

Driven Newton equations and separable time-dependent potentials

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(Received 14 March 2002; accepted 9 August 2002)

We present a class of time-dependent potentials in R^n that can be integrated by separation of variables: by embedding them into so-called cofactor pair systems of higher dimension, we are led to a time-dependent change of coordinates that allows the time variable to be separated off, leaving the remaining part in separable Stäckel form. © 2002 American Institute of Physics. [DOI: 10.1063/1.1514833]

I. INTRODUCTION

Newton’s law of force in mechanics leads to second order ordinary differential equations $\ddot{q} = M(q, \dot{q}, t)$, where $q = (q^1, \dots, q^n)$ are coordinates on some manifold Q , the configuration space of the system. Often the force M is derived from a potential $V(q, t)$ and the equations can be written in Lagrangian form

$$\frac{\partial L}{\partial q^i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} = 0, \quad L(q, \dot{q}, t) = \frac{1}{2} g_{ij} \dot{q}^i \dot{q}^j - V(q, t),$$

or, via the Legendre transformation, in Hamiltonian form

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = - \frac{\partial H}{\partial q^i}, \quad H(q, p, t) = \frac{1}{2} g^{ij} p_i p_j + V(q, t).$$

Here g_{ij} is the metric tensor on Q , with inverse g^{ij} , and (q^i, p_j) are (adapted) coordinates on the cotangent bundle T^*Q .

Powerful techniques have been developed for solving such equations; in particular the well-known Hamilton–Jacobi method, where one tries to find new coordinates $u = u(q)$ on Q , in terms of which the Hamilton–Jacobi equation corresponding to H can be solved by separation of variables. If this succeeds, the mechanical system can be integrated by quadratures.

We will restrict ourselves to Euclidean n -space, i.e., $Q = R^n$ and $g_{ij} = \delta_{ij}$. The coordinates will be written with lower indices in this case, and regarded as a column vector $q = (q_1, \dots, q_n)^T$, the T denoting matrix transposition.

Consider a Newton system which does not contain time t or velocity \dot{q} explicitly,

$$\ddot{q} = M(q).$$

If there is a potential, the system takes the form

$$\ddot{q} = - \nabla V(q), \quad \nabla = \frac{\partial}{\partial q} = \left(\frac{\partial}{\partial q_1}, \dots, \frac{\partial}{\partial q_n} \right)^T,$$

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and then the energy $E = \frac{1}{2}\dot{q}^T\dot{q} + V(q)$ is conserved ($\dot{E} = 0$). The separability theory for such time-independent potentials in Euclidean space is highly developed. It is known that separation of the corresponding Hamilton–Jacobi equation can only take place in so-called generalized elliptic coordinates or some degeneration thereof.⁴ There even exists an effective algorithm for determining whether or not a given potential $V(q)$, expressed in Cartesian coordinates, is separable, and if so, in which coordinate system.¹⁰

Less is known in the time-dependent case. One of the aims of this paper is to show how certain Newton systems in R^n with time-dependent potential can be integrated by viewing them as driven systems in R^N , with $N > n$, as the following example illustrates.

Example 1: Consider the time-dependent potential

$$V(x_1, x_2, t) = \frac{1}{x_1 x_2 - t} \tag{1}$$

and the corresponding Newton system in R^2 :

$$\begin{aligned} \ddot{x}_1 &= -\frac{\partial V}{\partial x_1} = \frac{x_2}{(x_1 x_2 - t)^2}, \\ \ddot{x}_2 &= -\frac{\partial V}{\partial x_2} = \frac{x_1}{(x_1 x_2 - t)^2}. \end{aligned} \tag{2}$$

In order to integrate this system, we introduce the following auxiliary Newton system in R^3 , where the first equation drives the other two:

$$\begin{aligned} \ddot{q}_1 &= 0, \\ \ddot{q}_2 &= \frac{q_3}{(q_2 q_3 - q_1)^2}, \\ \ddot{q}_3 &= \frac{q_2}{(q_2 q_3 - q_1)^2}. \end{aligned} \tag{3}$$

We think of the q coordinates as partitioned into *driving* coordinates y and *driven* coordinates x :

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \equiv \begin{pmatrix} y \\ x_1 \\ x_2 \end{pmatrix}.$$

The particular solution $y(t) \equiv q_1(t) = t$ clearly gives rise to the system (2) under the identification $x_1 = q_2$, $x_2 = q_3$. The Newton system (3) in R^3 is what we call a *cofactor system* (see Sec. II), which means that it has the form

$$\ddot{q} = -A(q)^{-1} \nabla W(q) = -\frac{1}{\det G(q)} G(q) \nabla W(q),$$

where $A = \text{cof } G = (\det G)G^{-1}$ is the cofactor matrix of a symmetric matrix $G(q)$ of the form

$$G_{ij}(q) = \alpha q_i q_j + \beta_i q_j + \beta_j q_i + \gamma_{ij}.$$

Equivalently, $\frac{1}{2}\dot{q}^T A(q)\dot{q} + W(q)$ is an integral of motion (of *cofactor type*) for the system.

In this specific case, as is easily verified, the system (3) can be written as $\ddot{q} = -G\nabla W/(\det G)$ with

$$G(q) = \begin{pmatrix} 2q_1 & q_2 & q_3 \\ q_2 & 0 & 1 \\ q_3 & 1 & 0 \end{pmatrix}, \quad W(q) = -\frac{q_2^2 + q_3^2}{q_2 q_3 - q_1}.$$

According to the general theory to be developed in this paper, such a driven cofactor system can be integrated using a time-dependent change of coordinates

$$u_1 = \lambda_1(t, x_1, x_2),$$

$$u_2 = \lambda_2(t, x_1, x_2),$$

where $\lambda_1(q)$ and $\lambda_2(q)$ are the roots of the equation $\det(G(q) - \lambda \tilde{G}) = 0$, with $\tilde{G} = \text{diag}(0, 1, 1)$.

It turns out that by defining corresponding momenta s_1 and s_2 appropriately, the equations of motion for (u_1, u_2) can be put in Hamiltonian form with a time-dependent *separable* Hamiltonian. Consequently, $u_1(t)$ and $u_2(t)$ can be found using a variant of the Hamilton–Jacobi method. Changing back to old coordinates, we find $x_1(t)$ and $x_2(t)$, and the problem is solved.

We will fill in the details of this example in Sec. V, after explaining the method in general.

II. QUASIPOTENTIAL NEWTON SYSTEMS OF COFACTOR TYPE

The general framework in which we are working was developed in Refs. 9, 6, and 7. It has been extended² to cover also the case of Riemannian manifolds, but here we will restrict ourselves to Euclidean space. We will now quote the definitions and results needed here, some of which have already been hinted at above.

We use the shorthand $\partial_i = \partial / \partial q_i$. The notation $\text{cof } X$ means the cofactor matrix of a square matrix X . If X is nonsingular, then $\text{cof } X = (\det X)X^{-1}$.

Proposition 2: The “energy-like” function

$$E(q, \dot{q}) = \frac{1}{2} \sum_{i,j=1}^n A_{ij}(q) \dot{q}_i \dot{q}_j + W(q) = \frac{1}{2} \dot{q}^T A(q) \dot{q} + W(q), \quad (4)$$

with $A(q)$ a symmetric $n \times n$ matrix, is an integral of motion of the Newton system $\ddot{q} = M(q)$ in R^n if and only if

(1) The matrix entries $A_{ij}(q)$ satisfy the cyclic conditions

$$\partial_i A_{jk} + \partial_j A_{ki} + \partial_k A_{ij} = 0, \quad i, j, k = 1, \dots, n. \quad (5)$$

[The general solution of these equations is a subspace, of dimension $\frac{1}{12}n(n+1)^2(n+2)$, of the vector space of symmetric matrices whose entries are polynomials of degree at most two in q_1, \dots, q_n .]

(2) The force $M(q)$ satisfies $A(q)M(q) + \nabla W(q) = 0$.

Definition 3 (quasipotential system): A Newton system of the form

$$\ddot{q} = -A(q)^{-1} \nabla W(q),$$

where the matrix A satisfies the cyclic conditions (5), is called a *quasipotential* system. By the proposition above, $E = \frac{1}{2} \dot{q}^T A \dot{q} + W$ is an integral of motion for the system, and it is said to generate the system, since the system is completely determined by $A(q)$ and $W(q)$, and hence by E . (Special case: if $A = I$ is the identity matrix, then W is a potential for the system and E is the usual energy.)

Definition 4 (elliptic coordinates matrix G): A symmetric matrix of the form

$$G_{ij}(q) = \alpha q_i q_j + \beta_i q_j + \beta_j q_i + \gamma_{ij}, \quad i, j = 1, \dots, n, \tag{6}$$

is called an *elliptic coordinates matrix*. Using matrix multiplication, $G(q)$ can be written

$$G(q) = \alpha q q^T + \beta q^T + q \beta^T + \gamma, \tag{7}$$

with α a scalar, q and β column vectors, and γ a symmetric matrix.

Set briefly, the eigenvalues $u_1(q), \dots, u_n(q)$ of $G(q)$ give the change of coordinates from Cartesian coordinates q to elliptic coordinates $u = u(q)$. See Ref. 6 for a more detailed explanation.

Definition 5 (associated vector N): To a given elliptic coordinate matrix G we associate the column vector $N = \alpha q + \beta = \frac{1}{2} \nabla \text{tr } G$.

Proposition 6: If G is an elliptic coordinates matrix, N the associated vector, and $A = \text{cof } G$, then

$$\nabla \det G = 2AN. \tag{8}$$

The preceding proposition is frequently useful. It implies, for example, that $A = \text{cof } G$ satisfies

$$(\det G) \partial_k A_{ij} = 2[AN]_k A_{ij} - [AN]_i A_{kj} - [AN]_j A_{ik}, \tag{9}$$

from which the following remarkable property of elliptic coordinates matrices follows.

Proposition 7: If $G(q)$ is an elliptic coordinates matrix, then $A(q) = \text{cof } G(q)$ satisfies the cyclic conditions (5).

Corollary 8: If $G(q)$ and $\tilde{G}(q)$ are elliptic coordinates matrices, then the matrices $A^{(0)}(q), \dots, A^{(n-1)}(q)$ defined by the generating function

$$\text{cof}(G + \mu \tilde{G}) = \sum_{k=0}^{n-1} A^{(k)} \mu^k \tag{10}$$

all satisfy the cyclic conditions (5).

Remark 9: Note that $A^{(0)} = \text{cof } G$ and $A^{(n-1)} = \text{cof } \tilde{G}$.

We will also need a proposition that does not occur in Ref. 6.

Proposition 10: With G , N , and $A = \text{cof } G$ as above,

$$\nabla(N^T A N) = 2\alpha A N. \tag{11}$$

Proof: Equation (9) implies that $\sum_{i,j} (\partial_k A_{ij}) N_i N_j = 0$, from which the statement follows easily. □

Definition 11 (cofactor system): A cofactor system is a quasipotential Newton system of the special form

$$\ddot{q} = -A(q)^{-1} \nabla W(q) = -\frac{1}{\det G(q)} G(q)^{-1} \nabla W(q),$$

where $A = \text{cof } G$, and G is a nonsingular elliptic coordinates matrix. The integral of motion $E = \frac{1}{2} \dot{q}^T A \dot{q} + W = \frac{1}{2} \dot{q}^T (\text{cof } G) \dot{q} + W$ is said to be of cofactor type.

Definition 12 (cofactor pair system): A cofactor pair system is a Newton system which has two independent integrals of motion of cofactor type,

$$E = \frac{1}{2} \dot{q}^T (\text{cof } G) \dot{q} + W \quad \text{and} \quad \tilde{E} = \frac{1}{2} \dot{q}^T (\text{cof } \tilde{G}) \dot{q} + \tilde{W}.$$

Equivalently, it is a system which can be written as

$$\ddot{q} = -A^{-1} \nabla W = -\tilde{A}^{-1} \nabla \tilde{W}, \tag{12}$$

where $A = \text{cof } G$ and $\tilde{A} = \text{cof } \tilde{G}$.

Theorem 13 (two implies n): A cofactor pair system $\ddot{q} = M(q)$ in R^n has n integrals of motion

$$E^{(k)} = \frac{1}{2} \dot{q}^T A^{(k)} \dot{q} + W^{(k)}, \quad k = 0, \dots, n-1, \tag{13}$$

where the matrices $A^{(k)}$ are given by (10) and the quasipotentials $W^{(k)}$ are determined (up to irrelevant additive constants) by $\nabla W^{(k)} = -A^{(k)}M$.

