FIXED POINTS OF ZIRCON AUTOMORPHISMS

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Abstract. A *zircon* is a poset in which every principal order ideal is finite and equipped with a so-called special matching. We prove that the subposet induced by the fixed points of any automorphism of a zircon is itself a zircon. This provides a natural context in which to view recent results on Bruhat orders on twisted involutions in Coxeter groups.

1. Background and results

Let P be a partially ordered set (*poset*). A *matching* on P is an involution $M : P \to P$ such that $M(p) \lhd p$ or $p \lhd M(p)$ for all $p \in P$, where $\lhd$ denotes the covering relation of P. In other words, M is a graph-theoretic (complete) matching on the Hasse diagram of P.

Definition 1.1. *Suppose M is a matching on a poset P. Then, M is called* special *if for all $p, q \in P$ with $p \lhd q$, we either have $M(p) = q$ or $M(p) < M(q)$.*

The term "special matching" was coined by Brenti [3, 4]. In the context of an Eulerian poset, a special matching is another way to think of a *compression labelling* as defined by du Cloux [6].

Definition 1.2. *A poset P is a* zircon *if for any non-minimal element $x \in P$, the subposet induced by the principal order ideal $\{p \in P \mid p \le x\}$ is finite and has a special matching.*

Zircons were defined by Marietti in [12]. Actually, his definition differs somewhat from ours, but Proposition 2.3 below shows that they are equivalent. We have chosen to use our definition because it is typically more convenient to check the finiteness condition in Definition 1.2 rather than finding a rank function as required by the definition in [12].

The motivation to introduce zircons comes from the fact that they mimic the behaviour of Coxeter groups ordered by the Bruhat order. More precisely, the Bruhat order ideal below a non-identity element w in a Coxeter group has a special matching given by multiplication with any descent of w. The finiteness condition in Definition 1.2 is trivially satisfied, implying that the Bruhat order on any Coxeter group is a zircon.

In this note, we study the fixed points of automorphisms of zircons. Our main results are the next theorem and its corollary. The proofs are postponed to Section 2.

Say that a poset is *bounded* if it has unique maximal and minimal elements.

Theorem 1.3. *Suppose P is a finite, bounded poset equipped with a special matching M. Let ϕ be an automorphism of P. Then, the subposet of P induced by the fixed points of ϕ has a special matching.*

Corollary 1.4. *The fixed points of any automorphism of a zircon induce a subposet which is itself a zircon.*

Now, we briefly describe our reasons for being interested in results of this kind. We refer to [11] or [1] for a thorough account of the theory of Coxeter groups and their Bruhat orders.

Let (W, S) be a finitely generated Coxeter system. For $X \subseteq W$, let $\mathrm{Br}(X)$ denote the subposet of the Bruhat order on W which is induced by X. A fundamental result due to Björner and Wachs [2] asserts that the (order complexes[1] of the) open intervals in $\mathrm{Br}(W)$ are homeomorphic to spheres.

An interesting subposet of $\mathrm{Br}(W)$ is induced by the involutions in W. More generally, if $\theta : W \to W$ is an involutive group automorphism which preserves the generating set S, the set of *twisted involutions* is

$$\mathfrak{I}(\theta) = \{w \in W \mid \theta(w) = w^{-1}\}.$$

The ordinary involutions are obtained by taking θ to be the trivial automorphism. Richardson and Springer [15, 16] showed that $\mathrm{Br}(\mathfrak{I}(\theta))$ is of importance to the study of orbit decompositions of certain symmetric varieties.

In [10] it was shown that $\mathrm{Br}(\mathfrak{I}(\theta))$, just as $\mathrm{Br}(W)$, has the property that every open interval is a sphere. The method of proof was to show that $\mathrm{Br}(\mathfrak{I}(\theta))$, too, is a zircon (although this terminology was not used), and then observing that a result of Dyer [7] implies that every open interval in any zircon is homeomorphic to a sphere.

Earlier, another approach to the topology of $\mathrm{Br}(\mathfrak{I}(\theta))$ was followed in [9] where it was observed that $\mathrm{Br}(\mathfrak{I}(\theta))$ is the subposet of $\mathrm{Br}(W)$ induced by the fixed points of the involutive poset automorphism given by $w \mapsto \theta(w^{-1})$. Invoking Smith theory on automorphisms of spheres [17], this permitted the conclusion that the intervals in $\mathrm{Br}(\mathfrak{I}(\theta))$ are homology spheres over the integers modulo 2.

