Beyond the representation given\(^1\) –
The parabola and historical metamorphoses of meanings\(^2\)

**Abstract**

Tracking the history of a mathematical object reveals a double influence on its development - from inside and outside the pure mathematical realm. From a semiotic point of view, a chain of metamorphoses of meanings is created, which becomes critical as the mathematical object by the social process of didactic transposition is turned into a presented object of teaching and learning. As an example, by representing the mathematical object “parabola” by means of its historical progression within classical geometry, analytic geometry, and dynamic geometry software, it is seen, by using analytic tools from semiotics, anthropological theory of didactics, and embodied cognition, how and why students’ concept images and problem solving techniques may become disconnected and instrumental.

**Introduction**

In today’s school mathematics concepts and methods are often treated in isolation, with only surface level theoretical embedding within a mathematical sub-domain. In addition, this treatment is most often, at least in Swedish secondary school, more or less dominated by algebraic tools. However, many students have significant problems to handle and appreciate this particular semiotic tool, both regarding the meaning of symbols used and the ways to handle them. As a consequence, an image of mathematics as disconnected and difficult to understand may emerge. Applications used in teaching to enhance motivation then easily get the character of decorations rather than parts of an integrated knowledge structure. One example that well illustrates this didactic phenomenon is the parabola. Problems and techniques related to this ‘classical’ graph are ubiquitous in school mathematics:

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\(^1\) The title alludes to Bruner (1973) who argues that “when one goes beyond the information given, one does so by virtue of being able to place the present given in a more generic coding system and that one essentially reads off from the coding system additional information either on the basis of learned contingent probabilities or learned principles of relating material” (p. 224).

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• the concept of square root, often used as a basis for the extension of the number concept from rational to real numbers,
• the solution of second degree equations, often used as a basis for the extension of the number concept from real to complex numbers,
• a prototype for polynomials and the connection between zeros and factoring of polynomials,
• second degree polynomials often serve as a prototype for the study of derivatives and optimisation problems,
• a prototypical example for mathematical modelling, e.g. of reflections (parabolic antennas) and projectile motion.

It is obvious that a student’s conceptualisation of a parabola is coloured by the way it is described, defined, treated, used, and so forth. The term concept image has been coined to capture the (non-static) outcome of this process (Tall & Vinner, 1981). A key role is here taken by the representations used, and how the object in focus (in this case the parabola) is being objectified by the mediation of these representations (Radford, 2003). However, also at the cultural level of the mathematical community the representations used influence the views and the development of the object, which in turn have an influence on the development of mathematics itself. In today’s school mathematics these different ways of looking at a mathematical object, such as the parabola, all live in a mixed world of mathematical ideas and tools, not always treated by a systematic approach, as seen from the students’ perspective.

In this paper, a short account of the historical progression of the mathematical object ”parabola”, within classical geometry, analytic geometry, and dynamic geometry software will be given. From a semiotic point of view, a chain of metamorphoses of meanings will be described, and how this becomes critical as the mathematical object by the social process of didactic transposition is turned into a presented object of teaching and learning. The purpose of the paper is to open up for questions, by linking different approaches, rather than to try to answer them.

**Historical metamorphoses**

Tracking the history of a mathematical object reveals a double influence on its development - from inside and outside the pure mathematical realm. In the works of Euclid, Archimedes and Apollonius the parabola is a geometric object, defined rhetorically and analysed by the tools of constructive and deductive geometry. In the famous treatise by Archimedes of the quadrature of the parabola, he speaks of the parabola as ”a section of a right angled cone” (Archimedes, 1952, p. 527)
and refers to "the elementary propositions in conics which are of service in the proof" (p. 527), three of which are referenced to the work (now lost) by Euclid and Aristaeus (p. 528). By this definition a parabola is the section produced by a plane cutting a right angled cone at a right angle to an element of the cone. Historically, the conic sections were invented, it is believed, by the attempts of Menaechmus (approx. 350 B.C.) to solve the problem of duplicating the cube (Eves, 1983, p. 80). One of the "elementary propositions" listed by Archimedes (1952, p. 528) is of special interest here:

If from a point on a parabola a straight line be drawn which is either itself the axis or parallel to the axis, as $PV$, and if from two other points $Q$, $Q'$ on the parabola straight lines be drawn parallel to the tangent at $P$ meeting $PV$ in $V$, $V'$ respectively, then $PV : PV' = QV^2 : Q'V'^2$.

