(\iota(\theta))$, where $\iota(\theta) = \{\theta(w^{-1})w \mid w \in W\} \subseteq \mathcal{I}(\theta)$ is the set of *twisted identities* and $\mathrm{Br}(\cdot)$ denotes induced subposet of $\mathrm{Br}(W)$. In Section 3, we shall make explicit under what circumstances this fairly restrictive assumption holds. Now suppose that φ is such an isomorphism and let $\overline{\mathcal{O}_w}$, $w \in \iota(\theta)$, denote the closure of the orbit $\mathcal{O}_w = \varphi^{-1}(w)$. In this article we express the rationally singular locus of the symmetric variety $\overline{\mathcal{O}_w}$ in terms of the combinatorics of $\iota(\theta)$.

With each $w \in \iota(\theta)$, we shall associate a *Bruhat graph* $\mathrm{BG}(w)$ with vertex set $I_w = \{u \in \iota(\theta) \mid u \le w\}$. Our first main result, Theorem 5.9, states that $\overline{\mathcal{O}_w}$ is rationally smooth at $\mathcal{O}_u$ if and only if v is contained in exactly $\rho(w)$ edges for all $u \le v \le w$, where $\rho(w)$ is the rank of w in $\mathrm{Br}(\iota(\theta))$. In particular, $\overline{\mathcal{O}_w}$ is rationally smooth if and only if $\mathrm{BG}(w)$ is $\rho(w)$-regular. This latter statement also turns out to be equivalent to the principal order ideal $\mathrm{Br}(I_w)$ being rank-symmetric; see Theorem 5.10 below.

The assertions just stated generalise celebrated criteria due to Carrell and Peterson [6] for rational smoothness of Schubert varieties. We recover their results in the special case where $G = G' \times G'$ and $\theta(x, y) = (y, x)$.

The main brushstrokes of our proofs are completely similar to those of Carrell and Peterson. Below the surface, however, their results rely on delicate connections between Kazhdan-Lusztig polynomials and the combinatorics of (ordinary) Bruhat graphs. Our chief contribution is to extend these properties to a more general setting. Very roughly, here is what we do:

First, properties of $\iota(\theta)$ are established that combined with results of Brion [5] imply a bound on the degrees in $\mathrm{BG}(w)$ that generalises "Deodhar's inequality" for degrees in ordinary Bruhat graphs of Weyl groups.

Second, an explicit procedure, in terms of the combinatorics of $\iota(\theta)$, for computing the "R-polynomials" of [12, 21] is extracted from the correspondence $V \leftrightarrow \iota(\theta)$. Using this procedure we establish several properties of these polynomials (and therefore of Kazhdan-Lusztig-Vogan polynomials) and relate them to degrees in the graphs $\mathrm{BG}(w)$. This generalises well known properties of ordinary Kazhdan-Lusztig polynomials and R-polynomials and how they are related to ordinary Bruhat graphs.

The most prominent example where our results say something which is not contained in [6] is $G = \mathrm{SL}_{2n}(\mathbb{C})$, $K = \mathrm{Sp}_{2n}(\mathbb{C})$. For this setting, we prove the stronger statement (Corollary 6.7) that the degree of the bottom vertex alone suffices to decide rational smoothness. That is, $\overline{\mathcal{O}_w}$ is rationally smooth at $\mathcal{O}_u$ if and only if the degree of u in $\mathrm{BG}(w)$ is $\rho(w)$. This is analogous to a corresponding result for type A Schubert varieties which is due to Deodhar [7]. Again, that result is contained in ours as a special case.

Remark 1.1. After a preliminary version of this article was circulated, McGovern [13] has applied our results in order to deduce a criterion for (rational) smoothness in the case $G = \mathrm{SL}_{2n}(\mathbb{C})$, $K = \mathrm{Sp}_{2n}(\mathbb{C})$ in terms of pattern avoidance among fixed point free involutions. Moreover, he proved that in this case the rationally singular loci in fact coincide with the singular loci.

Closures of symmetric orbits are interesting objects in their own right, but another important reason to study their singularities is their impact on representation theory. We outline this connection while describing one of our main tools, Kazhdan-Lusztig-Vogan polynomials, in the next section.

In Section 3, we make precise the assumptions on θ for which our results are valid. Thereafter, the Bruhat graphs $\mathrm{BG}(w)$ are introduced in Section 4. Our Carrell-Peterson type criteria for rational smoothness are deduced in Section 5. Finally, in Section 6, we prove that the bottom vertex alone suffices to decide rational smoothness when $G = \mathrm{SL}_{2n}(\mathbb{C})$, $K = \mathrm{Sp}_{2n}(\mathbb{C})$.

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2. KLV polynomials and representation theory

In the present paper, the principal method for detecting rational singularities of symmetric orbit closures is via Kazhdan-Lusztig-Vogan polynomials. Here, we briefly review some of their properties and establish notation. For more information we refer the reader to [12] or [21]. Our terminology chiefly follows the latter reference.

Let $\mathcal{D}$ denote the set of pairs $(\mathcal{O}, \gamma)$, where $\mathcal{O} \in K\backslash X$ and γ is a K-equivariant local system on $\mathcal{O}$. The choice of γ is equivalent to the choice of a character of the component group of the stabiliser K_x of a point $x \in \mathcal{O}$. In particular, γ is unique if K_x is connected. Since $\mathcal{O}$ is determined by γ, we may abuse notation and write γ for $(\mathcal{O}, \gamma)$. With each pair $\gamma, \delta \in \mathcal{D}$, we associate polynomials $R_{\gamma,\delta}, P_{\gamma,\delta} \in \mathbb{Z}[q]$. The R-polynomials can be computed using a recursive procedure which we refrain from stating in full generality here; see [21, Lemma 6.8] for details. A special case sufficient for our purposes is formulated in Proposition 5.1 below.

Let $\mathcal{M}$ denote the free $\mathbb{Z}[q, q^{-1}]$ module with basis $\mathcal{D}$. For fixed $\delta \in \mathcal{D}$, we have in $\mathcal{M}$ the identity

$$q^{-l(\delta)} \sum_{\gamma \le \delta} P_{\gamma,\delta}(q)\gamma = \sum_{\beta \le \gamma \le \delta} (-1)^{l(\beta)-l(\gamma)} q^{-l(\gamma)} P_{\gamma,\delta}(q^{-1}) R_{\beta,\gamma}(q)\beta \tag{1}$$

which subject to the restrictions $P_{\gamma,\gamma} = 1$ and $\deg(P_{\gamma,\delta}) \le (l(\delta) - l(\gamma) - 1)/2$ uniquely determines the *Kazhdan-Lusztig-Vogan (KLV)* polynomials $P_{\gamma,\delta}$ [21, Corollary 6.12].[1] Here, $l(\cdot)$ indicates the dimension of the corresponding orbit, and the order on $\mathcal{D}$ is the *Bruhat $\mathcal{G}$-order* [21, Definition 5.8].