Remark 14: Note that the original integrals of motion $E = E^{(0)}$ and $\tilde{E} = E^{(n-1)}$ of cofactor type sit at either end of this ‘‘cofactor chain’’ of integrals.

Remark 15: It is sometimes convenient to handle the integrals of motion using a generating function

$$E_\mu = \sum_{k=0}^{n-1} E^{(k)} \mu^k = \frac{1}{2} \dot{q}^T \text{cof}(G + \mu \tilde{G}) \dot{q} + W_\mu, \tag{14}$$

where $W_\mu = \sum_{k=0}^{n-1} W^{(k)} \mu^k$.

Remark 16: For W to be well defined by $\nabla W = -AM$, the compatibility conditions $\partial_i[AM]_j = \partial_j[AM]_i$ have to be satisfied for all i and j . This, of course, is the reason that not every Newton system $\ddot{q} = M(q)$ has a potential V , and also that not every Newton system has a quasipotential W , even though by allowing $A(q) \neq I$ we enlarge the class of systems under consideration.

Now, for $\ddot{q} = M(q)$ to be a cofactor pair system, two sets of compatibility conditions need to be satisfied simultaneously; $\partial_i[AM]_j = \partial_j[AM]_i$ and $\partial_i[\tilde{A}M]_j = \partial_j[\tilde{A}M]_i$. For given G and \tilde{G} , this is a rather strong restriction on M . In fact, according to the theorem, it is so strong that if $\partial_i[A^{(k)}M]_j = \partial_j[A^{(k)}M]_i$ holds for $A^{(0)} = A = \text{cof } G$ and $A^{(n-1)} = \tilde{A} = \text{cof } \tilde{G}$, then it must hold for all the matrices $A^{(k)}$.

Definition 17 (fundamental equations): The fundamental equations associated to a pair (G, \tilde{G}) of elliptic coordinates matrices is the following set of $\binom{n}{2}$ second order linear PDEs:

$$0 = \sum_{r,s=1}^n (G_{ir} \tilde{G}_{js} - G_{jr} \tilde{G}_{is}) \partial_{rs} K + 3 \sum_{r=1}^n (G_{ir} \tilde{N}_j + \tilde{G}_{jr} N_i - G_{jr} \tilde{N}_i - \tilde{G}_{ir} N_j) \partial_r K + 6(N_i \tilde{N}_j - N_j \tilde{N}_i) K, \quad i, j = 1, \dots, n. \tag{15}$$

Here $N = \alpha q + \beta$ is the vector associated to G , with the same parameters α and β as in $G = \alpha q q^T + \beta q^T + q \beta^T + \gamma$, and similarly for \tilde{N} .

Theorem 18: Let

$$\ddot{q} = -(\text{cof } G)^{-1} \nabla W = -(\text{cof } \tilde{G})^{-1} \nabla \tilde{W} \tag{16}$$

be a cofactor pair system. Then the functions $K_1 = W/\det G$ and $K_2 = \tilde{W}/\det \tilde{G}$, while in general different, both satisfy the fundamental equations (15) associated to the pair (G, \tilde{G}) .

Conversely, if K satisfies (15) and we set $W = K \det G$, then there is a function \tilde{W} such that (16) holds. And if we set $\tilde{W} = K \det \tilde{G}$, then there is a function W such that (16) holds (but these W and \tilde{W} are in general not the same as those in the previous sentence).

Remark 19: Once again, this is all about compatibility conditions. If G , \tilde{G} , and W are given, then \tilde{W} is well defined by (16) if and only if

$$\partial_i[(\text{cof } \tilde{G})(\text{cof } G)^{-1} \nabla W]_j = \partial_j[(\text{cof } \tilde{G})(\text{cof } G)^{-1} \nabla W]_i$$

for all i and j . This is a system of $\binom{n}{2}$ second order linear PDEs for W , with coefficients depending in a complicated way on G and \tilde{G} . Substituting $K = W/\det G$ and forming suitable linear combinations of the equations simplifies this system to precisely the fundamental equations (15). These being completely antisymmetric with respect to coefficients with and without the tilde, the result is the same if we go the other way around, interchanging the roles of W and \tilde{W} .

Remark 20: This theorem leads to a recursive procedure for explicitly constructing infinite families of cofactor pair systems. See Ref. 6 for details.

In Ref. 6 it was shown, using the theory of bi-Hamiltonian systems, that cofactor pair systems generically are completely integrable, but it was not clear if they admit some kind of separation of variables. The special case $\tilde{G} = I$ corresponds to conservative systems with an extra integral of motion of cofactor type. Such systems are precisely those with potentials separable in the elliptic (or parabolic) coordinates given by the eigenvalues of $G(q)$, so in that case we have a concrete method of integration. Reference 8, which appeared recently, deals with separation of variables for generic cofactor pair systems, with both G and \tilde{G} nonsingular (and nonconstant, in general). Here, we study the very degenerate case of cofactor pair systems with $\tilde{G} = \text{diag}(0, \dots, 0, 1, \dots, 1)$. As we will see in the next section, these systems admit a somewhat nonstandard integration by separation of variables, and there is a surprising connection with time-dependent potentials.

III. DRIVEN SYSTEMS

From now on we fix positive integers m and n , and let $N = m + n$. Hopefully there is no risk of confusing this integer N with the vector $N(q)$ associated to an elliptic coordinates matrix $G(q)$. Let us begin by defining some notation.

Definition 21 (block notation): If X is an $N \times N$ matrix, with $N = m + n$, then we use arrow subscripts to denote blocks in X , as follows:

$$X = \begin{pmatrix} X_{\setminus} & X_{\nearrow} \\ X_{\swarrow} & X_{\searrow} \end{pmatrix} \quad \text{with sizes} \quad \begin{bmatrix} m \times m & m \times n \\ n \times m & n \times n \end{bmatrix}. \tag{17}$$

Similarly, if Y is a column vector in R^N , then

$$Y = \begin{pmatrix} Y_{\uparrow} \\ Y_{\downarrow} \end{pmatrix} \quad \text{with sizes} \quad \begin{bmatrix} m \\ n \end{bmatrix}. \tag{18}$$

So, for instance, $[X_{\nearrow}]_{ij} = X_{i,m+j}$.

We will consider *driven* Newton systems in R^N , where $N = m + n$. By this we mean that the first m equations depend only on the first m variables, so that they form a Newton system in R^m on their own:

$$\begin{aligned} \ddot{q}_1 &= M_1(q_1, \dots, q_m), \\ &\vdots \\ \ddot{q}_m &= M_m(q_1, \dots, q_m), \\ \ddot{q}_{m+1} &= M_{m+1}(q_1, \dots, q_m; q_{m+1}, \dots, q_{m+n}), \\ &\vdots \\ \ddot{q}_{m+n} &= M_{m+n}(q_1, \dots, q_m; q_{m+1}, \dots, q_{m+n}). \end{aligned} \tag{19}$$

This notion is not new; for example Kossowski and Thompson⁵ use tangent bundle geometry to study *submersive* systems, which are second order ODEs on manifolds, containing a subsystem depending on fewer variables (possibly after a change of coordinates).

Here, however, our purpose is to investigate what happens when a system is at the same time a driven system and a cofactor system. In this initial stage of investigation we have restricted ourselves to Euclidean space and assume that the systems splits as above in Cartesian coordinates. We hope that further research will clarify the relation between our present results and the geometric picture of Refs. 5 and 2.

Definition 22 (vectors x and y): Since we will consider the time evolution of q_{\uparrow} and q_{\downarrow} separately, we write $y = q_{\uparrow}$ and $x = q_{\downarrow}$ to simplify the notation.

With this definition, the system (19) can be written as

$$\begin{aligned} \ddot{y} &= M_{\uparrow}(y), \\ \ddot{x} &= M_{\downarrow}(y, x). \end{aligned} \tag{20}$$

As in example 1, (y_1, \dots, y_m) are called *driving* variables and (x_1, \dots, x_n) are called *driven* variables. The system $\ddot{y} = M_{\uparrow}(y)$ is called the *driving* system, since its solution $y = y(t)$, when fed into $\ddot{x} = M_{\downarrow}(y(t), x)$, drives the evolution of the x variables.

An important observation is that if

$$G = \alpha q q^T + \beta q^T + q \beta^T + \gamma$$

is an $N \times N$ elliptic coordinates matrix, then

$$G_{\searrow} = \alpha y y^T + \beta_{\uparrow} y^T + y (\beta_{\uparrow})^T + \gamma_{\searrow},$$

so that $G_{\searrow}(y)$ is an $m \times m$ elliptic coordinates matrix in the y variables. [Similarly for $G_{\swarrow}(x)$, but we will not use that here.]

The major part of this paper is devoted to proving the following theorem.

Theorem 23 (driven cofactor systems): Suppose that a driven Newton system in R^{m+n} is of cofactor type

$$\ddot{q} = \begin{pmatrix} M_{\uparrow}(y) \\ M_{\downarrow}(y, x) \end{pmatrix} = -(\text{cof } G(q))^{-1} \frac{\partial W}{\partial q}(q). \tag{21a}$$

Suppose also that G is not constant (i.e., that α and β are not both zero), that $\det G_{\searrow} \neq 0$, and that there is a potential $V(y, x)$, with y occurring parametrically, such that

$$M_{\downarrow}(y, x) = -\frac{\partial V}{\partial x}(y, x). \tag{21b}$$

Then the driving system is a cofactor system in R^m . Namely, there is a function $w(y)$ such that

$$\ddot{y} = -(\text{cof } G_{\searrow}(y))^{-1} \frac{\partial w}{\partial y}(y). \tag{22}$$

Moreover, for any given solution $y = y(t)$ of the driving system $\ddot{y} = M_{\uparrow}(y)$, the system

$$\ddot{x} = M_{\downarrow}(y(t), x) = -\frac{\partial V}{\partial x}(y(t), x), \tag{23}$$

given by the time-dependent potential $V(y(t), x)$, has n (time-dependent) integrals of motion. Under some technical assumptions, stated in definition 28, its solution $x(t)$ can be found by quadratures.

The proof is quite lengthy, so we have divided it into subsections labeled A through F. First we show that a driven cofactor system can be viewed as a degenerate form of cofactor pair system, with $\tilde{G} = \text{diag}(0, \dots, 0, 1, \dots, 1)$. The integrals of motion are given by a cofactor chain that terminates prematurely. We introduce a new system of coordinates, which is given by the m driving Cartesian coordinates together with the n roots of the equation $\det(G(q) - u\tilde{G}) = 0$. This is similar to defining elliptic coordinates implicitly as the eigenvalues of G . When the integrals of motion are transformed into these new coordinates, which is the most technical part of this paper, it turns out that they take a form similar to that known from classical separability theory (Stäckel systems). This suggests that the system should be solvable by separation of variables. We show that this is indeed the case, since the equations of motion are Hamiltonian and the variables can be separated in the time-dependent Hamilton–Jacobi equation. Perhaps surprisingly, the Hamiltonian does not involve the potential $V(y, x)$ in any direct way, but is instead given by one of the integrals of motion in the cofactor chain, divided by the determinant $\det G$, all expressed in new coordinates.

A. Driven cofactor systems as cofactor pair systems

Definition 24 (matrix J): Let J denote the $N \times N$ diagonal matrix

$$J = \text{diag}(0, \dots, 0, 1, \dots, 1), \tag{24}$$

with m zeros and n ones along the diagonal ($N = m + n$).

Proposition 25: A system of the form (21) is a cofactor pair system with

$$\tilde{G}(q) = \lambda G(q) + J =: \tilde{G}_\lambda(q), \tag{25}$$

for any λ such that $\det \tilde{G}_\lambda \neq 0$. Conversely, any such cofactor pair system has the form (21).

We note that since G is assumed nonsingular by the definition of cofactor system, $\det(\lambda G(q) + J)$ cannot vanish identically, so there are λ such that $\det \tilde{G}_\lambda \neq 0$. The reason for taking $\tilde{G} = \tilde{G}_\lambda$ instead of just $\tilde{G} = J$ is that the theorems we use about cofactor pair systems require both G and \tilde{G} to be nonsingular. However, many of the results will be the same as if applying the theorems formally with $\tilde{G} = J$ directly, so we will regard such systems as cofactor pair systems associated with the pair (G, J) .