To summarise, we have a zircon (namely $\mathrm{Br}(W)$) whose intervals are spheres. We construct an involution on it whose induced subposet of fixed points (namely $\mathrm{Br}(\mathfrak{I}(\theta))$) also turns out to form a zircon. Therefore, the intervals in this fixed point poset are not only $\mathbb{Z}_2$-homology spheres (as implied by Smith theory) but actual spheres.

Corollary 1.4 explains this behaviour by showing that, in fact, any automorphism of any zircon has a zircon as induced fixed point poset.

Remark 1.5. Let θ and W be as above. Then, θ is a poset automorphism of $\mathrm{Br}(W)$. Moreover, it is known [8, 13, 18] that the fixed points $\mathrm{Fix}(\theta)$ themselves form a Coxeter group. It was shown in [9], and independently by Nanba [14], that the subposet of $\mathrm{Br}(W)$ induced by $\mathrm{Fix}(\theta)$ coincides with the Bruhat order on $\mathrm{Fix}(\theta)$. In particular, this is another situation where the fixed points of a zircon automorphism are known to form a zircon.

2. Proofs

The next lemma provides an extremely useful tool when dealing with posets with special matchings. For Bruhat orders it was established by Deodhar [5, Theorem 1.1]. Brenti [4, Lemma 4.2] proved the general case under the assumption that P

[1]The order complex of a poset is the (abstract) simplicial complex whose simplices are the totally ordered subsets.

is graded. This assumption is, however, not essential to the proof. For the reader's convenience, we restate Brenti's proof with slight adjustments.

A poset is *locally finite* if every interval is finite.

Lemma 2.1 (Lifting property)**.** *Suppose P is a locally finite poset with a special matching M. Choose $x, y \in P$ with $x < y$ and $M(y) < y$. Then,*

(i) $M(x) \leq y$.
(ii) $M(x) < x \Rightarrow M(x) < M(y)$.

Proof. We proceed by induction on the length of a shortest non-refinable chain c in $[x, y]$, both statements following directly from the definition of special matchings if $x \lhd y$.

For the first assertion, we may assume $M(x) > x$. Choose $z \in c$ such that $x \lhd z$. If $M(x) = z$, we are done. Otherwise, we have $M(z) > M(x)$ since M is a special matching. By the induction assumption, we may therefore conclude $M(z) \leq y$, so that $M(x) \leq y$.

To prove the second claim, suppose $M(x) < x$. Pick $z \in c$ with $z \lhd y$. In case $M(y) = z$, there is nothing to prove. Otherwise, because M is special, $M(z) < M(y)$. Moreover, $M(z) < z$ since otherwise we would have $z < M(z) < M(y) < y$ contradicting the choice of z. By induction, $M(x) < M(z)$, and the proof is complete. □

We are now ready to prove the main results.

Proof of Theorem 1.3. The automorphism ϕ is of finite order N since it is a permutation of a finite set. Each automorphism ϕ^k, $k \in [N] = \{1, \ldots, N\}$, transforms M into a special matching M_k on P. In particular, $M = M_N$.

Given $p \in P$, let

$$C(p) = \{q \in P \mid q = M_{i_t} \circ M_{i_{t-1}} \circ \cdots \circ M_{i_1}(p) \text{ for some } i_1, \ldots, i_t \in [N]\}.$$

In other words, $C(p)$ consists of the elements in the same connected component as p in the graph we obtain from the Hasse diagram of P by throwing away the edges that are not used by any of the matchings M_k. By abuse of notation, we also let $C(p)$ denote the subposet of P induced by this set.

Given $q \in C(p)$, we may write $q = M_{i_t} \circ \cdots \circ M_{i_1}(p)$ for suitably chosen $i_j \in [N]$. Now define $q' \in C(p)$ by $q' = M'_{i_t} \circ \cdots \circ M'_{i_1}(p)$, where we recursively have defined

$$M'_{i_j} = \begin{cases} M_{i_j} & \text{if } M_{i_j} \circ M'_{i_{j-1}} \cdots \circ M'_{i_1}(p) < M'_{i_{j-1}} \circ \cdots \circ M'_{i_1}(p), \\ \mathrm{id} & \text{otherwise.} \end{cases}$$