KLV polynomials serve as measures of the singularities of symmetric orbit closures; cf. [21, Theorem 1.12]. In particular, their coefficients are nonnegative. Another consequence is the following:

Proposition 2.1. *Let $\le$ denote the order relation in V, i.e. containment among orbit closures. Given orbits $\mathcal{P}, \mathcal{O} \in K\backslash X$ with $\mathcal{P} \le \mathcal{O}$, let $\delta = (\mathcal{O}, \mathbb{C}_{\mathcal{O}})$, where $\mathbb{C}_{\mathcal{O}}$ is the trivial local system. Then, $\overline{\mathcal{O}}$ is rationally smooth at some (equivalently, every) point in $\mathcal{P}$ if and only if*

$$P_{\gamma,\delta} = \begin{cases} 1 & \text{if } L = \mathbb{C}_{\mathcal{Q}}, \\ 0 & \text{if } L \ne \mathbb{C}_{\mathcal{Q}}, \end{cases}$$

for all $\gamma = (\mathcal{Q}, L) \in \mathcal{D}$ with $\mathcal{P} \le \mathcal{Q} \le \mathcal{O}$.

The gadgets just described are fundamental ingredients in the representation theory of $G_{\mathbb{R}}$. Fix an infinitesimal character for $G_{\mathbb{R}}$. Then, $\mathcal{D}$ is in bijective correspondence with two families of $G_{\mathbb{R}}$-representations with this infinitesimal character. Given $\gamma \in \mathcal{D}$, there is the standard $(\mathfrak{g}, K_{\mathbb{R}})$-module $X(\gamma)$ induced from a discrete series representation, and there is the irreducible $(\mathfrak{g}, K_{\mathbb{R}})$-module $\overline{X}(\gamma)$. The transition between the two families is governed by the KLV polynomials. Namely, one

[1] Note that there is a typo which has an impact on the cited result. We are grateful to D. A. Vogan for pointing out that the displayed formula in the statement of [21, Lemma 6.8] should read

$$D(\delta) = u^{-l(\delta)} \sum_{\gamma} (-1)^{l(\gamma)-l(\delta)} R_{\gamma,\delta}(u)\gamma.$$

has

$$\overline{\Theta}(\gamma) = \sum_{\gamma'} (-1)^{l(\gamma)-l(\gamma')} P_{\gamma',\gamma}(1)\Theta(\gamma'),$$

where $\Theta(\gamma)$ and $\overline{\Theta}(\gamma)$ denote the characters of $X(\gamma)$ and $\overline{X}(\gamma)$, respectively [12, 21].

3. Restricting the involution

Consider the set $\mathcal{V} = \{g \in G \mid \theta(g^{-1})g \in N\}$. The set of orbits $K\backslash\mathcal{V}/T$ parametrises $K\backslash X$. In this way, the map $\mathcal{V} \to W$ given by $g \mapsto \theta(g^{-1})gT$ induces the map $\varphi : V \to W$ which was mentioned in the introduction. Observe that the image of φ is contained in $\mathcal{I}(\theta)$.

Throughout this paper we shall only allow certain choices of θ. More precisely, *we from now on assume that θ obeys the following condition:*

Hypothesis 3.1. *The fixed point subgroup K is connected. Moreover, $\varphi : V \to W$ satisfies $\varphi(v_0) \in \iota(\theta)$, where $v_0 \in V$ is the maximum element, i.e. the dense orbit.*

Remark 3.2. If G is semisimple and simply connected, then K is necessarily connected. This result is due to Steinberg [20, Theorem 8.1]. In some sense, the general situation can be reduced to the study of semisimple simply connected G; see [15].

Several consequences are collected in the next proposition. We let Φ denote the root system of G, T and write $R \subset W$ for the corresponding set of reflections.

Proposition 3.3. *Hypothesis 3.1 implies the following:*

(i) *The map φ yields a poset isomorphism $V \to \mathrm{Br}(\iota(\theta))$.*
(ii) *There is a unique K-equivariant local system, namely $\mathbb{C}_{\mathcal{O}}$, on each orbit $\mathcal{O} \in K\backslash X$. In particular, the sets $\mathcal{D}$, $K\backslash X$ and $\iota(\theta)$ may be identified, and the Bruhat $\mathcal{G}$-order on $\mathcal{D}$ coincides with V and $\mathrm{Br}(\iota(\theta))$.*
(iii) *Let $\alpha \in \Phi$ and denote by $G_\alpha \subseteq G$ the corresponding rank one semisimple group. Then, we are in one of the following two situations:*
  (a) *The root α is* compact imaginary. *That is, $G_\alpha \subseteq K$.*
  (b) *The root α is* complex *(meaning $\theta(\alpha) \neq \alpha$) and $\theta(\alpha) + \alpha \notin \Phi$.*
(iv) *If $r \in R$, then $\theta(r)r = r\theta(r)$.*
(v) *The poset $\mathrm{Br}(\iota(\theta))$ is graded with rank function ρ being half the ordinary Coxeter length. Moreover, $\rho(w) = l(\mathcal{O}_w) - l(\mathcal{O}_{\mathrm{id}})$.*

Proof. Assertion (i) follows from Richardson and Springer's [15, Proposition 9.16].

For (ii), the local system on Kx, $x \in X$, is unique if the isotropy subgroup K_x is connected. Under Hypothesis 3.1, Springer's [17, Proposition 4.8] implies that K_x is connected if the torus fixed point group T^θ is connected. Since K is connected, this follows from [14, Lemma 5.1].

In order to prove (iii), suppose $\theta(\alpha) = \alpha$ but $G_\alpha \not\subseteq K$. Then, the corresponding reflection $r_\alpha \in R$ is in the image of φ by [18, Lemma 2.5(i)]. This image is, however, $\iota(\theta)$ which does not contain any reflections.

If $\theta(\alpha) \neq \alpha$ and $\beta = \alpha + \theta(\alpha) \in \Phi$, then $\theta(\beta) = \beta$ and $G_\beta \not\subseteq K$ by [17, Lemma 2.6]. This once again leads to the above contradiction.

Concerning (iv), assuming $\theta(r) \neq r$ [17, Lemma 2.5] implies that the dihedral group generated by r and $\theta(r)$ is either of type $A_1 \times A_1$ or of type A_2. If the latter were true, we would have $\theta(\alpha) + \alpha \in \Phi$, where α is the positive root corresponding to r. This contradicts part (iii), and the claim is established.

Finally, the first part of (v) follows from (iv) in conjunction with [10, Theorem 4.6]. The second is then immediate from [15, Theorem 4.6]. □

The following example allows us to consider many of our results as generalisations of statements about Schubert varieties.