The property identified here is what relates the parabola, in modern language and within another mathematical domain, to the quadratic function (see e.g. Charbonneau, 1997, p. 20, however discussing Apollonius). The basic idea in the classic constructions involved in relating the parabola to a square, in line with the proposition above, builds on the Pythagorean method of applications of rectilinear areas. The construction in figure 1 is based on Euclid I.43 (see Thompson, 1991, p. 494).

Figure 1. Area application

Figure 2. Definition of parabola
(Apollonius, 1952)

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3 The power notation on $QV$ is used by the editor to refer to the square with each of its sides equal to $QV$.

4 The names that Apollonius introduced for the conic sections (ellipse, parabola, hyperbola) have their origin in such applications; see e.g. Eves, 1983, p. 128).
In figure 1, OC is a given segment (parameter), apply the square on the (variable) segment OA to OC by the method in Euclid I.43. This will produce the point P, choose P’ so that BP’ and BP are equal. As the point A is moving horizontally the points P and P’ will move along a parabola. Using algebraic notation, with OB = x, OA = y, and OC = 2p, we have \( y^2 = 2px \). In the more general setting of Apollonius, the parabola is defined, along with a proposition (Conics I.11) similar to the one given above, by the following wordings after the proof (see figure 2):

And let such a section be called a parabola, and let HF be called the straight line to which the straight lines drawn ordinatewise to the diameter FG are applied in square [...], and let it also be called the upright side [...]. (Apollonius, 1952, p. 616)

Apollonius defines the diameter of a section as the line passing through the midpoints of parallel chords, and the axis of the section as the diameter which is perpendicular to the parallel chords. The focal property is not discussed for the parabola but may be inferred as limit cases from such discussions about the hyperbola and the ellipse (see e.g. the note in Apollonius, 1952, p. 788). The terminology used, based on drawings where the size of segments are supposed to vary, may be seen as predecessors of both the coordinate system of analytic geometry and the concepts of variable and parameter as used in the study of functions (see e.g. Charbonneau, 1997). The semiotic register found in the works of Euclid, Archimedes and Apollonius consists of natural language (i.e. symbols) and diagrams (drawings with letters indicating points, i.e. icons and indices).

After the event of analytic geometry, most notably by Descartes and Fermat, where the more modern algebraic notation in Descartes’ book La géomètrie from 1637 was the most influential, a line (curve) could be described by the relation between coordinates (referring to distances along given directions) and studied by algebraic treatment. By representing a point on a parabola by an algebraic equation of the coordinates at the point, it was, by a historical metamorphosis, transformed from a geometrical object into an algebraic object: from being described by a rhetoric sequence (referring to a configuration) or a