For brevity, define $a_j = M_{i_j} \circ \cdots \circ M_{i_1}(p)$ and $a'_j = M'_{i_j} \circ \cdots \circ M'_{i_1}(p)$. We claim that $a'_j \leq a_j$ for all $j \in \{0, \ldots, t\}$. To see this, we assume by induction that $a'_{j-1} \leq a_{j-1}$. If $a_j > a_{j-1}$, the desired conclusion is immediate. Otherwise, we either have $a'_j = M_{i_j}(a'_{j-1})$ or $a'_j = a'_{j-1}$. In the former case, $a'_j \leq a_j$ by the lifting property. In the latter, we may apply the lifting property with (using the notation of Lemma 2.1) $x = M_{i_j}(a'_{j-1})$ and $y = a_{j-1}$, again concluding $a'_j \leq a_j$. The claim is established. In particular, $q' \leq q$.

By construction, $p \geq q'$. Furthermore, if q is a minimal element in $C(p)$, we have $q = q'$. Thus, $C(p)$ has a unique minimal element. A completely analogous argument, where we reverse the inequality in the definition of M'_{i_j}, shows that $C(p)$ also has a unique maximal element. Moreover, the same line of reasoning shows that if p is neither minimal nor maximal in $C(p)$, then there exist $i, j \in [N]$ such that $M_i(p) \lhd p$ and $M_j(p) \rhd p$.

Let P^ϕ be the subposet of P induced by the fixed points of ϕ. Any $p \in P^\phi$ is either minimal or maximal in $C(p)$, because $M_i(p) \lhd p$ either holds for all i or for none of the i. Furthermore, for any given $p \in P$, $\min C(p)$ belongs to P^ϕ if and only if $\max C(p)$ does; this happens if and only if $\phi(C(p)) = C(p)$.

Assume $p \in P^\phi$ is the maximal element in $C(p)$, and let $q \in P^\phi$ denote the minimal element in $C(p)$. We claim that p covers q in P^ϕ. Indeed, suppose $r < p$ for some $r \in P^\phi$ with $r = \min C(r)$. Choose an expression $q = M_{i_t} \circ \cdots \circ M_{i_1}(p)$ with t as small as possible. Repeated application of the lifting property shows $M_{i_j} \circ \cdots \circ M_{i_1}(p) \geq r$ for all $j \in [t]$. Hence, $q \geq r$. A similar argument shows that if $q < r$ and $r = \max C(r)$, then $r \geq p$. The claim is established.

The above shows that we have a well-defined matching M^ϕ on P^ϕ given by

$$M^\phi(p) = \begin{cases} \min C(p) & \text{if } p = \max C(p), \\ \max C(p) & \text{otherwise.} \end{cases}$$

It remains to show that M^ϕ is special, so suppose p covers q in P^ϕ. First, we assume $p = \max C(p)$. If $\min C(p) = q$, there is nothing to show. Otherwise, $q = \max C(q)$ as was shown above. In this case, the above argument shows $\min C(p) > \min C(q)$, i.e. $M^\phi(p) > M^\phi(q)$ as required. The situation when $p = \min C(p)$ is completely analogous. □

Corollary 1.4 is now straightforward to establish.

Proof of Corollary 1.4. Let P be a zircon. Choose a non-minimal $p \in P$. The principal order ideal $P_{\leq p} = \{q \in P \mid q \leq p\}$ contains a unique minimal element $\min P_{\leq p}$. To see this, choose a special matching M on $P_{\leq p}$. If $p_0 \in P_{\leq p}$ is minimal, then $p_0 < M(p_0) \leq p$ and $p_0 \leq M(p)$ by the lifting property. Therefore, if $P_{\leq p}$ would contain more than one minimal element, the same would be true for $P_{\leq M(p)}$. Thus, we would obtain an infinite descending sequence in $P_{\leq p}$ contradicting its finiteness.

Choose an automorphism ϕ of P, and let P^ϕ denote the subposet of P induced by the fixed points. If $p \in P^\phi$, then $\min P_{\leq p} \in P^\phi$, too. Thus, the principal order ideal $P^\phi_{\leq p}$ coincides with the fixed point set of the restriction of ϕ to the interval $[\min P_{\leq p}, p]$. By Theorem 1.3, $P^\phi_{\leq p}$ has a special matching. □

Earlier, we claimed that our definition of zircons coincides with the one given by Marietti. We conclude by verifying this assertion.

