Higher-dimensional integrable Newton systems with quadratic integrals of motion

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Abstract

Newton systems $\ddot{q} = M(q)$, $q \in R^n$, with integrals of motion quadratic in velocities, are considered. We show that if such a system admits two quadratic integrals of motion of so-called *cofactor type*, then it has in fact n quadratic integrals of motion and can be embedded into a $(2n+1)$-dimensional bi-Hamiltonian system, which under some non-degeneracy assumptions is completely integrable. The majority of these *cofactor pair* Newton systems are new, but they include also conservative systems with elliptic and parabolic separable potentials, as well as many integrable Newton systems previously derived from soliton equations. We explain the connection between cofactor pair systems and solutions of a certain system of second order linear PDEs (the *fundamental equations*), and use this to recursively construct infinite families of cofactor pair systems.

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1 Introduction

Conservative Newton systems, i.e., systems of differential equations of the form

$$\ddot{q} = -\nabla V(q), \tag{1.1}$$

are of fundamental importance in classical mechanics. Here $q = (q_1, \dots, q_n)^T$ are Cartesian coordinates on R^n, dots denote derivatives with respect to time t, and $\nabla = (\partial_1, \dots, \partial_n)^T$ is the gradient operator. (We use ∂_i, or sometimes ∂_{q_i}, as an abbreviation for $\partial/\partial q_i$, and X^T denotes the transpose of a matrix X. Thus, we regard elements of R^n as column vectors. We will only consider systems on R^n, not on general manifolds.) A large mathematical machinery has been built up for integrating such systems. We will here quickly review some well-known facts. For a system of the form (1.1), the energy $E = \frac{1}{2}\dot{q}^T\dot{q} + V(q)$ is always an integral of motion. There are the standard Lagrangian and Hamiltonian formulations. The system is called *completely integrable* if it has n Poisson commuting integrals of motion, in which case the Liouville–Arnol'd theorem says (among other things) that it can, in principle, be integrated by quadrature. A powerful method for finding solutions analytically is the Hamilton–Jacobi method, which is applicable if the potential V is such that the Hamilton–Jacobi equation $\frac{1}{2}\sum_1^n(\partial_i F(q))^2 + V(q) = E$ can be solved by separation of variables in some suitable coordinate system. In that case the potential is said to be *separable*, and the n integrals of motion of the system will all depend quadratically on the momenta $p_i = \dot{q}_i$. It is known through the work of many people, beginning with classical results by Stäckel, Levi-Civita and Eisenhart, that in R^n such separation can only occur in so-called generalized elliptic coordinates or some degeneration thereof. There exist criteria for determining if, and in that case in which system of coordinates, a given potential V is separable. For $n = 2$, the condition is that $V(q_1, q_2)$ must satisfy the Bertrand–Darboux equation [1, Sec. 152]

$$\begin{aligned} 0 =& (\alpha q_1 q_2 + \beta_1 q_2 + \beta_2 q_1 + \gamma_{12})(\partial_{22}V - \partial_{11}V) \\ &+ (\alpha(q_1^2 - q_2^2) + 2\beta_1 q_1 - 2\beta_2 q_2 + \gamma_{11} - \gamma_{22})\partial_{12}V \\ &- 3(\alpha q_2 + \beta_2)\partial_1 V + 3(\alpha q_1 + \beta_1)\partial_2 V \end{aligned} \tag{1.2}$$

for some constants α, β_1, β_2, γ_{12}, $\gamma_{11} - \gamma_{22}$, not all zero. Depending on the values of these parameters, the characteristic coordinates of the Bertrand–Darboux equation are either elliptic, polar, parabolic, or Cartesian coordinates, and this determines the coordinate system in which the Hamilton–Jacobi equation separates. The extra integral of motion is $F = (\alpha q_2^2 + 2\beta_2 q_2 + \gamma_{22})\dot{q}_1^2 - 2(\alpha q_1 q_2 + \beta_1 q_2 + \beta_2 q_1 + \gamma_{12})\dot{q}_1\dot{q}_2 + (\alpha q_1^2 + 2\beta_1 q_1 + \gamma_{11})\dot{q}_2^2 + k(q_1, q_2)$ for some function k. Similar results are known for $n > 2$. These will be described in section 7.