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\(^5\) Apollonius defined a conic surface and a cone as follows: "1. If from a point a straight line is joined to the circumference of a circle which is not in the same plane with the point, and the line is produced in both directions, and if, with the point remaining fixed, the straight line being rotated about the circumference of the circle returns to the same place from which it began, then the generated surface composed of the two surfaces lying vertically opposite one another, each of which increases indefinitely as the generating line is produced indefinitely, I call a conic surface, and I call the fixed point the vertex, and the straight line drawn from the vertex to the center of the circle the axis. 2. And the figure contained by the circle and by the conic surface between the vertex and the circumference of the circle I call a cone, and the point which is also the vertex of the surface I call the vertex of the cone, and the straight line drawn from the vertex to the center of the circle the axis, and the circle the basis of the cone." (Apollonius, 1952, p. 604)
diagram (drawing), it showed itself as an algebraic expression such as \( y = x^2 \). By defining the geometrical properties of a curve, it was possible by inserting a coordinate system to find an algebraic representation\(^6\), provided the algebraic manipulations could be coped with. As an example, from the defining property for a parabola as the loci for points with the same distance to a fixed point (focus) as to a fixed line (directrix)\(^7\), a simple calculation gives the equation \( x^2 = 4ay \) if the focus point is in \((0,a)\) and the directrix is \( y = -a \), using an ON system. The fact that Fermat reverted this process by finding the geometric counterparts starting with given algebraic equations (Eves, 1983, p. 265), shows how the power of representations through semiotic invention and chaining may produce new and unexpected knowledge. Curves never before thought of, such as the ‘parabolas of Fermat’ (given by the equations \( y^n = ax^m \)), or in modern time fractals, could be described and pictured. Descartes performs in the second book of his *Géomètrie* a classification of all possible curves produced by a quadratic equation in two variables, and opens the way for higher order equations. His analytic method is clearly expressed in these lines:

I could give here several other ways of tracing and conceiving a series of curved lines, each curve more complex than any preceding one, but I think the best way to group together all such curves and then classify them in order, is by recognizing the fact that all points of those curves which we call "geometric", that is, those which admit of precise and exact measurement, must bear a definite relation to all points on a straight line, and that this relation must be expressed by means of a single equation. If this equation contains no term of higher degree than the rectangle of two unknown quantities, or the square of one, the curve belongs to the first and simplest class, which contains only the circle, the parabola, the hyperbola, and the ellipse; but when the equation contains one or more terms of the third or fourth degree in one or both of the unknown quantities (for it requires two unknown quantities to express the relation between two points) the curve belongs to the second class; and if the equation contains a term of the fifth or sixth degree in either or both of the unknown quantities the curve belongs to the third class, and so on indefinitely. (Descartes, 1954, p. 48)\(^8\)

Thus, curves corresponding to the equation \( y^n = px \), \( n > 2 \), are considered parabolas of higher order (Eves, 1983, p. 264). When the properties of geometric

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\(^6\) In the embodied cognition literature this phenomenon is termed a "conceptual blend" (Lakoff & Nuñez, 2000, p. 48).

\(^7\) In the *Conics* of Apollonius we find the corresponding properties of the hyperbola and the ellipse in propositions III.51 and III.52, respectively.

\(^8\) It can be observed here how Descartes still sticks to the tradition when interpreting second order terms like \( xy \) or \( xx \) geometrically ("rectangle", "square"), but calls higher order terms more freely as "third or fourth degree", and writes them with exponential notation.
objects be studied by work in the algebraic domain, the latter may serve as a means to better understand, or gain new knowledge, of the former (Bolea et al., 1999). The symbolic language of algebra can thus be seen as a didactic tool to learn about the other domain, in this case geometry, fundamentally different (Bergsten, 2003).

As an example of Descartes’ algebraic work, in performing the classification of the second degree equation, the following short passage gives a flavour of both his skill and enthusiasm for his new method:

Again, for the sake of brevity, put \(-\frac{2mn}{z} + \frac{bcfgl}{ez^3 - cgz^2}\) equal to \(o\), and \(\frac{n^2}{z^2} - \frac{bcfg}{ez^3 - cgz^2}\) equal to \(\frac{p}{m}\); for these quantities being given, we can represent them in any way we please. Then we have

\[ y = m - \frac{n}{z}x + \sqrt{m^2 + ox} + \frac{p}{m}x^2. \] (Descartes, 1954, p. 63)

As soon as the sometimes heavy geometric analysis had been complemented with smooth algebraic calculus, the development of mathematics exploded. Not only the ancient conics, now called second degree curves, but also corresponding 3D second degree surfaces could be systematically studied (as for example by Euler, who introduced coordinate transformations to find the canonical form representation; see Kline, 1972, p. 545-547), also in the general setting of quadric forms, which by matrix notation and eigenvalue theory were given a unified and systematic treatment during the 19th century (see e.g. Kline, 1972, pp. 799-812). The formal notation of analytic geometry was easy to expand into any dimension, as shown by the first printed publication focused on higher-dimensional point geometry by Cayley in 1843 and other early independent work by Grassmann and Schäfli (Eves, 1983, p. 415).