A poset P is *ranked* if there is a *rank function* $\rho : P \to \mathbb{N}$ satisfying $\rho(x) = \rho(y) - 1$ whenever $x \lhd y$.

Definition 2.2 (Marietti [12])**.** *A* zircon *is a locally finite, ranked poset in which every non-trivial principal order ideal has a special matching.*

Proposition 2.3. *Definition 1.2 and Definition 2.2 define the same class of posets.*

Proof. In this proof, say that a poset satisfying Definition 1.2 is of the *first kind*, whereas Definition 2.2 defines posets of the *second kind*.

First, suppose P is of the second kind. It was shown in [12] that the principal order ideals in P are intervals, i.e. have unique minimal elements. They are finite since P is locally finite. Hence, P is of the first kind.

For the converse, we now assume P to be of the first kind. Since every interval is contained in a principal order ideal, P is locally finite. To show that P is ranked, it suffices to verify that for any $p \in P$, the maximal chains in $P_{\leq p}$ all have the same length.

Assume, in order to deduce a contradiction, that $P_{\leq p}$ is a minimal non-ranked principal order ideal. Let M be a special matching on $P_{\leq p}$. Choose $q \lhd p$ such that the maximal chains in $P_{\leq q}$ differ in length from those in $P_{\leq M(p)}$. Since $M(q) < M(p)$, this implies that there is some z with $M(q) \lhd z < M(p)$. By the lifting property, $M(z) \in P_{\leq p}$. However, this means that $q \not< M(z)$, contradicting the fact that M is special. Hence, P is of the second kind. □

References

[1] A. Björner and F. Brenti, *Combinatorics of Coxeter groups*, Graduate Texts in Mathematics, vol. 231, Springer, New York, 2005.

[2] A. Björner and M. Wachs, Bruhat order of Coxeter groups and shellability, *Adv. Math.* **43** (1982), 87–100.

[3] F. Brenti, Kazhdan-Lusztig polynomials: History, problems, and combinatorial invariance, *Sémin. Lothar. Combin.* **49** (2003), B49b, 30pp.

[4] F. Brenti, The intersection cohomology of Schubert varieties is a combinatorial invariant, *European J. Combin.* **25** (2004), 1151–1167.

[5] V. V. Deodhar, Some characterizations of Bruhat ordering on a Coxeter group and determination of the relative Möbius function, *Invent. Math.* **39** (1977), 187–198.

[6] F. du Cloux, An abstract model for Bruhat intervals, *European J. Combin.* **21** (2000), 197–222.

[7] M. J. Dyer, *Hecke algebras and reflections in Coxeter groups*, Ph. D. thesis, University of Sydney, 1987.

[8] J.-Y. Hée, Systèmes de racines sur un anneau commutatif totalement ordonné, *Geom. Dedicata* **37** (1991), 65–102.

[9] A. Hultman, Fixed points of involutive automorphisms of the Bruhat order, *Adv. Math.* **195** (2005), 283–296.

[10] A. Hultman, The combinatorics of twisted involutions in Coxeter groups, *Trans. Amer. Math. Soc.* **359** (2007), 2787–2798.

[11] J. E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Univ. Press, 1990.

[12] M. Marietti, Algebraic and combinatorial properties of zircons, *J. Algebraic Combin.* **26** (2007), 363–382.

[13] B. Mühlherr, Coxeter groups in Coxeter groups, *Finite Geometry and Combinatorics (Deinze 1992)*, 277–287, London Math. Soc. Lecture Note Ser., vol. 191, Cambridge Univ. Press, Cambridge, 1993.

[14] M. Nanba, Bruhat order on the fixed-point subgroup by a Coxeter graph automorphism, *J. Algebra* **285** (2005), 470–480.

[15] R. W. Richardson and T. A. Springer, The Bruhat order on symmetric varieties, *Geom. Dedicata* **35** (1990), 389–436.

[16] R. W. Richardson and T. A. Springer, Complements to: The Bruhat order on symmetric varieties, *Geom. Dedicata* **49** (1994), 231–238.

[17] P. A. Smith, Transformations of finite period, *Ann. of Math.* **39** (1938), 127–164.

[18] R. Steinberg, Endomorphisms of linear algebraic groups, *Mem. Amer. Math. Soc.* **80** (1968), 1–108.

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