Example 3.4. *If G' is a connected reductive complex linear algebraic group and $G = G' \times G'$, the involution θ which interchanges the two factors makes K the diagonal subgroup. In this case, $\iota(\theta) = \mathcal{I}(\theta)$, so Hypothesis 3.1 is satisfied. The poset $\mathrm{Br}(\iota(\theta)$ coincides with $\mathrm{Br}(W')$, where W' is the Weyl group of G'. There is a one-to-one correspondence between K-orbits in X and Schubert cells in the Bruhat decomposition of the flag variety of G' which preserves a lot of structure including the property of having rationally smooth closure at a given orbit.*

In addition to the setting in Example 3.4 there are a few more cases that satisfy Hypothesis 3.1. They are denoted $A\,II$, $D\,II$ and $E\,IV$ in the classification of symmetric spaces $G_\mathbb{R}/K_\mathbb{R}$ given e.g. in Helgason [9].[2] The corresponding Weyl groups are A_{2n+1}, D_n and E_6, respectively, with θ in each case restricting to the Weyl group as the unique nontrivial Dynkin diagram involution. Types D and E could in principle be handled separately. In the former case, $\iota(\theta)$ has a very simple structure (cf. [10, proof of Theorem 5.2]), whereas the latter admits a brute force computation. Thus, the main substance lies in the A_{2n+1} case where $\mathrm{Br}(\iota(\theta))$ is an incarnation of the containments among closures of $\mathrm{Sp}_{2n}(\mathbb{C})$ orbits in the flag variety $\mathrm{SL}_{2n}(\mathbb{C})/B$; see [15, Example 10.4] for a discussion of this case. Nevertheless, we have opted to keep our arguments type independent regarding all assertions that are valid in the full generality of Hypothesis 3.1. There are two reasons. First, the natural habitat for Theorems 5.9 and 5.10 is the general setting; no simplicity would be gained by formulating the arguments in type A specific terminology. Second, we hope that the less specialised viewpoint shall prove suitable as point of departure for generalisations beyond Hypothesis 3.1.

4. "Bruhat graphs"

Let $*$ denote the θ-twisted right conjugation action of W on itself, i.e. $u * w = \theta(w^{-1})uw$ for $u, w \in W$. Then $\iota(\theta)$ is the orbit of the identity element $\mathrm{id} \in W$.

Recall that $I_w = \{u \in \iota(\theta) \mid u \leq w\}$.

Definition 4.1. *Given $w \in \iota(\theta)$, let $\mathrm{BG}(w)$ be the graph with vertex set I_w and an edge $\{u, v\}$ whenever $u = v * t \neq v$ for some reflection $t \in R$.*

Notice that $\mathrm{BG}(u)$ is an induced subgraph of $\mathrm{BG}(w)$ if $u \leq w$. See Figure 1 for an illustration.

We shall refer to graphs of the form $\mathrm{BG}(w)$ as *Bruhat graphs*, because in the setting of Example 3.4, they coincide with (undirected versions of) the ordinary Bruhat graphs in W' introduced by Dyer [8].

Our next goal is to show that (the first part of) Brion's [5, Theorem 2.5] implies lower bounds for the degrees in a Bruhat graph. This essentially amounts to a reformulation of the relevant parts of [5] using our terminology.

[2] The "usual" construction of $D\,II$ would yield $G = \mathrm{SO}_{2n}(\mathbb{C})$, $K = \mathrm{S}(\mathrm{O}_{2n-1}(\mathbb{C}) \times \mathrm{O}_1(\mathbb{C})) \cong \mathrm{O}_{2n-1}(\mathbb{C})$ so that K is disconnected. However, passing to the fundamental cover, we have $G = \mathrm{Spin}_{2n}(\mathbb{C})$, $K = \mathrm{Spin}_{2n-1}(\mathbb{C})$ in agreement with Hypothesis 3.1.

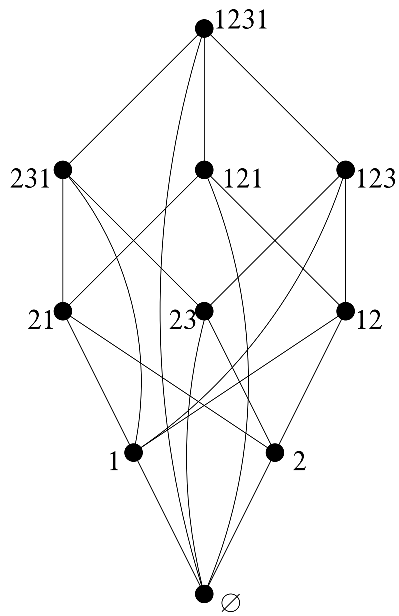

FIGURE 1. A picture of the Bruhat graph $\mathrm{BG}(w)$ where $w = s_5s_3s_4s_5s_1s_2s_3s_1 \in \iota(\theta) \subset A_5$. Here, s_i denotes the simple reflection $(i, i+1)$ in the usual manifestation of A_5 as the symmetric group S_6. The involution θ sends s_{6-i} to s_i. A vertex $u \in I_w$ is labelled by the indices of a sequence of simple reflections whose product x satisfies $u = \theta(x^{-1})x$. The straight edges indicate the covering relation of $\mathrm{Br}(\iota(\theta))$.

Lemma 4.2. *Let $w \in \iota(\theta)$ and $u, v \in I_w$, $u \neq v$. Write $u = \theta(x^{-1})x$ for $x \in W$. The following are equivalent:*

(i) *$\{u, v\}$ is an edge in* $\mathrm{BG}(w)$.
(ii) *There are exactly two distinct reflections $t \in R$ such that $u * t = v$.*
(iii) *There are exactly two distinct reflections $t \in R$ such that $\theta(x^{-1})\theta(t)tx = v$. If t is one of these reflections, then $\theta(t)$ is the other.*

Proof. The implication (ii) ⇒ (i) is obvious.

We have (iii) ⇒ (ii), since $\theta(x^{-1})\theta(t)tx = v$ if and only if $u*r = v$ for $r = x^{-1}tx$.

In order to show (i) ⇒ (iii), assume $v = \theta(x^{-1})\theta(r)rx = \theta(x^{-1})\theta(t)tx$, for $r, t \in R$. In particular, $t\theta(t) = r\theta(r)$. Dyer's [8, Lemma 3.1] shows that $\langle r, \theta(r), t, \theta(t)\rangle$ is a dihedral reflection subgroup of W. Since W is simply laced (which e.g. follows from part (iv) of Proposition 3.3 and inspection of finite type Dynkin diagrams), this subgroup must be of type $A_1 \times A_1$; A_2 is not possible since $t\theta(t) = \theta(t)t$. Hence, $\{r, \theta(r)\} = \{t, \theta(t)\}$, and these two reflections are the possible candidates for t. □

We are now in position to bound the degrees of a Bruhat graph. Combining the first part of Brion's [5, Theorem 2.5] with part (iii) of Proposition 3.3 shows that the rank of a vertex $v = \theta(x^{-1})x$ in $\mathrm{BG}(w)$ is at most half the number of complex reflections (i.e. reflections that correspond to complex roots) $t \in R$ such that $\theta(x^{-1})\theta(t)tx \leq w$. By Lemma 4.2, this is precisely the degree of v in $\mathrm{BG}(w)$. We thus have the following fact:

Theorem 4.3. *For $w \in \iota(\theta)$, the degree of each vertex in $\mathrm{BG}(w)$ is at least $\rho(w)$.*

Remark 4.4. In the setting of Example 3.4, Theorem 4.3 specialises to "Deodhar's inequality" in W'; see [1, §6] and the references cited there.