With the computers of present time, performing high speed complex matrix and numerical calculations, the user can study the geometrical properties of the parabola (and other curves) as points on a screen, by doing simple hand movements (using the ”drag mode” of a dynamic geometry software). The geometric ( iconic) object is again in focus and this time directly available, manipulative like the

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9 This enthusiasm also echoes in the very last words of the treatise: “I hope that posterity will judge me kindly, not only as to the things which I have explained, but also to those which I have intentionally omitted so as to leave to others the pleasure of discovery.” (Descartes, 1954, p. 240)

10 Descartes used the notation \(xx\) for \(x^2\) and a special sign for equality.

11 Already Fermat had in 1643 anticipated coordinate descriptions of these 3D surfaces (Kline, 1972, p. 321).
algebraic equations used to describe it and put it on the screen, i.e. the algebraic register being used but hidden. By a historical metamorphosis the parabola has reappeared as a *dynamic object* on a computer screen. Mathematics has given itself, provided the hardware technology, one more *didactic tool*, possibly to better understand itself.

**The parabola and school mathematics**

*A theoretical framework*

The anthropological theory of didactics, based on the work of Yves Chevallard, identifies two inseparable aspects of mathematical activity, the *practical block* and the *knowledge block*. The former consists of types of problems that are studied and techniques used to solve them, while the latter is formed by the corresponding discursive environment, i.e. issues of *technology* (the word is here used in the sense of the discourse/ingredients related to the techniques) and *theory* (deeper justification). Types of problems, techniques, technologies and theories form *mathematical organisations* or *praxeologies*, the term indicating that the two blocks are inseparable. (Barbé et al, in press).

As an example, the problem of finding the midpoint of a given circle may be solved by the technique of constructing midpoint perpendiculars to chords in the circle, thus using technological ingredients such as chords to circles, a solution process justified by theorems in Euclidean geometry. Mathematical organisations can be considered at different levels, where a *punctual* mathematical organisation of a special type of problems and techniques used to solve it can be embedded in a *local* mathematical organisation with the technology available for using those techniques. Some local mathematical organisations may build on the same theoretical discourse to form a *regional* mathematical organisation. (*ibid.*)

By the process of didactical transposition, different levels of determination form didactic constraints of classroom activity: from society to school, pedagogy, discipline, area, sector, theme to question. The activity of the teacher is more or less restricted to the last two levels, with the mathematical scope of the three preceding levels. (*ibid.*)

Given this theoretical framework, students’ first acquaintance and work with the parabola is limited by the local mathematical organisations previously explored by the curriculum. The available level of justification varies with the kinds of problems that are studied.
Examples of didactic transpositions

For the upper secondary science student in Sweden in the 1960s, a parabola was the geometric loci of points equidistant to a given point (focus) and a straight line (directrix), a property which was immediately elaborated by the algebraic register of analytic geometry to the formula $y^2 = 4ax$, upon which a systematic treatment of a number of geometric and algebraic properties was based (e.g. Sjöstedt & Thörnqvist, 1963, pp. 62-76). The study of the parabola was embedded in a local mathematical organisation of analytic geometry, including techniques of Euclidean geometry and elementary algebraic equation solving. Techniques from the study of functions, such as the derivate, were not used.

Later, in the 1990s, for the same student the parabola was presented as an algebraically defined curve of second degree, $y = x^2$, the shape of which was plotted, based on a table of values (e.g. Björk et al., 1990, pp. 264-274). There was only little or no discussion about geometric properties of the parabola (such as the equidistance or reflection properties), other than the obvious symmetry, and the vertex point. Tangents were treated by means of the derivative. The study of the parabola was embedded in a local mathematical organisation of functions. Techniques from Euclidean geometry were used only casually, as well as analytic geometry other than the basic idea of coordinate representation.