The proof of proposition 25 uses the following lemma, which follows from the algebraic properties of an elliptic coordinates matrix G .

Lemma 26: If $M = -(\det G)^{-1} G \nabla W$, then

$$-\partial_j M_i = \sum_{r=1}^N G_{ir} \partial_{rj} K + 3N_i \partial_j K \quad (i \neq j),$$

where $K(q) = W(q) / \det G(q)$.

Proof: Proposition 6 implies that $-M = G \nabla (K \det G) / \det G = 2KN + G \nabla K$. Differentiating $-M_i = 2KN_i + \sum_{r=1}^N G_{ir} \partial_r K$ we obtain the result immediately, since for $j \neq i$ we have $\partial_j G_{ir} = \delta_{jr} N_i$ and $\partial_j N_i = 0$. □

Proof of proposition 25: By construction, the given cofactor system

$$\dot{q} = M(q) = -(\text{cof } G)^{-1} \nabla W = -(\det G)^{-1} G \nabla W,$$

has an integral of motion of cofactor type $E = \frac{1}{2} \dot{q}^T (\text{cof } G) \dot{q} + W$. Now fix some constant λ such that $\det \tilde{G}_\lambda \neq 0$. Theorem 18 says that the system is a cofactor pair system with $\tilde{G} = \tilde{G}_\lambda$, i.e., admits an additional integral of motion of cofactor type

$$\tilde{E}_\lambda = \frac{1}{2} \dot{q}^T (\text{cof } \tilde{G}_\lambda) \dot{q} + \tilde{W}_\lambda,$$

if and only if $K = W/\det G$ satisfies the fundamental equations (15) associated to the pair (G, \tilde{G}_λ) .

The antisymmetry of the fundamental equations shows that any pair $(G, \lambda G + J)$ gives rise to the same fundamental equations as the pair (G, J) , so we simply plug $\tilde{G} = J$ into the fundamental equations (15) (with n replaced by $m + n$). To begin with, since J is diagonal and constant (so that $\tilde{N} = 0$), we obtain

$$0 = \sum_{r=1}^{m+n} G_{ir} J_{jj} \partial_{rj} K - \sum_{r=1}^{m+n} G_{jr} J_{ii} \partial_{ri} K + 3(J_{jj} N_i \partial_j K - J_{ii} N_j \partial_i K), \quad i, j = 1, \dots, m+n. \quad (26)$$

Now $J_{ii} = 0$ or 1 as $i \leq m$ and $i > m$, respectively. From this it is immediate that (26) is identically satisfied if $i, j \leq m$. Using lemma 26 to express the remaining equations (26) for K in terms of $M = -(\det G)^{-1} G \nabla(K \det G)$ gives $0 = \partial_j M_i$ for $i \leq m < j$, and $0 = \partial_i M_j - \partial_j M_i$ for $m < i, j$. Clearly, these equations are equivalent to M having the block structure

$$M(q) = \begin{pmatrix} M_\uparrow(y) \\ M_\downarrow(y, x) \end{pmatrix}$$

and (at least locally) a ‘‘partial potential’’ V such that $M_\downarrow = -\partial V / \partial x$. □

B. Integrals of motion

Proposition 27: The system (21) has $n+1$ integrals of motion $E^{(0)}, \dots, E^{(n)}$ given by the generating function

$$E_\mu = \sum_{k=0}^n E^{(k)} \mu^k = \sum_{k=0}^n \left(\frac{1}{2} \dot{q}^T A^{(k)} \dot{q} + W^{(k)} \right) \mu^k = \frac{1}{2} \dot{q}^T \operatorname{cof}(G + \mu J) \dot{q} + W_\mu \quad (27)$$

for some functions $W^{(k)}$. The integral $E^{(n)}$ has the form

$$E^{(n)}(y, \dot{y}) = \frac{1}{2} \dot{y}^T \operatorname{cof} G_{\setminus} (y) \dot{y} + w(y), \quad (28)$$

and is an integral of motion of the driving system $\ddot{y} = M_\uparrow(y)$, of cofactor type in the y variables.

Proof: According to theorem 13, our cofactor pair system should have a chain of $N = m + n$ integrals of motion. Here, however, that number is reduced since some of them will be linearly dependent. More specifically, for arbitrary λ such that $\det \tilde{G}_\lambda \neq 0$, theorem 13 gives us integrals $E_\lambda^{(0)}, \dots, E_\lambda^{(N-1)}$ which we write using a generating function

$$E_{\lambda, \mu} = \sum_{k=0}^{m+n-1} E_\lambda^{(k)} \mu^k = \frac{1}{2} \dot{q}^T \operatorname{cof}(G + \mu \tilde{G}_\lambda) \dot{q} + W_{\lambda, \mu} \quad (29)$$

as in (14). By construction, $\dot{E}_{\lambda, \mu} = 0$ for all values of μ and all λ such that $\det \tilde{G}_\lambda \neq 0$. But $E_{\lambda, \mu}$ depends polynomially on λ and μ , since $\operatorname{cof}(G + \mu \tilde{G}_\lambda) = \operatorname{cof}(G + \mu(\lambda G + J)) = \operatorname{cof}((1 + \mu\lambda)G + \mu J)$ does. Hence, $\dot{E}_{\lambda, \mu} = 0$ identically. In particular, if we set $\lambda = 0$ we extract the constant term with respect to λ , which is just the E_μ of (27), a polynomial in μ whose coefficients are integrals of motion.

The reason why E_μ is only of degree n (instead of $m + n - 1$) is that the matrix J has so few nonzero elements that the expansion of $\operatorname{cof}(G + \mu J)$ in powers of μ terminates ‘‘prematurely’’ (the details in this expansion are explained below, after the proof):

$$\begin{aligned} \text{cof}(G + \mu J) &= \text{cof } G \\ &+ \dots + \begin{pmatrix} A_{\swarrow}^{(n-1)} & -(\text{cof } G_{\swarrow})G_{\nearrow} \\ -((\text{cof } G_{\swarrow})G_{\nearrow})^T & (\det G_{\swarrow})I_{n \times n} \end{pmatrix} \mu^{n-1} + \begin{pmatrix} \text{cof } G_{\swarrow} & 0_{m \times n} \\ 0_{n \times m} & 0_{n \times n} \end{pmatrix} \mu^n \\ &=: \sum_{k=0}^n A^{(k)} \mu^k. \end{aligned} \tag{30}$$

All the coefficients in the generating function $E_{\lambda, \mu}$ in (29) are linear combinations of these $n + 1$ basic integrals $E^{(0)}, \dots, E^{(n)}$, so even though one can obtain a seemingly longer chain (with $N = m + n$ integrals) by taking $\lambda \neq 0$, it would not contain any essentially new integrals of motion. (Note also that the polynomial E_{μ} is what we would have obtained by applying theorem 13 formally with the singular matrix $\tilde{G} = J$ instead of \tilde{G}_{λ} .)

The integral $E^{(n)}$ has the form

$$E^{(n)} = \frac{1}{2} \begin{pmatrix} \dot{y}^T & \dot{x}^T \end{pmatrix} \begin{pmatrix} \text{cof } G_{\swarrow}(y) & 0_{m \times n} \\ 0_{n \times m} & 0_{n \times n} \end{pmatrix} \begin{pmatrix} \dot{y} \\ \dot{x} \end{pmatrix} + W^{(n)}(y, x) = \frac{1}{2} \dot{y}^T \text{cof } G_{\swarrow}(y) \dot{y} + w(y), \tag{31}$$

where clearly $W^{(n)} = w(y)$ cannot depend on x if $E^{(n)}$ is to be an integral of motion. Consequently, $E^{(n)}(y, \dot{y})$ must be an integral of motion of the driving system $\ddot{y} = M_{\uparrow}(y)$, and it is of cofactor type in the y variables. \square

In (30) we have written out some blocks in the matrices $A^{(n-1)}$ and $A^{(n)}$ for future reference (in the proof of proposition 36). These can be found either by analyzing the cofactor expansion directly or by writing the identity

$$(G + \mu J)\text{cof}(G + \mu J) = \det(G + \mu J)I_{N \times N}$$

as

$$JA^{(n)}\mu^{n+1} + (JA^{(n-1)} + GA^{(n)})\mu^n + \dots = (0\mu^{n+1} + (\det G_{\swarrow})\mu^n + \dots)I_{N \times N}$$

and identifying coefficients block-wise at μ^{n+1} and μ^n , using that the matrices $A^{(i)}$ are symmetric. The block $A_{\swarrow}^{(n-1)}$ does not enter into this identity until at the power μ^{n-1} , and depends on G in a more complicated way. Fortunately, the only information about $A_{\swarrow}^{(n-1)}$ that we will need is that $A^{(n-1)}$ satisfies the cyclic conditions (5) which connect derivatives of $A_{\swarrow}^{(n-1)}$ to derivatives of the other blocks, which are known explicitly.

We have now completed the proof of the first statement of theorem 23, namely, that the driving system is a cofactor system in the y variables.

Moreover, for any given solution $y = y(t)$ of the driving system, we can consider $E^{(0)}, \dots, E^{(n-1)}$ as functions of (x, \dot{x}, t) , and these constitute n time-dependent integrals of motion of the driven system (23) given by the time-dependent potential $V(y(t), x)$. These are the integrals referred to at the end of theorem 23.

C. Separation coordinates

Our remaining task (which is much more complicated) is to show how to integrate the driven system $\ddot{x} = -(\partial V / \partial x)(y(t), x)$, given a solution $y(t)$ of the driving system $\ddot{y} = M_{\uparrow}(y)$. This will be accomplished using a change of variables $(y, x) \mapsto (v, u)$ on R^{m+n} defined as follows:

Definition 28 (variables u and v , roots λ): Let $v_i = y_i$ for $i = 1, \dots, m$. Let $u_j = \lambda_j(y, x)$ for $j = 1, \dots, n$, where $\lambda_1, \dots, \lambda_n$ are the roots of the n th degree polynomial equation

$$\det(G(y, x) - \lambda J) = 0. \tag{32}$$

(We assume that this really defines a coordinate system. This requires, to begin with, that all the roots λ_j are nonconstant as functions of q . Moreover, the gradients of the v_i and u_j must be linearly independent. Because of lemma 31 below, this holds at least in a neighborhood of any point where all $\lambda_j(q)$ are distinct.)

Definition 29 [polynomial $U(\mu)$]. Let

$$U(\mu) = (u_1 - \mu)(u_2 - \mu) \cdots (u_n - \mu). \tag{33}$$

It follows from the definition of the u_k as roots of the polynomial $\det(G - \mu J)$, which has the leading term $(-\mu)^n \det G_{\setminus}$, that

$$\det(G - \mu J) = U(\mu) \det G_{\setminus}. \tag{34}$$

Our aim is to express the integrals of motion $E^{(0)}, \dots, E^{(n)}$ in terms of the new coordinates v and u , and likewise for the equations of motion for the system [although for that purpose we view $x \mapsto u = \lambda(y(t), x)$, where $y(t)$ is a given solution of the driving system, as a time-dependent change of variables in R^n ; more about that later]. The remainder of this subsection contains technical preparations for these tasks.

Definition 30 (matrix Ψ): Let Ψ denote the $N \times N$ matrix of partial derivatives of v and u with respect to y and x , arranged so that the columns of Ψ are the gradients of v and u with respect to $q = \begin{pmatrix} y \\ x \end{pmatrix}$:

$$\Psi = (\nabla v_1 \cdots \nabla v_m \quad \nabla u_1 \cdots \nabla u_n) = (e_1 \cdots e_m \quad \nabla \lambda_1 \cdots \nabla \lambda_n), \tag{35}$$

where e_i is the column vector with 1 in position i and 0 elsewhere. [In the block notation of (17), $\Psi_{\setminus} = I_{m \times m}$ and $\Psi_{/} = 0_{n \times m}$.]

With this definition we have

$$\begin{pmatrix} \dot{v} \\ \dot{u} \end{pmatrix} = \Psi^T \dot{q}, \quad \dot{q} = \frac{1}{\det \Psi} (\text{cof } \Psi^T) \begin{pmatrix} \dot{v} \\ \dot{u} \end{pmatrix}, \tag{36}$$

and also

$$\nabla = \begin{pmatrix} \partial/\partial y \\ \partial/\partial x \end{pmatrix} = \Psi \begin{pmatrix} \partial/\partial v \\ \partial/\partial u \end{pmatrix}. \tag{37}$$

(Note that $\partial/\partial y \neq \partial/\partial v$ even though $y = v$, hence the need for the different names.)