Lemma 4.5. *If $\{u, v\}$ is an edge in $\mathrm{BG}(w)$, then either $u < v$ or $v < u$. Furthermore, v has exactly $\rho(v)$ neighbours u such that $u < v$.*

Proof. Suppose $u = \theta(x^{-1})x \neq v = u * t$ for some $x \in W$, $t \in R$. Define reflections $r = u^{-1}\theta(t)u$ and $\tau = xtx^{-1}$. Using part (iv) of Proposition 3.3, we compute

$$urt = v = \theta(x^{-1})\theta(\tau)\tau x = \theta(x^{-1})\tau\theta(\tau)x = \theta(x^{-1})\tau\theta(x)ur = \theta(x^{-1})\tau xr = utr.$$

Thus, t and r commute. Hence, $\{u, ut, \theta(t)u, v\} = u\langle r, t\rangle$ and, by Dyer's [8, Theorem 1.4] the subgraph of the (ordinary) Bruhat graph on W induced by these four vertices is isomorphic to the Bruhat graph of the dihedral group on four elements. In particular, all pairs $\{a, b\} \subset \{u, ut, \theta(t)u, v\}$ except at most one satisfy $a \leq b$ or $b \leq a$. Since the map $y \mapsto \theta(y^{-1})$ is a poset automorphism of the Bruhat order which sends $\theta(t)u$ to ut, $\{\theta(t)u, ut\}$ is the only incomparable pair. This proves the first assertion.

The second assertion follows from Brion's [5, Theorem 2.5] using that an orbit closure is rationally smooth at the orbit. Alternatively, the above argument implies

$$\{t \in R \mid vt < v\} \supseteq \{t \in R \mid v * t < v\}.$$

In fact equality holds, for it is well known that the left hand side has $\ell(v) = 2\rho(v)$ elements, so Lemma 4.2 and Theorem 4.3 conclude the proof. □

Note that the last argument of the preceding proof in particular implies the useful fact that $v < vs$ whenever $v * s = v$ for $v \in \iota(\theta)$ and a simple reflection s. This is also immediate from properties of the rank function of $\mathcal{I}(\theta)$; see [10].

5. A criterion for rational smoothness

In general, the recursion for the R-polynomials mentioned in Section 2 is technically rather involved. Since we are assuming Hypothesis 3.1, however, the situation is simpler. Proposition 3.3 allows us to identify the indexing set $\mathcal{D}$ with $\iota(\theta)$. With $D_{\mathrm{R}}(v)$ denoting the *descent set* of $v \in \iota(\theta)$, i.e. the set of simple reflections s such that $vs < v$, or equivalently $v * s < v$, the recursion takes the following explicit form:

Proposition 5.1. *Suppose $s \in D_{\mathrm{R}}(v)$. Then, the R-polynomials satisfy*

$$R_{u,v}(q) = \begin{cases} R_{u*s,v*s}(q) & \text{if } u * s < u, \\ qR_{u*s,v*s}(q) + (q-1)R_{u,v*s}(q) & \text{if } u * s > u, \\ -R_{u,v*s}(q) & \text{if } u * s = u. \end{cases}$$

Proof. Consider the free $\mathbb{Z}[q, q^{-1}]$ module $\mathcal{M}$ with basis $\iota(\theta)$. The definition of the map $T_s : \mathcal{M} \to \mathcal{M}$ formulated in [21, Definition 6.4] boils down to

$$T_s w = \begin{cases} qw & \text{if } w * s = w, \\ w * s & \text{if } w * s > w, \\ qw * s + (q-1)w & \text{if } w * s < w, \end{cases}$$

for $w \in \iota(\theta)$ (the relevant cases being (a), (b1) and (b2), respectively). Equating coefficients in the identity

$$\sum_{u \in \iota(\theta)} (-1)^{\rho(u)} R_{u,w}(q) u = -\sum_{u \in \iota(\theta)} (-1)^{\rho(u)} R_{u,w*s}(q)(T_s + 1 - q)u$$

(see [21, proof of Lemma 6.8]) now yields

$$R_{u,w}(q) = \begin{cases} R_{u*s,w*s}(q) - R_{u,w*s}(q)(q-1+1-q) & \text{if } u*s < u, \\ qR_{u*s,w*s}(q) - (1-q)R_{u,w*s}(q) & \text{if } u*s > u, \\ -R_{u,w*s}(q) & \text{if } u*s = u, \end{cases}$$

if $s \in S$ satisfies $w * s < w$.

□

Together with the "initial values" $R_{u,u}(q) = 1$ and $R_{u,v}(q) = 0$ if $u \not\leq v$, we may calculate any $R_{u,v}$ using Proposition 5.1. Rather than working with the actual R-polynomials, we shall however find it more convenient to use the following simple variation:

Definition 5.2. *For $u, v \in \iota(\theta)$, let $Q_{u,v}(q) = (-q)^{\rho(v)-\rho(u)} R_{u,v}(q^{-1})$.*

One readily verifies the following recursion:

Proposition 5.3. *For $s \in D_{\mathrm{R}}(v)$, we have*

$$Q_{u,v}(q) = \begin{cases} Q_{u*s,v*s}(q) & \text{if } u*s < u, \\ qQ_{u*s,v*s}(q) + (q-1)Q_{u,v*s}(q) & \text{if } u*s > u, \\ qQ_{u,v*s}(q) & \text{if } u*s = u. \end{cases}$$

In particular, the $Q_{u,v}(q)$ are polynomials. In the setting of Example 3.4, both the $R_{u,v}(q)$ and the $Q_{u,v}(q)$ coincide with the classical Kazhdan-Lusztig R-polynomials introduced in [11]. The three lemmata coming up next hint that the $Q_{u,v}(q)$ may provide the more useful generalisation. To establish them we need the following fact from [10]:

Proposition 5.4 (Lifting property)**.** *Let $u, w \in \iota(\theta)$ with $u \leq w$. Suppose $s \in D_{\mathrm{R}}(w)$. Then,*

(i) *$u * s \leq w$ and $us \leq w$.*
(ii) *$s \in D_{\mathrm{R}}(u) \Rightarrow u * s \leq w * s$.*
(iii) *$s \notin D_{\mathrm{R}}(u) \Rightarrow u \leq w * s$.*