For university students in Sweden the acquaintance with (3D) second degree surfaces, and the corresponding plane curves, is accomplished by the study of quadratic forms, embedded in the domain of linear algebra. However, the application of integrals on rotational volumes for the same mathematical objects, in the calculus course, is done in isolation within the local mathematical organisation of calculus.

Will the students of the new millennium meet the parabola as a dynamic object on a computer screen, the properties of which are explored by experimentation, within the iconic transformations made possible by the specific software, and validated by geometric or algebraic tools where “possible”? If so, in which mathematical domain (local mathematical organisation) will this study be embedded?

Objectification of knowledge

In many educational situations, the student is presented objects of learning previously not known, as might be the case with the parabola in a mathematics class. For the student to become aware of this object the teacher might use

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12 The approach of Apollonius was also included in this textbook, for optional deepening studies.
“semiotic means of objectification – e.g. objects, artifacts, linguistic devices and signs that are intentionally used by the individuals in social processes of meaning production” (Radford, 2002, p. 14). The drawing of a parabola may seem the optimal such means but is by itself void of meaning besides its appearance. It is necessary to go beyond the representation given to find meaning. The same remark holds, of course, for the algebraic representation \( y = x^2 \) or \( y^2 = 4ax \). In the case of the icon, a defining property such as the equidistance property (focus-directrix) gives not only the shape of the icon (the curve) but also a basis for further explorations and analysis of the object. Using another defining non-algebraic property, such as the application of area in figure 1 above, gives the same shape but offers another basis for further explorations. The objectification process for the learner diverges between the two options, even more so by providing an algebraic formula as the main means of objectification, leading into a different mathematical organisation (see below) by the different techniques available.

As a way to enhance the objectification process, what may be called linking tasks can be used, provided the techniques and technology of the different mathematical organisations are available. An obvious example of a linking task is to show that the curve defined algebraically by the equation \( y = x^2 \) has the equidistance property, and vice versa. As a more demanding linking task, investigate if the two (non-algebraic) constructions shown in figures 3 and 4 produce the same curve. Figure 3 is based on the equidistance property and figure 4 is a simplified version for the method of area application.

Another linking task is to demonstrate the reflective property of a parabola, as defined by figure 3, and as defined algebraically by the equation \( y = x^2 \). In the latter case, the technology of the mathematical organisation of the study of functions may include the derivative. Linking tasks may also be designed to use the history of mathematics in teaching by the “voices and echoes game” (Boero et al, 1997).
Discussion

The historical development of the parabola may be seen as an illustration to the idea of algebraization of a mathematical organisation (or work): “a *mathematical work is algebraized* if it can be considered as an *algebraic model* of another mathematical work, the *system to be modelled*” (Bolea et al., 1999, p. 142). This kind of modelling should respond to all the techniques and technological questioning in the organisation being modelled as a whole (ibid.). This seems exactly what Descartes aimed at with his analytic method (cf. quotation on pages 4-5 above, and footnote 8). His method proved self-generating for the development of mathematics.

By the "parabolic" cases in 2 and 3 dimensions, i.e. $y = x^2$ and $z = x^2 + y^2$, the obvious (in some sense) "parabolic" extension to $n$ dimensions would be

$$y = x_1^2 + x_2^2 + x_3^2 + \ldots + x_{n-1}^2,$$

where $\bar{x} = (x_1, x_2, x_3, \ldots, x_{n-1}) \in R^{n-1}$. From the concrete action on an object – cutting a cone – the iconic "section" of a parabola by a process of metamorphosis returned in the shape of a symbolic representation in the semiotic register of algebra. By the development of this register, hitherto unknown objects were hidden beyond transformation, variation and generalisation by elaborations on this representation. With the new eyes new things were

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13 To the segment between the (fixed) focus point F and a (variable) point A on the (fixed) directrix d, draw the perpendicular at the midpoint M, intersecting at the point P the perpendicular to d at A. The point P is on a parabola by the equidistance definition. Note that the distance from F to d is the parameter.