The following lemma will give us information about the last n columns in the matrix Ψ (or, equivalently, about the blocks $\Psi_{/}$ and Ψ_{\setminus}).

Lemma 31 (eigenvalues and eigenvectors): Let $G(q)$ and $\tilde{G}(q)$ be elliptic coordinates matrices. If $\lambda = \lambda(q)$ is a simple root of $\det(G - \lambda \tilde{G}) = 0$, then $\nabla \lambda(q)$ is the corresponding ‘‘eigenvector’’:

$$(G(q) - \lambda(q) \tilde{G}(q)) \nabla \lambda(q) = 0. \tag{38}$$

If λ_1 and λ_2 are two different such roots, then

$$(\nabla \lambda_1)^T \tilde{G} \nabla \lambda_2 = 0. \tag{39}$$

Proof: Let $G_r = G - r \tilde{G}$ and $p(r) = \det G_r$. For each r , G_r is an elliptic coordinates matrix, with associated vector $N_r = N - r \tilde{N}$, where $N = \alpha q + \beta$ and $\tilde{N} = \tilde{\alpha} q + \tilde{\beta}$. If we apply proposition 6 to G_r we get $\nabla p(r) = 2(\text{cof } G_r) N_r$. Now compute the gradient of $p(\lambda(q)) \equiv 0$:

$$0 = (\nabla p)(\lambda(q)) + p'(\lambda(q)) \nabla \lambda(q) = 2 \text{cof}(G - \lambda(q) \tilde{G})(N - \lambda(q) \tilde{N}) + p'(\lambda(q)) \nabla \lambda(q). \tag{40}$$

Multiplying this by $G - \lambda(q)\tilde{G}$ yields, since $\det(G - \lambda(q)\tilde{G}) = 0$ by definition of λ ,

$$0 = p'(\lambda(q)) (G - \lambda(q)\tilde{G}) \nabla \lambda(q).$$

But $p'(\lambda(q)) \neq 0$ since $\lambda(q)$ is assumed to be a simple root of p . The first statement follows.

The second statement comes from the simple observation that if $G X_1 = \lambda_1 \tilde{G} X_1$ and $G X_2 = \lambda_2 \tilde{G} X_2$, then, since G and \tilde{G} are symmetric,

$$0 = (G X_1)^T X_2 - X_1^T (G X_2) = (\lambda_1 - \lambda_2) X_1^T \tilde{G} X_2.$$

□

Lemma 31, with $\tilde{G} = J$, says that

$$G \nabla u_k = u_k J \nabla u_k, \tag{41}$$

and that $\nabla u_1, \dots, \nabla u_n$ (which are the last n columns of Ψ) are “ J -orthogonal,”

$$(\nabla u_j)^T J (\nabla u_k) = 0, \quad \text{if } j \neq k. \tag{42}$$

Thus, the columns $(\nabla u_j)_\downarrow$ of the lower right $n \times n$ block Ψ_{\searrow} in Ψ are orthogonal in R^n in the ordinary Euclidean sense, with squared lengths $\Delta_1, \dots, \Delta_n$, where

$$\Delta_k = ((\nabla u_k)_\downarrow)^T (\nabla u_k)_\downarrow = \sum_{i=1}^n (\Psi_{n+i, n+k})^2. \tag{43}$$

Consequently, since the first m columns in Ψ are just e_1, \dots, e_m , the interpretation of an $n \times n$ determinant as a volume in R^n shows that

$$(\det \Psi)^2 = \Delta_1 \Delta_2 \cdots \Delta_n. \tag{44}$$

It also follows that, with $\Delta = \text{diag}(\Delta_1, \dots, \Delta_n)$ and $\mathcal{U} = \text{diag}(u_1, \dots, u_n)$,

$$\Psi^T J \Psi = \begin{pmatrix} 0_{m \times m} & 0_{m \times n} \\ 0_{n \times m} & \Delta \end{pmatrix} \tag{45}$$

and

$$\Psi^T G \Psi = \begin{pmatrix} G_{\searrow} & 0_{m \times n} \\ 0_{n \times m} & \mathcal{U} \Delta \end{pmatrix}. \tag{46}$$

D. Integrals of motion in separation coordinates

Now we will transform the integrals of motion $E^{(0)}, \dots, E^{(n)}$ given by (27) to the new coordinates (v, u) .

Kinetic part: We begin with the “kinetic” part $\dot{q}^T \text{cof}(G + \mu J) \dot{q}$. Write $G_\mu = G + \mu J$ for simplicity. Equation (36) gives

$$\dot{q}^T (\text{cof } G_\mu) \dot{q} = \frac{1}{(\det \Psi)^2} (v^T \quad \dot{u}^T) \text{cof}(\Psi^T G_\mu \Psi) \begin{pmatrix} \dot{v} \\ \dot{u} \end{pmatrix}.$$

Equations (45) and (46) show that

$$\Psi^T G_\mu \Psi = \begin{pmatrix} G_{\searrow} & 0_{m \times n} \\ 0_{n \times m} & \mathcal{U}_\mu \Delta \end{pmatrix},$$

where

$$\mathcal{U}_\mu = \mathcal{U} + \mu I_{n \times n} = \text{diag}(u_1 + \mu, \dots, u_n + \mu). \tag{47}$$

This, together with (44), gives

$$\frac{1}{(\det \Psi)^2} \text{cof}(\Psi^T G_\mu \Psi) = \begin{pmatrix} \det \mathcal{U}_\mu \text{cof} G_{\setminus} & 0_{m \times n} \\ 0_{n \times m} & (\det G_{\setminus}) \Delta^{-1} \text{cof} \mathcal{U}_\mu \end{pmatrix}.$$

Sandwiching this between $(\dot{v}^T \dot{u}^T)$ and $\begin{pmatrix} \dot{v} \\ \dot{u} \end{pmatrix}$, we finally obtain

$$\dot{q}^T (\text{cof} G_\mu) \dot{q} = (\det \mathcal{U}_\mu) \dot{v}^T (\text{cof} G_{\setminus}) \dot{v} + (\det G_{\setminus}) \dot{u}^T (\Delta^{-1} \text{cof} \mathcal{U}_\mu) \dot{u}. \tag{48}$$

Note that $\det \mathcal{U}_\mu = \prod_1^n (u_i + \mu)$ is the generating function for the elementary symmetric polynomials in the n variables $\{u_1, \dots, u_n\}$, while the k th entry in the diagonal matrix $\text{cof} \mathcal{U}_\mu$ generates the elementary symmetric polynomials in the $n - 1$ variables $\{u_1, \dots, u_n\} \setminus \{u_k\}$.

Structure of Δ_k : Next we prove a statement about how Δ_k , defined by (43), depends on u and v . This result is important for showing separability later.

Proposition 32: The quantities $\Delta_1, \dots, \Delta_n$ satisfy

$$\Delta_k(u, v) U'(u_k) \det G_{\setminus}(v) = f_k(u_k), \quad k = 1, \dots, n, \tag{49}$$

where each of the functions f_1, \dots, f_n depends on one variable only, as indicated. [But $U'(u_k)$, which is just the derivative of $U(\mu) = \prod(u_i - \mu)$ evaluated at $\mu = u_k$, depends on all the variables u_i .]

Proof: Recall that $\Delta = \text{diag}(\Delta_1, \dots, \Delta_n) = (\Psi_{\setminus})^T \Psi_{\setminus}$, by (45). Since the columns ∇u_k make up the blocks Ψ_{\nearrow} and Ψ_{\setminus} , the ‘‘upper part’’ of (41) shows that

$$G_{\setminus} \Psi_{\nearrow} + G_{\nearrow} \Psi_{\setminus} = 0_{m \times n}. \tag{50}$$

Recall from (34) that

$$\det(G - \mu J) = U(\mu) \det G_{\setminus} = \det G_{\setminus} \left((-\mu)^n + (-\mu)^{n-1} \left(\sum u_i \right) + \dots \right).$$

By proposition 6,

$$\nabla \det(G - \mu J) = 2 \text{cof}(G - \mu J) N = 2 \left((-\mu)^n A^{(n)} + (-\mu)^{n-1} A^{(n-1)} + \dots \right) N.$$

(Note that N is the vector associated to $G - \mu J$ as well as to G , since J is constant.) Hence, in particular,

$$2 A^{(n-1)} N = \nabla \left((\det G_{\setminus}) \sum u_i \right).$$

Now, $(\nabla \det G_{\setminus})_{\downarrow} = 0$ since G_{\setminus} depends only on the y variables, and consequently

$$2 (A^{(n-1)} N)_{\downarrow} = (\det G_{\setminus}) \sum (\nabla u_i)_{\downarrow} = (\det G_{\setminus}) \Psi_{\setminus} 1_n,$$

where $1_n \in R^n$ is the column vector with all ones. If we use what we know from (30) about the block structure of $A^{(n-1)}$ and divide by $\det G_{\setminus}$, this takes the form

$$2 \begin{pmatrix} -G_{\setminus}^{-1} G_{\nearrow} \\ I_{n \times n} \end{pmatrix}^T N = \Psi_{\setminus} 1_n. \tag{51}$$

Combining (50) and (the transpose of) (51), we find

$$2 N^T \begin{pmatrix} \Psi \nearrow \\ \Psi \searrow \end{pmatrix} = 2 N^T \begin{pmatrix} -G^{-1} G \nearrow \Psi \searrow \\ \Psi \searrow \end{pmatrix} = (\Psi \searrow 1_n)^T \Psi \searrow = 1_n^T \Delta = (\Delta_1 \quad \Delta_2 \dots \Delta_n).$$

In other words,

$$\Delta_k = 2 N^T \nabla u_k, \quad k = 1, \dots, n. \tag{52}$$

As a special case of (40), with $\tilde{G} = J$, $\tilde{N} = 0$, $p(\mu) = \det(G - \mu J) = U(\mu) \det G \searrow$, and $\lambda = u_k$, we have

$$U'(u_k) (\det G \searrow) \nabla u_k = -2 \operatorname{cof}(G - u_k J) N, \tag{53}$$

which, because of (52), when multiplied from the left by $2 N^T$ yields

$$U'(u_k) (\det G \searrow) \Delta_k = -4 N^T \operatorname{cof}(G - u_k J) N, \tag{54}$$

The left-hand side here is what we claim depends on u_k only, and we will prove this by showing that the gradient of the right hand side is proportional to ∇u_k . [Clearly, a function $f(v, u)$ depends on u_k alone iff $(\partial f / \partial u_k) \nabla u_k$ is the only contribution when computing ∇f with the chain rule.]

Proposition 10, applied to $G - \mu J$ (which has the same α and N as G), shows that

$$\nabla(N^T \operatorname{cof}(G - \mu J) N) = 2 \alpha \operatorname{cof}(G - \mu J) N.$$

Hence, by the chain rule,

$$\nabla(N^T \operatorname{cof}(G - u_k J) N) = 2 \alpha \operatorname{cof}(G - u_k J) N + \frac{d}{d\mu} [N^T \operatorname{cof}(G - \mu J) N]_{\mu=u_k} \nabla u_k.$$

It is manifest that the second term is proportional to ∇u_k , and so is in fact also the first term, because of (53). This finishes the proof of proposition 32. \square

Remark 33: In all the examples we have computed, it turns out that $f_i(q_i) = f(q_i)$ for a single function f , but we have no proof that this is always true. In any case, it is not needed for proving separability here.

Solution of the fundamental equations: We previously (in the proof of proposition 25) investigated the fundamental equations associated to the pair (G, J) :

$$0 = \frac{\partial M_i}{\partial q_j} \quad \text{for } i \leq m < j, \tag{55}$$

$$0 = \frac{\partial M_i}{\partial q_j} - \frac{\partial M_j}{\partial q_i} \quad \text{for } m < i, j, \tag{56}$$

where

$$M = - \frac{G \nabla(K \det G)}{\det G}$$

is the right-hand side in the cofactor pair system $\dot{q} = M(q)$ generated by $E^{(0)} = \frac{1}{2} \dot{q}^T (\operatorname{cof} G) \dot{q} + K \det G$.