Lemma 5.5. *For $u, v \in \iota(\theta)$, we have*

$$Q'_{u,v}(1) = \begin{cases} 1 & \text{if } u < v \text{ and } \{u,v\} \text{ is an edge in } \mathrm{BG}(v), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Suppose $s \in D_{\mathrm{R}}(v)$. Differentiating the equation in Proposition 5.3 with respect to q, and using that $Q_{u,v}(1) = R_{u,v}(1) = \delta_{u,v}$ (Kronecker's delta), it follows that

$$Q'_{u,v}(1) = Q'_{u*s,v*s}(1) + \delta_{u,v*s}.$$

It is clear that $\{u*s, v*s\}$ is an edge in $\mathrm{BG}(v)$ if and only if the same is true about $\{u, v\}$. Employing induction on $\rho(v)$, it thus suffices to show that $u * s < v * s$ if $v * s \neq u < v$ and $\{u, v\}$ is an edge. Lemma 4.5 shows that $u * s$ and $v * s$ are comparable in this situation. The assertion $u*s > v*s$ would contradict the lifting property, and we are done. □

Lemma 5.6. *Denote by* μ *the Möbius function*[3] *of* $\mathrm{Br}(\iota(\theta))$. *Then,* $\mu(u,v) = Q_{u,v}(0)$ *for all* $u, v \in \iota(\theta)$.

Proof. Let us induct on $\rho(v)$. The assertion holds for $\rho(v) = 0$ because $Q_{\mathrm{id},\mathrm{id}}(q) = R_{\mathrm{id},\mathrm{id}}(q) = 1$. We shall demonstrate that $\mu(u,v)$ satisfies the recursion for $Q_{u,v}(0)$ derived from Proposition 5.3.

Borrowing terminology from [10], call $[u,v]$ *full* if every twisted involution in the interval $[u,v]$ is in fact a twisted identity. Combining Philip Hall's theorem (see e.g. [19, Proposition 3.8.5]) with [10, Theorem 4.12] shows that

$$\mu(u,v) = \begin{cases} (-1)^{\rho(v)-\rho(u)} & \text{if } [u,v] \text{ is full,} \\ 0 & \text{otherwise.} \end{cases}$$

Pick $s \in D_{\mathrm{R}}(v)$. In case $u*s = u$, us is a twisted involution which belongs to $[u,v]$ by the lifting property. Thus, $[u,v]$ is not full, and $\mu(u,v) = 0$ as desired. If $u*s > u$, it follows from [10, Lemma 4.10] that $[u, v*s]$ is full if and only if $[u,v]$ is full. Thus, $\mu(u,v) = -\mu(u, v*s)$, and we are done. Finally, suppose $u*s < u$. If $[u*s, v*s]$ is full then $[u,v]$ is also full, again by [10, Lemma 4.10]. On the other hand, [10, Theorem 4.9] implies that $\mu(u*s, v) = -\mu(u,v)$, so if $[u*s, v*s]$ (and therefore $[u*s, v]$) is not full, then $[u,v]$ cannot be full either. Completing the proof, we conclude $\mu(u,v) = \mu(u*s, v*s)$. □

Lemma 5.7. *For all* $v \in \iota(\theta)$,

$$\sum_{u \leq v} Q_{u,v}(q) = q^{\rho(v)}.$$

Proof. We prove the lemma using induction on $\rho(v)$. Given $s \in D_{\mathrm{R}}(v)$, so that $\rho(v*s) = \rho(v) - 1$, let us partition I_v into three sets:

$$\begin{aligned} A &= \{u \leq v \mid u*s < u\}, \\ B &= \{u \leq v \mid u*s > u\}, \\ C &= \{u \leq v \mid u*s = u\}. \end{aligned}$$

By the lifting property, the map $u \mapsto u*s$ is a bijection between A and B. The recursion in Proposition 5.3 therefore yields

$$\begin{aligned} \sum_{u \leq v} Q_{u,v}(q) &= \sum_{u \in A} Q_{u*s,v*s}(q) \\ &\quad + \sum_{u \in B} (qQ_{u*s,v*s}(q) + (q-1)Q_{u,v*s}(q)) \\ &\quad + \sum_{u \in C} qQ_{u,v*s}(q) \\ &= \sum_{\substack{u \in A \\ u \leq v*s}} qQ_{u,v*s}(q) + \sum_{\substack{u \in B \\ u \leq v*s}} (1+q-1)Q_{u,v*s}(q) + \sum_{\substack{u \in C \\ u \leq v*s}} qQ_{u,v*s}(q) \\ &= q \sum_{u \leq v*s} Q_{u,v*s}(q), \end{aligned}$$

[3] See e.g. [19] for the definition.

proving the claim. ☐

Lemma 5.8. *We have $P_{u,v}(0) = 1$ whenever $u \leq v$ in $\iota(\theta)$.*

Proof. The assertion is clear if $u = v$, and we employ induction on $\rho(v) - \rho(u)$.

Equation (1) translates to

$$q^{\rho(v)-\rho(u)} P_{u,v}(q^{-1}) = \sum_{u \leq w \leq v} Q_{u,w}(q) P_{w,v}(q). \tag{2}$$

The left hand side is a polynomial with zero constant term. Hence, Lemma 5.6 and the induction assumption imply

$$P_{u,v}(0) = -\sum_{u < w \leq v} \mu(u,w) = \mu(u,u) = 1,$$

as desired.

☐

We are finally in position to prove the main results. Since all necessary technical prerequisites have been established, the corresponding arguments from [6] can now be transferred to our setting more or less verbatim.

The (ii) ⇒ (i) part of the following result can be readily deduced from Brion's [5, Theorem 2.5]. Still, we provide a proof of the full assertion for completeness; omitting said part would not save us more than a few lines.

Theorem 5.9. *Suppose $u, w \in \iota(\theta)$, $u \leq w$. The following conditions are equivalent:*

- (i) *The degree of v in $\mathrm{BG}(w)$ is $\rho(w)$ for all $u \leq v \leq w$.*
- (ii) *The KLV polynomials satisfy $P_{v,w}(q) = 1$ for all $u \leq v \leq w$. That is, the orbit closure $\overline{\mathcal{O}_w}$ is rationally smooth at $\mathcal{O}_u$.*

Proof. Define

$$f_{u,w}(q) = q^{\rho(w)-\rho(u)}(P_{u,w}(q^{-2}) - 1).$$

The P-polynomials have nonnegative coefficients. By Lemma 5.8, $f_{u,w}(q)$ too is a polynomial with nonnegative coefficients. Since it has vanishing constant term, $f'_{u,w}(1) = 0$ if and only if $f_{u,w}(q) = 0$ which, in turn, is equivalent to $P_{u,w}(q) = 1$.