14 AB is a given segment (parameter), BC equal to AD constructed equal to AE, where E is a (variable) point on AB (or AB produced). EG is parallel to BC, P the intersection of AC and EG. As the rectangle ABFH is equal to the square AEGD, the point P is on a parabola from the definition by area application.
found, and old things were looked at in new ways. By the human quest for understanding, when images from experience are missing (as in the case of multi-dimensional “geometry” in $\mathbb{R}^n$, $n > 3$), the phenomenon of metaphorical mapping (Lakoff & Nunes, 2000, p. 43) comes into play, witnessed by the choice of wordings and diagrams most often produced when communicating about these things. Examples include the expression “unit ball” when referring to the set \[ \{ \mathbf{x} \in \mathbb{R}^n : x_1^2 + x_2^2 + x_3^2 + \ldots + x_n^2 < 1 \} \], or the use of the word “distance” between functions as elements in a Hilbert space.

As a remark, the wordings by Apollonius used to define a conic surface (see footnote 4) display an image schematic character (the path schema; see Lakoff & Nunes, pp. 37-39 and 141) in being thought-of generalised bodily based images.

In a learning setting the situation is different, as the objectification processes evoked by work on the different representations may result in fundamentally different outcomes. The options to relate to other representations or other semiotic registers are constrained by the overall didactic situation, as a consequence of the didactical transposition that has taken place. By those limitations, students’ work will remain at the punctual level, resulting in concept images often too vague for successful problem solving in situations spanning over more than the restricted punctual or local mathematical organisation that has been covered in class.

Then what is a parabola, beyond the representation given? Is there a “truth” hidden at the horizon, in line with the Peircean dream? The semiotic perspective shows how meaning is transferred through the representational forms to what the interpreter within a given situation and background constructs from them. And the historical development of mathematics shows how the cultural meaning changes its face when new semiotic registers are developed and used on objects that used to be studied by other registers. Through a semiotic chain the intersection of a cone with a plane (i.e. an icon) has been linked to a quadric form (the notation of which is a symbol), represented by a symmetric matrix (symbol), and to electronic dots on a computer screen (together constituting an icon). However, new developments do not always replace old meanings with new but rather grow as a new layer of signification outside the old. A process of a similar kind, at the individual level, has been described by Presmeg (2002) by a Peircean nested triadic model of semiotic chaining. The model may be applied also to a historical-cultural development of meaning, as the one of the parabola sketched above.
An additional conception of Peirce may overarch and in the realm of reasoning and problem solving integrate the iconically based argumentation of Euclidean geometry and symbolically based in analytic geometry, i.e. diagrammatic reasoning:

... all deductive reasoning, even simple syllogism, involves an element of observation; namely, deduction consists in constructing an icon or diagram the relations of whose parts shall present a complete analogy with those parts of the object of reasoning, of experimenting upon this image in the imagination, and of observing the result so as to discover unnoticed and hidden relations among the parts (Peirce, quoted in Dörfler, 2004, p. 7)

Also the manipulation of algebraic formulas is a work with icons, the patterns of formulae: “These are patterns, which we have the right to imitate in our procedure, and are the icons par excellence of algebra.” (ibid.; cf. the concept of mathematical form in Bergsten, 1999). This kind of diagrammatic reasoning is of such generality that it may function, for the individual, as a linking force between local mathematical organisations, since it is not primarily focused on the referential function (or meaning) of the forms.

To conclude, it is a question of curriculum development to design mathematical schooling to facilitate students’ development of concept images and problem solving techniques not limiting students’ thinking to the punctual level. Only by linking the different representations of the objects of learning (e.g. the parabola) to the local mathematical organisations into which they can be embedded, the flexibility needed for solving more non-routine problems by the availability of adequate technology can be developed.

References


/ Christer Bergsten