Proposition 34: In terms of the separations coordinates (v, u) , the general solution of the fundamental equations(55) and (56) is

$$K(v, u) = \frac{1}{\det G_{\setminus}(v)} \left(w(v) + \sum_{k=1}^n \frac{g_k(u_k)/u_k}{U'(u_k)} \right), \tag{57}$$

where $g_1(u_1), \dots, g_n(u_n)$ are arbitrary functions of one variable, and $U'(u_k)$ is as in proposition 32.

Proof: Recall from (37) that

$$\nabla \equiv \begin{pmatrix} \partial_y \\ \partial_x \end{pmatrix} = \Psi \begin{pmatrix} \partial_v \\ \partial_u \end{pmatrix},$$

while (34) shows that $\det G = u_1 \dots u_n \det G_{\setminus}(v)$. Hence,

$$\begin{aligned} -M &= G \Psi \begin{pmatrix} \partial_v(K \det G) \\ \partial_u(K \det G) \end{pmatrix} / \det G = \frac{\begin{pmatrix} G_{\setminus} & 0 \\ G_{\setminus} & \Psi_{\setminus} \mathcal{U} \end{pmatrix} \begin{pmatrix} u_1 \dots u_n \partial_v(K \det G_{\setminus}) \\ (\det G_{\setminus}) \partial_u(u_1 \dots u_n K) \end{pmatrix}}{u_1 \dots u_n \det G_{\setminus}} \\ &= \begin{pmatrix} \frac{G_{\setminus} \partial_v(K \det G_{\setminus})}{\det G_{\setminus}} \\ \frac{G_{\setminus} \partial_v(K \det G_{\setminus})}{\det G_{\setminus}} + \Psi_{\setminus} \begin{pmatrix} \partial_{u_1}(u_1 K) \\ \vdots \\ \partial_{u_n}(u_n K) \end{pmatrix} \end{pmatrix}, \end{aligned}$$

where $G\Psi$ was computed using (41). Equation (55) says that the upper part

$$M_{\uparrow} = - \frac{G_{\setminus} \partial_v(K \det G_{\setminus})}{\det G_{\setminus}}$$

depends only on the y (or v) variables, which happens if and only if

$$K \det G_{\setminus} = w(v) + F(u).$$

The function $w(y)$ here is the same as in theorem 23, since the driving system $\ddot{y} = M_{\uparrow}$ is generated by $E^{(n)} = \frac{1}{2} \dot{y}^T (\text{cof } G_{\setminus}) \dot{y} + w(y)$.

The function $F(u)$ is then determined by (56), which obviously is only interesting if $i \neq j$. In this case, if we set $i = m + k$ and $j = m + l$, the first term in

$$M_i = M_{m+k} = - \frac{[G_{\setminus}]_{\text{row } k} \partial_v(K \det G_{\setminus})}{\det G_{\setminus}} - [\Psi_{\setminus}]_{\text{row } k} \begin{pmatrix} \partial_{u_1}(u_1 K) \\ \vdots \\ \partial_{u_n}(u_n K) \end{pmatrix}$$

does not depend on $q_j = x_l$, since row k of G_{\setminus} depends on x_k and y only. Then, since by the definition of Ψ

$$[\Psi_{\setminus}]_{\text{row } k} = \begin{pmatrix} \frac{\partial u_1}{\partial x_k} & \frac{\partial u_2}{\partial x_k} & \dots & \frac{\partial u_n}{\partial x_k} \end{pmatrix},$$

we find

$$\frac{\partial M_i}{\partial q_j} = - \frac{\partial}{\partial x_l} \sum_{s=1}^n \frac{\partial u_s}{\partial x_k} \partial_{u_s}(u_s K) = - \sum_{s=1}^n \frac{\partial^2 u_s}{\partial x_l \partial x_k} \partial_{u_s}(u_s K) - [\Psi_{\setminus}]_{\text{row } k} \frac{\partial}{\partial x_l} \begin{pmatrix} \partial_{u_1}(u_1 K) \\ \vdots \\ \partial_{u_n}(u_n K) \end{pmatrix}.$$

In the second term we substitute $K = (w(v) + F(u))/\det G_{\setminus}(v)$ and plug what we have into (56). The first term cancels out in the subtraction, leaving

$$0 = \frac{\partial M_i}{\partial q_j} - \frac{\partial M_j}{\partial q_i} = -\frac{1}{\det G_{\setminus}} \left([\Psi_{\setminus}]_{\text{row } k} \frac{\partial}{\partial x_l} \begin{pmatrix} \partial_{u_1}(u_1 F) \\ \vdots \\ \partial_{u_n}(u_n F) \end{pmatrix} - [\Psi_{\setminus}]_{\text{row } l} \frac{\partial}{\partial x_k} \begin{pmatrix} \partial_{u_1}(u_1 F) \\ \vdots \\ \partial_{u_n}(u_n F) \end{pmatrix} \right).$$

Now, since $\partial_x = \Psi_{\setminus} \partial_u$, this shows that

$$0 = [\Psi_{\setminus}]_{\text{row } l} \Omega [\Psi_{\setminus}^T]_{\text{column } k} - [\Psi_{\setminus}]_{\text{row } k} \Omega [\Psi_{\setminus}^T]_{\text{column } l},$$

where Ω (temporarily) denotes the $n \times n$ matrix with entries $\Omega_{ab} = \partial_{u_a} \partial_{u_b}(u_b F)$. In other words, $0 = \Psi_{\setminus}(\Omega - \Omega^T)\Psi_{\setminus}^T$, or, finally,

$$\frac{\partial^2}{\partial u_a \partial u_b} ((u_a - u_b)F(u)) = 0, \quad a, b = 1, \dots, n. \tag{58}$$

This equation occurs in classical separability theory in connection with separation in elliptic and parabolic coordinates. It is known to have the general solution

$$F(u) = \sum_{k=1}^n \frac{F_k(u_k)}{\prod_{\substack{j=1 \\ j \neq k}}^n (u_k - u_j)},$$

with arbitrary functions $F_1(u_1), \dots, F_n(u_n)$ depending on one variable each (see Lemma 1 and Lemma 2 in Ref. 3). Hence, we have the general solution

$$K(v, u) = \frac{1}{\det G_{\setminus}(v)} \left(w(v) + \sum_k \frac{F_k(u_k)}{\prod_{j \neq k} (u_k - u_j)} \right). \tag{59}$$

For our purposes, it turns out to be most convenient to write this in the form (57). □

Potential part: It remains to investigate the form of the ‘‘potential’’ parts $W^{(0)}, \dots, W^{(n)}$ in the (v, u) coordinates.

Proposition 35: The functions $W^{(0)}, \dots, W^{(n-1)}$ take the following form when expressed in the (v, u) coordinates:

$$W^{(a)}(v, u) = \sigma_{n-a}(u) w(v) + \sum_{k=1}^n \frac{\sigma_{n-a-1}(\check{u}_k) g_k(u_k)}{U'(u_k)}, \tag{60}$$

where $\sigma_b(u)$ denotes the elementary symmetric polynomial of degree b in the n variables $\{u_1, \dots, u_n\}$, and $\sigma_b(\check{u}_k)$ denotes the elementary symmetric polynomial of degree b in the $n - 1$ variables $\{u_1, \dots, u_n\} \setminus \{u_k\}$. As above, $g_1(u_1), \dots, g_n(u_n)$ are functions of one variable, and $U'(u_k)$ is as in proposition 32.

In particular, the function $W^{(n)}$ depends on the v coordinates only:

$$W^{(n)} = w(v). \tag{61}$$

Proof: We have seen that $W^{(n)} = w(y)$ depends only on y in the original coordinates, hence also $W^{(n)} = w(v)$. We also know that $K = W^{(0)}/\det G$ is a solution of the fundamental equations, so according to (57)

$$W^{(0)}(v, u) = \frac{\det G(v, u)}{\det G_{\setminus}(v)} \left(w(v) + \sum_{k=1}^n \frac{g_k(u_k)/u_k}{U'(u_k)} \right) = u_1 \cdots u_n w(v) + \sum_{k=1}^n \frac{\sigma_{n-1}(\check{u}_k) g_k(u_k)}{U'(u_k)}. \tag{62}$$

With M determined by $W^{(0)}$, the remaining $W^{(a)}$ are determined (up to irrelevant additive constants) by the relation $\nabla W^{(a)} = -A^{(a)}M$, or

$$\nabla W_{\mu} = \sum_{a=0}^n \nabla W^{(a)} \mu^a = - \left(\sum_{a=0}^n A^{(a)} \mu^a \right) M = \text{cof}(G + \mu J) \frac{G}{\det G} \nabla W^{(0)}.$$

We multiply by $(\det G)\Psi^T(G + \mu J)$ from the left and use (37), (45), and (46) to obtain the equivalent condition

$$(\det G) \begin{pmatrix} G_{\setminus} & 0 \\ 0 & (\mathcal{U} + \mu I)\Delta \end{pmatrix} \begin{pmatrix} \partial_v W_{\mu} \\ \partial_u W_{\mu} \end{pmatrix} = \det(G + \mu J) \begin{pmatrix} G_{\setminus} & 0 \\ 0 & \mathcal{U}\Delta \end{pmatrix} \begin{pmatrix} \partial_v W^{(0)} \\ \partial_u W^{(0)} \end{pmatrix}.$$

It is a tedious but fairly straightforward calculation, which we omit, to verify that this is satisfied by

$$W_{\mu} = \left(\prod_{i=1}^n (u_i + \mu) \right) w(v) + \sum_{k=1}^n \left(\prod_{\substack{j=1 \\ j \neq k}}^n (u_j + \mu) \right) \frac{g_k(u_k)}{U'(u_k)},$$

from which $W^{(a)}$ can be read off as the coefficient of μ^a . □

Summary: We have now determined the form of the integrals of motion in separation coordinates (v, u) . We have seen that

$$E^{(n)} = \frac{1}{2} \dot{v}^T (\text{cof } G_{\setminus}(v)) \dot{v} + w(v) \tag{63}$$

depends only on v , while the form of $E^{(0)}, \dots, E^{(n-1)}$ is obtained from (48) and (60):

$$E^{(a)} = \sigma_{n-a}(u) E^{(n)} + \sum_{k=1}^n \sigma_{n-a-1}(\check{u}_k) \left(\frac{1}{2} (\det G_{\setminus}) \frac{\dot{u}_k^2}{\Delta_k} + \frac{g_k(u_k)}{U'(u_k)} \right). \tag{64}$$

If we let $s_k = \dot{u}_k / \Delta_k$ and use proposition 32, we can write this as

$$E^{(a)} = \sigma_{n-a}(u) E^{(n)} + \sum_{k=1}^n \frac{\sigma_{n-a-1}(\check{u}_k)}{U'(u_k)} \left(\frac{1}{2} f_k(u_k) s_k^2 + g_k(u_k) \right). \tag{65}$$

Note in particular that

$$E^{(n-1)} = \left(\sum_{k=1}^n u_k \right) E^{(n)} + \sum_{k=1}^n \left(\frac{f_k(u_k) s_k^2 + g_k(u_k)}{U'(u_k)} \right). \tag{66}$$

E. The equations of motion are Hamiltonian

Given some solution $y = y(t)$ [or $v = v(t)$] of the driving system, we now consider $u = u(y(t), x)$ as a time-dependent change of variables in R^n . We want to express the driven system $\ddot{x} = -(\partial V / \partial x)(y(t), x)$ in terms of the u variables. Note that since $E^{(n)}$ is an integral of motion for the driving system, it can from now on be treated as simply a constant, the value of which is determined by which solution $y(t)$ is taken.