Now,

$$f'_{u,w}(1) = (\rho(w) - \rho(u))(P_{u,w}(1) - 1) - 2P'_{u,w}(1).$$

Using (2) and the fact $Q_{u,w}(1) = \delta_{u,w}$, we obtain

$$\begin{aligned} -2P'_{u,w}(1) &= \frac{d}{dq} P_{u,w}(q^{-2})|_{q=1} \\ &= 2(\rho(u) - \rho(w))P_{u,w}(1) + 2 \sum_{u \leq v \leq w} Q'_{u,v}(1) P_{v,w}(1) + 2P'_{u,w}(1). \end{aligned}$$

Hence,

$$f'_{u,w}(1) = \rho(u) - \rho(w) + \sum_{u \leq v \leq w} Q'_{u,v}(1) P_{v,w}(1).$$

To begin with, assume (ii) holds. Then,

$$\rho(w) - \rho(v) = \sum_{v \leq v' \leq w} Q'_{v,v'}(1)$$

for all $u \leq v \leq w$. Condition (i) now follows from Lemma 5.5 together with Lemma 4.5.

Finally, let us prove (i) $\Rightarrow$ (ii) by induction on $\rho(w) - \rho(u)$. Suppose $u < v \leq w$ in $\mathrm{Br}(\iota(\theta))$. By Lemma 5.5 and the induction assumption, $Q'_{u,v}(1)P_{v,w}(1)$ is one if $\{u, v\}$ is an edge in $\mathrm{BG}(w)$, zero otherwise. Since $\deg(u) = \rho(w)$, u has exactly $\rho(w) - \rho(u)$ neighbours v such that $u < v$. We conclude $f'_{u,w}(1) = 0$ as desired. □

We remark that an alternative route to proving Theorem 5.9 has been suggested by an anonymous referee. Namely, it is possible to establish (i) $\Rightarrow$ (ii) using an argument with Brion's [5, Theorem 1.4] as main ingredient.

Theorem 5.10. *For $w \in \iota(\theta)$, the following are equivalent:*

(i) *The interval $[\mathrm{id}, w] = \mathrm{Br}(I_w)$ has equally many elements of rank i as of rank $\rho(w) - i$.*
(ii) *The graph $\mathrm{BG}(w)$ is regular.*
(iii) *$P_{u,w}(q) = 1$ for all $u \leq w$.*

Proof. (i) $\Rightarrow$ (ii): Let $n(i)$ denote the number of elements of rank i in $[e, w]$. Now, using Lemma 4.5 and Theorem 4.3, we count the edges in $\mathrm{BG}(w)$ in two ways and obtain

$$\sum_{i=0}^{\rho(w)} n(i)i \geq \sum_{i=0}^{\rho(w)} n(i)(\rho(w) - i)$$

with equality if and only if $\mathrm{BG}(w)$ is $\rho(w)$-regular. However, if $n(i) = n(\rho(w) - i)$ for all i, then equality does hold.

(ii) $\Rightarrow$ (iii): This follows from Theorem 5.9.

(iii) $\Rightarrow$ (i): We claim that

$$F_w(q) = \sum_{u \leq w} P_{u,w}(q) q^{\rho(u)}$$

is a symmetric polynomial, i.e. $F_w(q) = q^{\rho(w)} F_w(q^{-1})$. If the P-polynomials are all 1, this means

$$\sum_{u \leq w} q^{\rho(u)} = \sum_{u \leq w} q^{\rho(w) - \rho(u)}.$$

It therefore remains to verify the claim. Observe that

$$\begin{aligned} q^{\rho(w)} F_w(q^{-1}) &= \sum_{u \leq w} q^{\rho(w) - \rho(u)} P_{u,w}(q^{-1}) \\ &= \sum_{u \leq w} \sum_{u \leq v \leq w} Q_{u,v}(q) P_{v,w}(q) \\ &= \sum_{v \leq w} P_{v,w}(q) \sum_{u \leq v} Q_{u,v}(q) \\ &= F_w(q), \end{aligned}$$

where the last equality follows from Lemma 5.7. □

To illustrate these results, consider Figure 1. The interval $[\mathrm{id}, w]$ has three elements of rank three but only two of rank $\rho(w) - 3 = 1$. By Theorem 5.10, $\overline{\mathcal{O}_w}$ is rationally singular. A more careful inspection of the graph shows that $s_5 s_1$ and e both have degree five whereas all other vertices have degree $\rho(w) = 4$. By Theorem 5.9, the rationally singular locus of $\overline{\mathcal{O}_w}$ therefore is $\mathcal{O}_{s_5 s_1} \cup \mathcal{O}_e$. Also, observe that the degree never decreases as we move down in the graph. This phenomenon is explained in the next section.

6. Sufficiency of the bottom vertex

In this final section, the criterion given in Theorem 5.9 is significantly improved in the special case $G = \mathrm{SL}_{2n}(\mathbb{C})$, $K = \mathrm{Sp}_{2n}(\mathbb{C})$. In that case, as we shall see, whether or not an orbit closure $\overline{\mathcal{O}_w}$ is rationally smooth at $\mathcal{O}_u$ is determined by the degree of u alone (Corollary 6.7 below). The corresponding statement for Schubert varieties is known to be true in type A [7] but false in general (see [4] for some elaboration on this). Necessarily, therefore, our arguments must be type specific since they cannot possibly be extended to the situation in Example 3.4 for arbitrary G'.

6.1. **Notation and preliminaries.** Let us spend a few lines fixing notation with respect to the combinatorics of symmetric groups.

We work in the set F_{2n} of fixed point free involutions on $\{1, \dots, 2n\}$. Let $\star$ denote the conjugation action from the right by the symmetric group S_{2n} on itself, i.e. $\sigma \star \pi = \pi^{-1}\sigma\pi$. Then, $F_{2n} = w_0 \star S_{2n}$, where w_0 is the reverse permutation $i \mapsto 2n+1-i$.

If $t = (a, b) \in S_{2n}$ is a transposition and $u \in F_{2n}$, then $u \star t = u$ if and only if t is a 2-cycle in the cycle decomposition of u. If $u \star t \neq u$, the decompositions into 2-cycles of u and $u \star t$ are as follows:

$$u = (a, u(a))(b, u(b)) \cdots ,$$
$$u \star t = (a, u(b))(b, u(a)) \cdots ,$$

where the dots denote the remaining 2-cycles (that both involutions have in common). In particular, there is exactly one transposition $t' \neq t$ such that $u \star t' = u \star t$, namely $t' = (u(a), u(b))$.

The Bruhat order on S_{2n} restricts to a partial order on F_{2n}. Let $\preceq$ denote the dual of this poset. Its bottom element is w_0. Observe that if $u \neq u \star t$, then $u \star t \succ u$ iff t is an *inversion* of u (meaning $t = (a, b)$ with $a < b$ and $u(a) > u(b)$). If $s = (i, i+1)$ is an adjacent transposition, then s is a *descent* if it is an inversion; otherwise s is an *ascent*.

For $u \in F_{2n}$, $1 \leq i, j \leq 2n$, define

$$u_{(i,j)} = |\{x \in \{1, \dots, 2n\} \mid x \leq i \text{ and } u(i) \geq j\}|.$$

Thus, $u_{(i,j)}$ is the number of dots weakly northwest of (i, j) in the permutation diagram of u.

Lemma 6.1 (Standard criterion; Theorem 2.1.5 in [3])**.** *For $u, w \in F_{2n}$, we have $u \preceq w$ iff $u_{(i,j)} \geq w_{(i,j)}$ for all (i, j).*

For $w \in F_{2n}$, define the *Bruhat graph* $\mathrm{BG}(w)$ as the graph whose vertex set is $I_w = \{u \in F_{2n} \mid u \preceq w\}$ and $\{u, v\}$ is an edge iff $u \neq v = u \star t$ for some transposition t. Thus, each edge has exactly two transpositions associated with it, and the graph is simple (no loops or multiple edges). If w is understood from the context and $u \preceq w$, let $\mathrm{out}(u)$ denote the set of edges incident to u in $\mathrm{BG}(w)$. Also, define $\deg(u) = |\mathrm{out}(u)|$.

Proposition 6.2. *Suppose $W = A_{2n-1} \cong S_{2n}$ with $\theta : W \to W$ given by the unique nontrivial involution of the Dynkin diagram. Then, $x \mapsto w_0 x$ defines a bijection $F_{2n} \to \iota(\theta)$. Moreover, the bijection is an isomorphism of Bruhat graphs, i.e. $u \preceq w \Leftrightarrow w_0 u \le w_0 w$ and $w_0(w \star t) = (w_0 w) * t$.*

Proof. This is immediate from the well known facts that $\theta(x) = w_0 x w_0$ and that $x \mapsto w_0 x$ is an antiautomorphism of $\mathrm{Br}(W)$. □

6.2. **An injective map.** Suppose $w \succeq u \neq w_0$ and let $r = (i, j)$, $i < j$, be a transposition such that $u \star r \prec u$. Let $a = u(i)$ and $b = u(j)$. Thus, $a < b \neq i$.

For a transposition $t = (x, y)$, we use the notation $\mathrm{supp}(t) = \{x, y\}$.

Definition 6.3. *A transposition t is called* compatible *(with respect to u and r) if either* $\mathrm{supp}(t) \cap \{a, b, i, j\} = \emptyset$ *or* $\mathrm{supp}(t) \cap \{i, j\} \neq \emptyset$.

Given an edge $e \in \mathrm{out}(u)$ there are precisely two transpositions t and $t' \neq t$ such that $e = \{u, u \star t\} = \{u, u \star t'\}$. At least one of them is compatible.

Definition 6.4. *For any edge $e \in \mathrm{out}(u)$, let t_e be a compatible transposition such that $e = \{u, u \star t_e\}$.*

Definition 6.5. *Given $e \in \mathrm{out}(u)$, define $\epsilon(e) = \{u \star r, u \star r\tau_e\}$, where*

$$\tau_e = \begin{cases} r t_e r & \text{if } u \star t_e r \preceq w, \\ t_e & \text{otherwise.} \end{cases}$$

The point of all this is the following:

Theorem 6.6. *Definition 6.5 defines an injective map $\epsilon : \mathrm{out}(u) \to \mathrm{out}(u \star r)$.*

Proof. This follows from Lemmata 6.10, 6.11 and 6.12 below. □

By Theorem 6.6, the degree can never decrease as we go down along edges in a Bruhat graph. In particular, if a vertex has the minimum possible degree, then so does every vertex above it:

Corollary 6.7. *We have $\deg(v) = \deg(w)$ for all $u \preceq v \preceq w$ if and only if $\deg(u) = \deg(w)$.*

Thus, to determine whether condition (i) of Theorem 5.9 is satisfied, it suffices to check the degree of u.

Remark 6.8. The set $\mathcal{S}_{2n} = \{w \in F_{2n} \mid i \le n \Rightarrow w(i) \ge n+1\}$ is in natural bijective correspondence with S_{2n} in a way which identifies $\mathrm{Br}(S_{2n})$ with $\preceq$. Restricted to $w \in \mathcal{S}_{2n}$, Corollary 6.7 specialises to a result of Deodhar [7] for type A Schubert varieties. In that setting, our arguments are closely related to work of Billey and Warrington [2, §6]

Remark 6.9. Observe that for $G = \mathrm{SL}_{2n}(\mathbb{C})$, $K = \mathrm{Sp}_{2n}(\mathbb{C})$, Theorem 4.3 follows directly from Theorem 6.6. Thus, we have reproven Brion's [5, Theorem 2.5] in this case.

6.3. **Proof of Theorem 6.6.**

Lemma 6.10. *The set $\epsilon(e)$ is well defined, i.e. independent of the choice of t_e.*

Proof. This is clear if $\mathrm{supp}(t) \cap \{a,b,i,j\} = \emptyset$. If not, the only case when both transpositions associated with e are compatible is when $e = \{u, u\star t\} = \{u, u\star t'\}$ for $\{t,t'\} = \{(i,b),(j,a)\}$. In this case, we have $u\star tr = u\star t'r$ and $u\star rt = u\star r = u\star rt'$. □

Lemma 6.11. *For every $e \in \mathrm{out}(u)$, we have $\epsilon(e) \in \mathrm{out}(u\star r)$.*

Proof. We must show that $u\star r \neq u\star r\tau_e \preceq w$.

First, assume $u\star r = u\star r\tau_e$. Then, $u\star t_e = u\star r\tau_e r t_e$. If $\tau_e = rt_er$, this means $u\star t_e = u$ which contradicts the fact that $e \in \mathrm{out}(u)$. If, on the other hand, $\tau_e = t_e$, then we conclude that r and t_e do not commute, hence that $rt_ert_er = t_e$. But then, $u\star t_er = u\star rt_ert_er = u\star t_e \preceq w$ which contradicts $\tau_e = t_e$. Thus, $u\star r \neq u\star r\tau_e$.

It remains to prove $u\star r\tau_e \preceq w$, i.e. that either $u\star t_er \preceq w$ or $u\star rt_e \preceq w$ (or both). There are a few cases:

Case 1. If $\mathrm{supp}(t_e)\cap\{i,j,a,b\} = \emptyset$, then $u\star t_e(i) < u\star t_e(j)$. Thus, $w \succeq u\star t_e \succ u\star t_er$.

Case 2. If $t_e = (i,j)$, then $u\star t_er = u \preceq w$.

Case 3. If $t_e = (i,b)$, then $u\star t_e(i) = j$ so that $u\star t_er = u\star t_e \preceq w$.

Case 4. If $t_e = (j,a)$, we again have $u\star t_e(i) = j$.

Case 5. If $t_e = (i,k)$ with $k \notin \{j,a,b\}$, then $rt_er = t_ert_e = (j,k)$ and $u\star rt_e = u\star t_ert_er$. Let $c = u(k)$.

We have $u\star t_e(i) = c$, $u\star t_e(j) = b$, $u\star t_e(k) = a$, $u\star r(i) = b$ and $u\star r(k) = c$. If $k < j$, it follows that $w \succeq u\star t_e \succ u\star t_ert_er = u\star rt_e$. Otherwise, $k > i$ and either $w \succeq u\star r \succ u\star rt_e$ (if $b < c$) or $u\star t_er \prec u\star t_e$ (if $b > c$).