Proposition 36: The equations of motion for the u variables can be put into canonical Hamiltonian form

$$\dot{u} = \frac{\partial h}{\partial s}(u, s, t),$$

$$\dot{s} = -\frac{\partial h}{\partial u}(u, s, t),$$

with momenta s_1, \dots, s_n defined by

$$s_k = \frac{\dot{u}_k}{\Delta_k} \tag{67}$$

(Δ_i as in proposition 32), and with the time-dependent Hamiltonian

$$h(u, s, t) = \frac{1}{\det G_{\setminus}(y(t))} \left(\left(\sum_{k=1}^n u_k \right) E^{(n-1)} + \sum_{k=1}^n \left(\frac{f_k(u_k) s_k^2 + g_k(u_k)}{U'(u_k)} \right) \right). \tag{68}$$

Proof: First we see from (66) that h is simply $E^{(n-1)}/\det G_{\setminus}$, expressed in terms of u, s , and t . Now, with $p = \dot{x}$ the system $\ddot{x} = -(\partial V/\partial x)(y(t), x)$ has a canonical Hamiltonian formulation

$$\dot{x} = \frac{\partial H}{\partial p}(x, p, t),$$

$$\dot{p} = -\frac{\partial H}{\partial x}(x, p, t),$$

where $H(x, p, t) = \frac{1}{2} p^T p + V(y(t), x)$. Consider the extended phase space R^{2n+1} with coordinates (x, p, t) . With $T = t$, the variables (u, s, T) constitute a different coordinate system on this space. The vector field in extended phase space that corresponds to the canonical phase flow is encoded in the 1-form $p^T dx - H dt$ (by spanning the kernel of its exterior derivative). It follows that the equations of motion are canonical in the new coordinates, with Hamiltonian h , if the two 1-forms,

$$p^T dx - H dt \quad \text{and} \quad s^T du - h dT,$$

have the same exterior derivative (see Sec. 45 in Ref. 1). Here we view dx and du as column vectors of 1-forms dx_i and du_i , in order to be consistent with our previous matrix notation. Since here we have $dT = dt$, the proof amounts to showing that

$$d(p^T dx - s^T du + (h - H) dt) = 0. \tag{69}$$

The computations will be performed in the (x, p, t) coordinates, and whenever we write y we mean the given function $y(t)$. Note also that since G_{\setminus} depends only on the y variables, it too will be a function of t only. In particular, $\det G_{\setminus}$ is a function of t only.

We need to express $s^T du$ and h in terms of the (x, p, t) coordinates. Recall that by the definition 30 of the matrix Ψ we have

$$(\nabla u_1, \dots, \nabla u_n) = \begin{pmatrix} \Psi \\ \Psi_{\setminus} \end{pmatrix}.$$

Since

$$du_i = \sum_{k=1}^m \frac{\partial u}{\partial y_k} \dot{y}_k dt + \sum_{k=1}^n \frac{\partial u}{\partial x_k} dx_k$$

we obtain

$$du = (\Psi_{\nearrow})^T \dot{y} dt + (\Psi_{\searrow})^T dx,$$

that is,

$$\dot{u} = (\Psi_{\nearrow})^T \dot{y} + (\Psi_{\searrow})^T \dot{x}.$$

If we transpose and multiply from the right by $\Delta^{-1} = \text{diag}(\Delta_k^{-1})$, we get

$$s^T = \dot{y}^T \Psi_{\nearrow} \Delta^{-1} + p^T \Psi_{\searrow} \Delta^{-1}.$$

Now we define an $m \times n$ matrix Ξ by

$$\Xi = \Psi_{\nearrow} (\Psi_{\searrow})^{-1}. \tag{70}$$

Since $\Delta = (\Psi_{\searrow})^T \Psi_{\searrow}$, it follows that

$$\begin{aligned} \Psi_{\searrow} \Delta^{-1} (\Psi_{\nearrow})^T &= \Xi^T, \\ \Psi_{\nearrow} \Delta^{-1} (\Psi_{\searrow})^T &= \Xi \Xi^T. \end{aligned}$$

Consequently,

$$s^T du = p^T dx + (\dot{y}^T \Xi \Xi^T \dot{y} + \dot{y}^T \Xi p) dt + \dot{y}^T \Xi dx. \tag{71}$$

Furthermore, (50) shows that $G_{\nearrow} = -G_{\searrow} \Xi$, so that the expression for the block $A_{\nearrow}^{(n-1)}$ from (30) can be written as

$$A_{\nearrow}^{(n-1)} = -(\text{cof } G_{\searrow}) G_{\nearrow} = (\det G_{\searrow}) \Xi. \tag{72}$$

Hence, since from (30) we also have $A_{\searrow}^{(n-1)} = (\det G_{\searrow}) I$, we find the following expression for h :

$$h = \frac{E^{(n-1)}}{\det G_{\searrow}} = \frac{1}{\det G_{\searrow}} \left(\frac{1}{2} (\dot{y}^T \ p^T) A_{\searrow}^{(n-1)} \begin{pmatrix} \dot{y} \\ p \end{pmatrix} + W^{(n-1)} \right) = \frac{1}{2} p^T p + \dot{y}^T \Xi p + \frac{\frac{1}{2} \dot{y}^T A_{\searrow}^{(n-1)} \dot{y} + W^{(n-1)}}{\det G_{\searrow}}. \tag{73}$$

So far we have

$$p^T dx - s^T du + (h - H) dt = \left(\frac{\frac{1}{2} \dot{y}^T A_{\searrow}^{(n-1)} \dot{y} + W^{(n-1)}}{\det G_{\searrow}} - V - \dot{y}^T \Xi \Xi^T \dot{y} \right) dt - \dot{y}^T \Xi dx,$$

and the exterior derivative of this is zero iff

$$\frac{\partial}{\partial x} \left(\frac{\frac{1}{2} \dot{y}^T A_{\searrow}^{(n-1)} \dot{y} + W^{(n-1)}}{\det G_{\searrow}} - V - \dot{y}^T \Xi \Xi^T \dot{y} \right) + \frac{\partial}{\partial t} (\Xi^T \dot{y}) = 0.$$

Now $(\partial/\partial t)(\Xi^T \dot{y}) = (\partial \Xi^T / \partial t) \dot{y} + \Xi^T \ddot{y}$, and from $\ddot{q} = -[A^{(n-1)}]^{-1} \nabla W^{(n-1)}$ it follows that

$$\frac{\partial W^{(n-1)}}{\partial x} = -[A^{(n-1)} \ddot{q}]_{\downarrow} = -(A_{\nearrow}^{(n-1)})^T \ddot{y} + A_{\searrow}^{(n-1)} \frac{\partial V}{\partial x} = (\det G_{\searrow}) \left(-\Xi^T \ddot{y} + \frac{\partial V}{\partial x} \right),$$

so it remains to show that

$$\frac{\partial}{\partial x} \left(\frac{\frac{1}{2} \dot{y}^T A_{\searrow}^{(n-1)} \dot{y}}{\det G_{\searrow}} - \dot{y}^T \Xi \Xi^T \dot{y} \right) + \frac{\partial \Xi^T}{\partial t} \dot{y} = 0.$$

To simplify the notation for this final computation, write

$$\det G_{\setminus} = D, \quad A_{\setminus}^{(n-1)} = (a_{ij}), \quad \text{and} \quad A_{\setminus}^{(n-1)} = (b_{ij}).$$

Then $(b_{ij}) = -(\text{cof } G_{\setminus})G_{\setminus} = D\Xi$, by (72). In this notation, what we must show is

$$\frac{1}{2D} \sum_{i,j=1}^m \frac{\partial a_{ij}}{\partial x_k} \dot{y}_i \dot{y}_j - \frac{1}{D^2} \sum_{i,j=1}^m \sum_{l=1}^n \frac{\partial (b_{il} b_{jl})}{\partial x_k} \dot{y}_i \dot{y}_j + \sum_{i=1}^m \frac{\partial}{\partial t} \left(\frac{b_{ik}}{D} \right) \dot{y}_i = 0. \tag{74}$$

To begin with, since G_{\setminus} is independent of x and G_{\setminus} is linear in x , we see that b_{ij} is linear in x . More precisely, since

$$\frac{\partial [G_{\setminus}]_{rj}}{\partial x_k} = \frac{\partial G_{r,m+j}}{\partial q_{m+k}} = \delta_{jk} N_r,$$

applying proposition 6 with y instead of q gives

$$\frac{\partial b_{ij}}{\partial x_k} = - \sum_{r=1}^m [\text{cof } G_{\setminus}]_{ir} (\delta_{jk} N_r) = - \delta_{jk} [(\text{cof } G_{\setminus}) N_{\uparrow}]_i = - \frac{\delta_{jk}}{2} \frac{\partial D}{\partial y_i}.$$

Furthermore,

$$\sum_{i=1}^m \frac{\partial}{\partial t} \left(\frac{b_{ik}}{D} \right) \dot{y}_i = \sum_{i,j=1}^m \frac{\partial}{\partial y_j} \left(\frac{b_{ik}}{D} \right) \dot{y}_i \dot{y}_j.$$

Finally, since $A^{(n-1)}$ satisfies the cyclic conditions,

$$\frac{\partial a_{ij}}{\partial x_k} = \frac{\partial A_{ij}^{(n-1)}}{\partial q_{m+k}} = - \frac{\partial A_{j,m+k}^{(n-1)}}{\partial q_i} = \frac{\partial A_{m+k,i}^{(n-1)}}{\partial q_j} = - \frac{\partial b_{jk}}{\partial y_i} = \frac{\partial b_{ik}}{\partial y_j}.$$

Plugging all this into (74), it is easy to verify that everything cancels out, which completes the proof. \square

F. Separation of the time-dependent Hamilton–Jacobi equation

The time-dependent Hamilton–Jacobi equation corresponding to the Hamiltonian $h(u, s, t)$ of proposition 36 is

$$h\left(u, \frac{\partial F}{\partial u}, t\right) + \frac{\partial F}{\partial t} = 0. \tag{75}$$

A complete solution $F(u, \alpha, t)$ can be obtained by separation of variables, as we will now show. We number the parameters $\alpha_0, \dots, \alpha_{n-1}$ since they will in fact be just the values of the integrals of motion $E^{(0)}, \dots, E^{(n-1)}$, as will be clear by comparing (78) below with (65).

To begin with, since the time variable t appears in K only in the overall multiplicative factor $1/(\det G_{\setminus})$, it can be separated off by assuming a solution for F of the form

$$F(u, \alpha, t) = S(u, \alpha) - \alpha_{n-1} \int \frac{1}{\det G_{\setminus}(y(t))} dt. \tag{76}$$

With the explicit expression for h from proposition 36 we get the following equation for $S(u, \alpha)$:

$$\left(\sum_{k=1}^n u_k \right) E^{(n)} + \sum_{k=1}^n \left(\frac{\frac{1}{2} f_k(u_k) (\partial S / \partial u_k)^2 + g_k(u_k)}{U'(u_k)} \right) = \alpha_{n-1}. \tag{77}$$

In order to find a complete solution, depending on all the parameters α_i , we will use Stäckel’s method. Consider the n equations

$$\sum_{k=1}^n \frac{\sigma_{n-a-1}(\check{u}_k)}{U'(u_k)} \left(\frac{1}{2} f_k(u_k) \left(\frac{\partial S}{\partial u_k} \right)^2 + g_k(u_k) \right) = \alpha_a - \sigma_{n-a}(u) E^{(n)}, \tag{78}$$

where $a=0, \dots, n-1$. If we can find a solution of this system, it will be a complete solution of (77), since it will depend on all α_i . [Of course it will solve (77) which is just the last equation of the system, corresponding to $a=n-1$.]

Now (78) is a linear system of equations for the expression in parentheses, and the matrix of coefficients is the inverse of a Stäckel matrix (similar to the one occurring when separating in elliptic or parabolic coordinates). In fact, the matrix can be inverted using known properties of symmetric polynomials, resulting in

$$\frac{1}{2} f_k(u_k) \left(\frac{\partial S}{\partial u_k} \right)^2 + g_k(u_k) = -P(-u_k), \quad k=1, \dots, n, \tag{79}$$

where the polynomial P is given by

$$P(z) = \alpha_0 + \alpha_1 z + \dots + \alpha_{n-1} z^{n-1} + E^{(n)} z^n. \tag{80}$$

It is now clear that the additive Ansatz

$$S(u, \alpha) = S_1(u_1, \alpha) + \dots + S_n(u_n, \alpha)$$

yields a separated solution, provided that each function S_k satisfies the separation ODE

$$\frac{1}{2} \left(\frac{dS_k}{du_k} \right)^2 = \frac{-g_k(u_k) - P(-u_k)}{f_k(u_k)}. \tag{81}$$

Consequently,

$$F(u, \alpha, t) = \sum_{k=1}^n \int \sqrt{-2 \frac{g_k(u_k) + P(-u_k)}{f_k(u_k)}} du_k - \alpha_{n-1} \int \frac{1}{\det G_{\check{\cdot}}(y(t))} dt \tag{82}$$

is a complete solution, and in the usual way it generates a canonical transformation to variables (β, α) , where $\beta_i = \partial F / \partial \alpha_i$. These new variables will be constant during the motion, with values determined by the initial condition. One can then (at least in principle) solve for $u = u(\beta, \alpha, t)$, and hence $x = x(\beta, \alpha, t)$. This finishes the proof of theorem 23.