Case 6. If $t_e = (j,k)$ with $k \notin \{i,a,b\}$, then $rt_er = t_ert_e = (i,k)$ and $u\star rt_e = u\star t_ert_er$. Again, let $c = u(k)$.

Now, $u\star t_e(i) = a$, $u\star t_e(j) = c$ and $u\star t_e(k) = b$ so that $u\star t_er \succ u\star t_e$ implies $c < a < b$. It follows that either $u\star rt_e = u\star t_ert_er \prec u\star t_e \preceq w$ (if $i < k$) or $u\star r \succ u\star rt_e$ (if $k < j$). □

Lemma 6.12. *If $e \neq e'$ for $e,e' \in \mathrm{out}(u)$, then $\epsilon(e) \neq \epsilon(e')$.*

Proof. Suppose $e,e' \in \mathrm{out}(u)$ and $e \neq e'$. There are three cases:

Case 1. If $\tau_e = rt_er$ and $\tau_{e'} = rt_{e'}r$, then $\epsilon(e) = \epsilon(e') \Leftrightarrow u\star t_er = u\star t_{e'}r \Leftrightarrow u\star t_e = u\star t_{e'}$ which contradicts $e \neq e'$.

Case 2. Suppose $\tau_e = t_e$ and $\tau_{e'} = t_{e'}$. Assume $\epsilon(e) = \epsilon(e')$, i.e. $u\star rt_e = u\star rt_{e'}$. If both t_e and $t_{e'}$ commute with r we argue as in the previous case. If not, since both t_e and $t_{e'}$ are compatible, we either have $\{t_e, t_{e'}\} = \{(i,b),(j,a)\}$ leading to the contradiction $u\star t_e = u\star t_{e'}$, or we have $\{t_e,t_{e'}\} = \{(i,a),(j,b)\}$ which implies the contradiction $u = u \star t_e$.

Case 3. Finally, assume $\tau_e = rt_er$ and $\tau_{e'} = t_{e'}$. Then, the assumption $\epsilon(e) = \epsilon(e')$ amounts to $u \star t_e = u \star rt_{e'}r$. Suppose $rt_{e'} \neq t_{e'}r$ and $rt_e \neq t_er$; otherwise we would be in Case 1 or 2, respectively. This implies that either $rt_{e'}r = t_e$ or $\{t_e, rt_{e'}r\} = \{(i,b),(j,a)\}$. The latter case, though, leads to $u \star r = u \star rrt_{e'}r = u \star t_{e'}r$ implying the contradiction $u = u \star t_{e'}$. Thus, $rt_{e'}r = t_e$, and therefore $\{t_e, t_{e'}\} = \{(i,k),(j,k)\}$ for some $k \notin \{i,j,a,b\}$. Let us suppose $t_e = (i,k)$; the other case is completely similar. A small computation shows that

$$\begin{aligned} u &= (i,a)(j,b)(k,c)\cdots, \\ u \star r &= (i,b)(j,a)(k,c)\cdots, \\ u \star t_e &= (i,c)(j,b)(k,a)\cdots, \\ u \star rt_e &= (i,c)(j,a)(k,b)\cdots, \\ u \star t_er &= (i,b)(j,c)(k,a)\cdots, \\ u \star rt_er &= (i,a)(j,c)(k,b)\cdots, \end{aligned}$$

where we have written down the 2-cycle decompositions of the various elements (the dots indicate the remaining cycles; they are equal for all six elements). In particular, the elements are all distinct, so $|u \star \langle r, t_e\rangle| = 6$.

Now observe that precisely five of the elements in $u \star \langle r, t_e\rangle$ belong to $\mathrm{BG}(w)$; the one which does not is $u \star rt_e = u \star t_{e'}r$, because $\tau_{e'} \neq rt_{e'}r$. A contradiction is now provided by Lemma 6.13 below. □

Lemma 6.13. *Suppose $|u \star \langle t_1, t_2\rangle \cap I_w| \geq 5$ for two elements $u, w \in F_{2n}$ and some transpositions t_1, t_2. Then, $|u \star \langle t_1, t_2\rangle \cap I_w| = 6$, i.e. $u \star \langle t_1, t_2\rangle \subseteq I_w$.*

Proof. The set of transpositions in the dihedral subgroup $\langle t_1, t_2\rangle$ is $\{t_1, t_2, t_1t_2t_1\} = \{(x_1,x_2),(x_2,x_3),(x_1,x_3)\}$ for some $1 \leq x_1 < x_2 < x_3 \leq 2n$. There are elements $1 \leq a_1 < a_2 < a_3 \leq 2n$, with $x_i \neq a_j$ for all i and j, such that $u \star \langle t_1, t_2\rangle$ consists of the six involutions with cycle decomposition of the form

$$(x_1, a_{i_1})(x_2, a_{i_2})(x_3, a_{i_3})\cdots,$$

dots denoting the 2-cycles in u with support disjoint from $\{x_1, x_2, x_3, a_1, a_2, a_3\}$.

In order to simplify notation, let

$$[i_1i_2i_3] = (x_1, a_{i_1})(x_2, a_{i_2})(x_3, a_{i_3})\cdots.$$

Since $[123]$ is the maximum element in $u \star \langle t_1, t_2\rangle$, it suffices to show that $w \succeq [123]$ whenever $w \succeq [213]$ and $w \succeq [132]$. To this end, consider the standard criterion. For $1 \leq \alpha, \beta \leq 2n$, let

$$\begin{aligned} [i_1i_2i_3]^+_{(\alpha,\beta)} &= |\{j \in \{1,2,3\} \mid x_j \leq \alpha \text{ and } a_{i_j} \geq \beta\}, \\ [i_1i_2i_3]^-_{(\alpha,\beta)} &= |\{j \in \{1,2,3\} \mid a_{i_j} \leq \alpha \text{ and } x_i \geq \beta\}. \end{aligned}$$

Then, the number of dots weakly northwest of (α, β) in the diagram of $[i_1i_2i_3]$ is

$$[i_1i_2i_3]_{(\alpha,\beta)} = [i_1i_2i_3]^+_{(\alpha,\beta)} + [i_1i_2i_3]^-_{(\alpha,\beta)} + D,$$

where D counts dots with coordinates outside $\{x_1, x_2, x_3, a_1, a_2, a_3\}$; this number is independent of i_1, i_2, i_3.

By the symmetry between x and a, it is sufficient to show

$$[123]^+_{(\alpha,\beta)} = \min\left([213]^+_{(\alpha,\beta)}, [132]^+_{(\alpha,\beta)}\right)$$

for all α, β. This statement, however, follows immediately from the observation that for all m, the first m letters in the string "123" are the same as the first m letters in one of the strings "213" and "132". □

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