IV. THE CASE OF ONE DRIVEN EQUATION

The case when only the last equation is driven by the other ones is easier to handle, since it does not require the Hamilton–Jacobi method, as we shall soon see. Specializing our previous results to this case by setting $n = 1$, we find the following. If a system of the form

$$\begin{aligned} \ddot{y}_1 &= M_1(y_1, \dots, y_m), \\ &\vdots \\ \ddot{y}_m &= M_m(y_1, \dots, y_m), \end{aligned} \tag{83}$$

$$\ddot{x} = - \frac{\partial V}{\partial x}(y_1, \dots, y_m; x)$$

has an integral of motion $E^{(0)}$ of cofactor type, then it must have an extra integral of motion $E^{(1)} = \frac{1}{2} \dot{y}^T \text{cof } G_{\setminus}(y) \dot{y} + w(y)$ depending only on the variables y . We change to new coordinates (v_1, \dots, v_m, u) , where $v = y$ and u is the zero of the first degree polynomial $\det(G - \lambda J)$. Here $J = \text{diag}(0, \dots, 0, 1)$, so $\det(G - \lambda J) = \det G - \lambda \det G_{\setminus}$, hence

$$u = \frac{\det G(y, x)}{\det G_{\setminus}(y)}.$$

In the new variables, $E^{(1)}$ remains unchanged (with v instead of y), while $E^{(0)}$ takes the form given by (64),

$$E^{(0)} = uE^{(1)} + \frac{1}{2} \frac{\det G_{\setminus}(v)}{\Delta} \dot{u}^2 + g(u),$$

where, according to (43) and proposition 32,

$$\Delta = \left(\frac{\partial u}{\partial x} \right)^2 = \frac{f(u)}{\det G_{\setminus}(v)}$$

for some function $f(u)$. Hence,

$$E^{(0)} = uE^{(1)} + \frac{1}{2} \frac{(\det G_{\setminus}(v))^2}{f(u)} \dot{u}^2 + g(u). \tag{84}$$

Now, for a given solution $v(t) = y(t)$ of the driving system, we write this as

$$\left(\det G_{\setminus}(v(t)) \frac{du}{dt} \right)^2 = 2 f(u) (E^{(0)} - uE^{(1)} - g(u)),$$

or

$$\frac{du}{\sqrt{2 f(u) (E^{(0)} - uE^{(1)} - g(u))}} = \frac{dt}{\det G_{\setminus}(v(t))},$$

which can be integrated by quadrature, since u and t are separated.

This procedure can be applied recursively to ‘‘triangular’’ systems, as in the following proposition. Note that for an arbitrary triangular system all we can do in general is to solve the first equation for $q_1(t)$. It is quite surprising that the existence of an integral of motion of cofactor type is enough to allow us to solve the system completely.

Proposition 37 (triangular cofactor systems): Suppose that the ‘‘triangular’’ Newton system

$$\begin{aligned} \ddot{q}_1 &= M_1(q_1), \\ \ddot{q}_2 &= M_2(q_1, q_2), \\ \ddot{q}_3 &= M_3(q_1, q_2, q_3), \\ &\vdots \\ \ddot{q}_N &= M_N(q_1, q_2, q_3, \dots, q_N), \end{aligned} \tag{85}$$

is of cofactor type. Suppose also that no upper left $k \times k$ block in G is constant or singular ($k = 1, \dots, N - 1$). Then the system can be integrated by quadratures.

Proof: The whole system is of the type considered above (driven, with $n=1$), so it can be integrated provided that the driving system, consisting of the $N-1$ first equations, can be integrated. By what we said above, the driving system must have an integral of motion of cofactor type, so it is itself a triangular cofactor system, of one dimension less. Since the first equation can be integrated (being one dimensional), the statement follows by induction. \square

In each step of the integration procedure one new variable $u = u_k$ is introduced. Denoting the determinant of the upper left $k \times k$ block in G by $D_k(q_1, \dots, q_k)$, we can write the separation variables (u_1, \dots, u_N) as

$$u_1 = q_1 \quad \text{and} \quad u_i = \frac{D_i}{D_{i-1}}, \quad i = 2, \dots, N.$$

V. EXAMPLES

Example 38 (example 1 continued): We can now fill in the missing details in our first example. We had

$$M(q) = \frac{1}{(q_2 q_3 - q_1)^2} \begin{pmatrix} 0 \\ q_3 \\ q_2 \end{pmatrix}.$$

With

$$G(q) = \begin{pmatrix} 2q_1 & q_2 & q_3 \\ q_2 & 0 & 1 \\ q_3 & 1 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we find from $A_\mu = \text{cof}(G + \mu J)$ that

$$A^{(0)} = \text{cof } G = \begin{pmatrix} -1 & q_3 & q_2 \\ q_3 & -q_3^2 & q_2 q_3 - 2q_1 \\ q_2 & q_2 q_3 - 2q_1 & -q_2^2 \end{pmatrix},$$

$$A^{(1)} = \begin{pmatrix} 0 & -q_2 & -q_3 \\ -q_2 & 2q_1 & 0 \\ -q_3 & 0 & 2q_1 \end{pmatrix}, \quad A^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The relation $\nabla W^{(k)} = -A^{(k)} M$ then yields

$$W^{(0)} = -\frac{q_2^2 + q_3^2}{q_2 q_3 - q_1}, \quad W^{(1)} = \frac{2q_1}{q_2 q_3 - q_1}, \quad W^{(2)} = 0.$$

We introduce new variables (v, u_1, u_2) , where $v = q_1$ and $u_{1,2}$ are the roots of

$$0 = \det(G - uJ) = 2(q_2 q_3 - q_1) + (q_2^2 + q_3^2)u + (2q_1)u^2.$$

With (y, x_1, x_2) instead of (q_1, q_2, q_3) , we see that $u_1 + u_2 = -(x_1^2 + x_2^2)/2y$ and $u_1 u_2 = 2(x_1 x_2 - y)/2y$, so that

$$\left(\frac{x_1+x_2}{\sqrt{2}}\right)^2 = v(1-u_1)(1-u_2),$$

$$\left(\frac{x_1-x_2}{\sqrt{2}}\right)^2 = -v(1+u_1)(1+u_2).$$
(86)

Except for the factor v , the new variables (u_1, u_2) are elliptic coordinates aligned along axes that are rotated $\pi/4$ relative to the Cartesian coordinates (x_1, x_2) . With $-1 < u_1 < 1 < u_2 < 1$, the coordinate curves are ellipses (for u_1) and hyperbolas (for u_2). The example (2) is obtained by taking the particular solution $y(t) = v(t) = q_1(t) = t$ of the driving equation $\dot{q}_1 = 0$, and in this case we get a factor t with the effect of expanding the entire coordinate web as time increases, so these coordinates might be called “expanding elliptic coordinates.”

We can express (u_1, u_2) in terms of (y, x_1, x_2) as

$$u_{1,2} = \frac{-1}{4y} (x_1^2 + x_2^2 \pm \sqrt{(x_1^2 + x_2^2)^2 - 16y(x_1x_2 - y)}),$$

and then a straightforward computation gives the quantities

$$\Delta_{1,2} = \left(\frac{\partial u_{1,2}}{\partial x_1}\right)^2 + \left(\frac{\partial u_{1,2}}{\partial x_2}\right)^2 = \frac{1}{2y^2} \left(x_1^2 + x_2^2 \pm \frac{(x_1^2 + x_2^2)^2 - 8yx_1x_2}{\sqrt{(x_1^2 + x_2^2)^2 - 16y(x_1x_2 - y)}} \right).$$

With $U(\mu) = (u_1 - \mu)(u_2 - \mu)$ we find that $(\det G_{\setminus})U'(u_1)\Delta_1 = 2y(u_1 - u_2)\Delta_1 = 4(1 - u_1^2)$ and $(\det G_{\setminus})U'(u_2)\Delta_2 = 2y(u_2 - u_1)\Delta_2 = 4(1 - u_2^2)$ depend only on one variable, as predicted by Proposition 32. So in this case we have $f_1 = f_2 = f$, where $f(u) = 4(1 - u^2)$.

The functions $W^{(k)}$, expressed in the new variables, take the form

$$W^{(0)} = 2 \frac{u_1 + u_2}{u_1 u_2} = u_2 \frac{-2/u_1}{U'(u_1)} + u_1 \frac{-2/u_2}{U'(u_2)},$$

$$W^{(1)} = \frac{2}{u_1 u_2} = \frac{-2/u_1}{U'(u_1)} + \frac{-2/u_2}{U'(u_2)},$$

$$W^{(2)} = 0,$$

in accordance with proposition 35.

We can now write down the integrals of motion $E^{(k)} = \frac{1}{2} \dot{q}^T A^{(k)} \dot{q} + W^{(k)}$ in terms of the variables (v, u_1, u_2) . With $s_i = \dot{u}_i / \Delta_i$, we find

$$E^{(2)} = \frac{\dot{v}^2}{2},$$

$$E^{(1)} = (u_1 + u_2)E^{(2)} + \frac{4(1 - u_1^2) \frac{s_1^2}{2} - \frac{2}{u_1}}{U'(u_1)} + \frac{4(1 - u_2^2) \frac{s_2^2}{2} - \frac{2}{u_2}}{U'(u_2)},$$

$$E^{(0)} = u_1 u_2 E^{(2)} + u_2 \frac{4(1 - u_1^2) \frac{s_1^2}{2} - \frac{2}{u_1}}{U'(u_1)} + u_1 \frac{4(1 - u_2^2) \frac{s_2^2}{2} - \frac{2}{u_2}}{U'(u_2)}.$$

The new Hamiltonian is $h = E^{(1)} / \det G_{\setminus}(v(t))$, or, with $v(t) = t$,

$$h(u, s, t) = \frac{1}{2t} \left(\frac{u_1 + u_2}{2} + \frac{4(1-u_1^2) \frac{s_1^2}{2} - \frac{2}{u_1}}{U'(u_1)} + \frac{4(1-u_2^2) \frac{s_2^2}{2} - \frac{2}{u_2}}{U'(u_2)} \right).$$

The time-dependent Hamilton–Jacobi equation $h(u, \partial F / \partial u, t) + \partial F / \partial t = 0$ admits a separated complete solution of the form

$$F(u_1, u_2, \alpha_1, \alpha_2, t) = S_1(u_1, \alpha_0, \alpha_1) + S_2(u_2, \alpha_0, \alpha_1) - \frac{\alpha_1}{2} \ln|t|,$$

where S_1 and S_2 satisfy the separation equations

$$\frac{1}{2} \left(\frac{dS_1}{du_1} \right)^2 = \frac{\frac{2}{u_1} - \alpha_0 + \alpha_1 u_1 - \frac{u_1^2}{2}}{4(1-u_1^2)},$$

$$\frac{1}{2} \left(\frac{dS_2}{du_2} \right)^2 = \frac{\frac{2}{u_2} - \alpha_0 + \alpha_1 u_2 - \frac{u_2^2}{2}}{4(1-u_2^2)}.$$

From $\beta_k = \partial F / \partial \alpha_k$ we finally obtain

$$\beta_1(u_1, u_2, t, \alpha_0, \alpha_1) = \int^{u_1} \frac{x}{2R} dx + \int^{u_2} \frac{x}{2R} dx - \frac{1}{2} \ln|t|,$$

$$\beta_0(u_1, u_2, t, \alpha_0, \alpha_1) = \int^{u_1} \frac{-1}{2R} dx + \int^{u_2} \frac{-1}{2R} dx,$$

where

$$R(x, \alpha_1, \alpha_2) = \sqrt{2(1-x^2) \left(\frac{2}{x} - \alpha_0 + \alpha_1 x - \frac{x^2}{2} \right)}.$$

This gives the solution $u(\beta, \alpha, t)$ in implicit form.

Example 39 (a triangular system): An interesting example of a triangular cofactor system appears when applying the recursive method for constructing cofactor pair systems given in Ref. 6 to the matrices

$$G = \begin{pmatrix} 0 & -1 & q_1 \\ -1 & 0 & q_2 \\ q_1 & q_2 & 2q_3 \end{pmatrix}, \quad \tilde{G} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Starting with $W^{(0)} = W^{(1)} = 0$ and $W^{(2)} = -1$, one obtains after four steps the system

$$\begin{aligned} \ddot{q}_1 &= -4q_1, \\ \ddot{q}_2 &= 6q_1^2 - 4q_2, \\ \ddot{q}_3 &= -10q_1^3 + 12q_1q_2 - 4q_3, \end{aligned} \tag{87}$$

which is a cofactor pair system with respect to the given matrices G and \tilde{G} . Since the third equation is driven by the first two, the system is also a cofactor pair system with respect to G and $J = \text{diag}(0,0,1)$. In fact, the most general matrix G for which the system has an integral of motion of the form $\frac{1}{2}\dot{q}^T(\text{cof } G)\dot{q} + W(q)$ is

$$c_1 \begin{pmatrix} 0 & -1 & q_1 \\ -1 & 0 & q_2 \\ q_1 & q_2 & 2q_3 \end{pmatrix} + c_2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + c_4 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

so it might be called a ‘‘cofactor quadruple system.’’ [The third matrix comes from the fact that there is a function $U(q)$ such that $M_2 = \partial_3 U$ and $M_3 = \partial_2 U$.] Anyway, we know from Sec. IV that the driving system is a cofactor system with respect to

$$G_{\setminus} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

Since this matrix is constant, we cannot use it for integrating the driving system, but it so happens that the driving system is a cofactor system with respect to any matrix of the form

$$c_1 \begin{pmatrix} -1 & q_1 \\ q_1 & 2q_2 \end{pmatrix} + c_2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

So, forgetting about (87) for the moment, we consider the two-dimensional driving system

$$\ddot{q} = \begin{pmatrix} -4q_1 \\ 6q_1^2 - 4q_2 \end{pmatrix} = -\frac{g \nabla w}{\det g}, \tag{88}$$

where now

$$g = \begin{pmatrix} -1 & q_1 \\ q_1 & 2q_2 \end{pmatrix}, \quad w = \frac{3}{2}q_1^4 + 2q_1^2q_2 - 2q_2^2.$$

[In this example, we use lowercase letters for quantities referring to the two-dimensional system (88).] In the new variables $v = q_1$ and $u = \det g / \det g_{\setminus} = q_1^2 + 2q_2$, we have the integrals of motion

$$e^{(1)} = \frac{1}{2}\dot{q}_1^2 + 2q_2^2 = \frac{1}{2}\dot{v}^2 + 2v^2$$

from the first equation, and (after a short calculation)

$$e^{(0)} = \frac{1}{2}\dot{q}^T(\text{cof } g)\dot{q} + w(q) = ue^{(1)} - \frac{\dot{u}^2}{8} - \frac{u^2}{2}.$$

The function $v(t) = q_1(t)$ is just a harmonic oscillation, whose amplitude determines the numerical value of $e^{(1)}$ (or the other way around):

$$q_1(t) = \sqrt{\frac{e^{(1)}}{2}} \sin 2(t - t_1). \tag{89}$$

The value of $e^{(0)}$ is determined by the initial conditions for q_1 and q_2 . Then $u(t)$, and hence $q_2(t) = (u(t) - v(t)^2)/2$, can be found from the separable ODE

$$\frac{du}{dt} = \sqrt{8 \left(ue^{(1)} - \frac{u^2}{2} - e^{(0)} \right)}.$$

This gives

$$u(t) = \sqrt{(e^{(1)})^2 - 2e^{(0)}} \sin 2(t - t_2) + e^{(1)},$$

so that

$$q_2(t) = \frac{1}{2}(\sqrt{(e^{(1)})^2 - 2e^{(0)}} \sin 2(t - t_2) + e^{(1)}(1 - \frac{1}{2}\sin^2 2(t - t_1))). \tag{90}$$

Having found $q_1(t)$ and $q_2(t)$, we return to the three-dimensional system (87):

$$\ddot{q} = \begin{pmatrix} -4q_1 \\ 6q_1^2 - 4q_2 \\ -10q_1^3 + 12q_1q_2 - 4q_3 \end{pmatrix} = -\frac{G\nabla W}{\det G},$$

where

$$G = \begin{pmatrix} 0 & -1 & q_1 \\ -1 & 0 & q_2 \\ q_1 & q_2 & 2q_3 \end{pmatrix}, \quad W = 6q_1^2q_2^2 - 4q_1^4q_2 + 4q_1q_2q_3 - 2q_3^2 - 4q_1^3q_3.$$

Here we take new variables $v_1 = q_1$, $v_2 = q_2$, and $u = \det G / \det G_{\setminus} = 2(q_1q_2 + q_3)$. The integrals of motion turn out to be

$$E^{(1)} = \frac{1}{2}\dot{v}^T \operatorname{cof} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \dot{v} + 4v_1v_2 - 2v_1^3$$

and

$$E^{(0)} = \frac{1}{2}\dot{q}^T(\operatorname{cof} G)\dot{q} + W(q) = uE^{(1)} - \frac{\dot{u}^2}{8} - \frac{u^2}{2},$$

so the equation for $u(t)$ can again be separated (in exactly the same way as above). After finding $u(t)$, we finally obtain $q_3(t) = u(t)/2 - q_1(t)q_2(t)$, that is

$$q_3(t) = \frac{1}{2}(\sqrt{E^{(1)} - 2E^{(0)}} \sin 2(t - t_3) + E^{(1)} - q_1(t)q_2(t)). \tag{91}$$

By inserting the expressions for $q_1(t)$ and $q_2(t)$ into the expression for $E^{(1)}$ we find that it depends on the previous integration constants $e^{(0)}$, $e^{(1)}$, t_1 , t_2 through the equation

$$E^{(1)} = \sqrt{2e^{(1)}}\sqrt{e^{(1)} - 2e^{(0)}} \cos 2(t_2 - t_1).$$

On the other hand, $E^{(0)}$ and t_3 are independent of the previous integration constants.

Example 40 (construction of driven systems): Given a cofactor system

$$\ddot{y} = M_{\uparrow}(y) = -(\operatorname{cof} g(y))^{-1} \frac{\partial w}{\partial y}(y),$$

how can it be extended to a driven system

$$\ddot{q} = \begin{pmatrix} M_{\uparrow}(y) \\ M_{\downarrow}(y, x) \end{pmatrix} = -(\operatorname{cof} G(q))^{-1} \frac{\partial W}{\partial q}(q)$$

of the type considered in this paper? First of all, the restriction that the elliptic coordinates matrix $G(q)$ must have $g(y) = G_{\setminus}(y)$ as its upper left block fixes α , β_{\uparrow} and γ_{\setminus} . The remaining entries of β and γ can be chosen at will (as long as G is nonsingular). Then we want to find some

extension M_{\downarrow} of the right-hand side which is compatible with the chosen matrix G [i.e., so that $W(q)$ exists]. In separation coordinates, this amounts to specifying the functions $g_k(u_k)$ in the corresponding solution of the fundamental equations (proposition 34), the function $w(v) = w(y)$ already being determined by the driving system. One can find a family of possible M_{\downarrow} in Cartesian coordinates directly by using the recursion formula from Ref. 6. As it stands, this formula requires \tilde{G} to be nonsingular, but taking $\tilde{G} = J$ can be justified like in the proof of proposition 27 (however, it only makes sense in the “downwards” recursion formula). We then find that if a driven system has integrals of motion given by the generating function $E_{\mu} = \frac{1}{2}\dot{q}^T A_{\mu}\dot{q} + W_{\mu}$ as in (27), then we obtain another driven system with integrals of motion $\frac{1}{2}\dot{q}^T A_{\mu}\dot{q} + U_{\mu}$ by setting

$$U_{\mu} = \frac{1}{\mu} \left(\frac{\det(G + \mu J)}{\det G} W^{(0)} - W_{\mu} \right). \tag{92}$$

It is clear that U_{μ} is a polynomial in μ of degree $n - 1$, not n , which means that the new system (and any system obtained by iterating this process) is driven in the trivial way ($\dot{y} = 0$). They correspond to solutions (57) of the fundamental equations with $w(y) = 0$. Adding

$$\frac{\det(G + \mu J)}{\det G_{\setminus}} w(y)$$

to U_{μ} gives a system with any $w(y)$ desired.

As an example, consider the two-dimensional Garnier potential $V = (q_1^2 + q_2^2)^2 - (\lambda_1 q_1^2 + \lambda_2 q_2^2)$. We will demonstrate how to find G and M_3 such that the system

$$\begin{aligned} \ddot{q}_1 &= -\partial_1 V(q_1, q_2), \\ \ddot{q}_2 &= -\partial_2 V(q_1, q_2), \\ \ddot{q}_3 &= M_3(q_1, q_2, q_3) \end{aligned} \tag{93}$$

is of cofactor type $\ddot{q} = -(G/\det G)\nabla W$. With $G_{\setminus} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, corresponding to $w = V$, we have $\alpha = \beta_1 = \beta_2 = 0$, so we choose, for example, $\beta_3 = 1$ and extend γ with zeros to get

$$G = \begin{pmatrix} 1 & 0 & q_1 \\ 0 & 1 & q_2 \\ q_1 & q_2 & 2q_3 \end{pmatrix}.$$

Applying (92) with $W_{\mu} = 1 + 0\mu$ and $J = \text{diag}(0,0,1)$ gives $U_{\mu} = (\det G)^{-1}$, corresponding to the trivially driven system

$$\ddot{q} = -\frac{G}{\det G} \nabla \left(\frac{1}{\det G} \right) = \frac{2}{(2q_3 - q_1^2 - q_2^2)^2} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

To keep things simple we stop the recursion after this first step, and let

$$E_{\mu} = \frac{1}{2}\dot{q}^T \text{cof}(G + \mu J)\dot{q} + (\det G)^{-1} + \frac{\det(G + \mu J)}{\det G_{\setminus}} V(q_1, q_2),$$

which then generates an extended system of the desired form

$$\ddot{q} = -\frac{G}{\det G} \nabla \left((\det G)^{-1} + (\det G)V \right) = \begin{pmatrix} -\partial_1 V \\ -\partial_2 V \\ 2(2q_3 - q_1^2 - q_2^2)^{-2} - 2V \end{pmatrix}. \tag{94}$$

Since the Garnier potential is separable in elliptic coordinates it admits an extra integral of motion of cofactor type. This gives us the possibility to instead take

$$G_{\searrow} = \begin{pmatrix} \lambda_1 - q_1^2 & -q_1 q_2 \\ -q_1 q_2 & \lambda_2 - q_2^2 \end{pmatrix},$$

corresponding to

$$w = \lambda_2 q_1^4 + \lambda_1 q_2^4 + (\lambda_1 + \lambda_2) q_1^2 q_2^2 - \lambda_1 \lambda_2 (q_1^2 + q_2^2).$$

Here $\alpha = -1$ and $\beta_{\uparrow} = 0$, and we can for example extend G_{\searrow} to

$$G = \begin{pmatrix} \lambda_1 - q_1^2 & -q_1 q_2 & -q_1 q_3 \\ -q_1 q_2 & \lambda_2 - q_2^2 & -q_2 q_3 \\ -q_1 q_3 & -q_2 q_3 & \lambda_3 - q_3^2 \end{pmatrix}.$$

In a similar way as above we get in this case (after some computation) the extended system

$$\ddot{q} = -\frac{G}{\det G} \nabla \left(\frac{\det G_{\searrow}}{\det G} + \frac{\det G}{\det G_{\searrow}} w \right) = \begin{pmatrix} -\partial_1 V \\ -\partial_2 V \\ \lambda_1 \lambda_2 q_3 \left(\frac{2w(q_1, q_2)}{(\det G_{\searrow})^2} - \frac{2}{(\det G)^2} + \frac{q_1(\partial_1 V)/\lambda_1 + q_2(\partial_2 V)/\lambda_2}{\det G_{\searrow}} \right) \end{pmatrix}, \quad (95)$$

where $\det G = \lambda_1 \lambda_2 \lambda_3 (1 - q_1^2/\lambda_1 - q_2^2/\lambda_2 - q_3^2/\lambda_3)$ and $\det G_{\searrow} = \lambda_1 \lambda_2 (1 - q_1^2/\lambda_1 - q_2^2/\lambda_2)$.

ACKNOWLEDGMENTS

We thank Claes Waksjö and Krzysztof Marciniak for interesting discussions. The research of Stefan Rauch-Wojciechowski has been supported by NFR Grant No. M 5105-20005093/2000, which he gratefully acknowledges here.

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