For general (nonconservative) Newton systems

$$\ddot{q} = M(q) \tag{1.3}$$

less is known. (In this article we use the term *Newton system* only for systems in which the right-hand side $M(q)$ does not depend on the velocity $\dot{q}$ or on time t.) In [2] we studied the class of systems of the form (1.3) which possess an "energy-like" integral of motion E which is quadratic in $\dot{q}_1, \dots, \dot{q}_n$. The theory originated from the following example.

Example 1.1 (Harry Dym stationary flow)**.** The system

$$\begin{aligned} \ddot{q}_1 &= \kappa q_1 - q_2/q_1^5, \\ \ddot{q}_2 &= 4\kappa q_2 - d, \end{aligned} \tag{1.4}$$

is equivalent, under the substitution $u = q_1^{-4}$, to the second stationary flow of the Harry Dym soliton hierarchy, and therefore it was suspected to be integrable in some sense. In addition to the integral of motion $F = \frac{1}{2}\dot{q}_2^2 - 2\kappa q_2^2 + dq_2$, which comes from the second equation alone, this system has another quadratic integral of motion

$$\begin{aligned} E =& - q_2\dot{q}_1^2 + q_1\dot{q}_1\dot{q}_2 - \kappa q_1^2 q_2 + \frac{q_2^2}{2q_1^4} + \frac{d}{2}q_1^2 \\ =& \begin{pmatrix} \dot{q}_1 & \dot{q}_2 \end{pmatrix} \begin{pmatrix} -q_2 & q_1/2 \\ q_1/2 & 0 \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} - \kappa q_1^2 q_2 + \frac{q_2^2}{2q_1^4} + \frac{d}{2}q_1^2 \\ =& \dot{q}^T A(q)\, \dot{q} + k(q). \end{aligned} \tag{1.5}$$

No Lagrangian formulation could be found for the system (1.4). However, it was discovered that it could be generated from E in a "quasi-Lagrangian" way by changing the minus sign in the Euler–Lagrange derivate δ to plus. Indeed, defining the quasi-Lagrangian operator $\delta^+ = (\delta_1^+, \dots, \delta_n^+)^T$ by

$$\delta_i^+ E = \frac{\partial E}{\partial q_i} + \frac{d}{dt}\frac{\partial E}{\partial \dot{q}_i}, \tag{1.6}$$

one finds immediately that the equation $0 = \delta^+ E$ yields

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \delta_1^+ E \\ \delta_2^+ E \end{pmatrix} = 2 \begin{pmatrix} -q_2 & q_1/2 \\ q_1/2 & 0 \end{pmatrix} \begin{pmatrix} \ddot{q}_1 - (\kappa q_1 - q_2/q_1^5) \\ \ddot{q}_2 - (4\kappa q_2 - d) \end{pmatrix},$$

which is clearly equivalent to (1.4). This proved to be a general feature of Newton systems with quadratic integrals of motion, so such systems were given the name *quasi-Lagrangian Newton systems*, or *QLN systems*. Expressed in terms of the matrix $A(q)$ and the function $k(q)$ in (1.5), the system (1.4) can be written

$$\ddot{q} = -\frac{1}{2}A(q)^{-1}\nabla k(q),$$

a result that also holds in general (see theorem 2.1 below). Clearly, this contains the conservative case (1.1) as the special case $A = I$ (identity matrix) and $k = 2V$.

The following nonstandard Hamiltonian formulation was found for the system (1.4):

$$\frac{d}{dt}\begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \\ d \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -q_1/2 & p_1 \\ & 0 & -q_1/2 & -q_2 & p_2 \\ & & 0 & p_1/2 & \kappa q_1 - q_2/q_1^5 \\ & * & & 0 & 4\kappa q_2 - d \\ & & & & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \Pi \overline{\nabla} d, \tag{1.7}$$

where the star denotes entries determined by antisymmetry of the matrix Π, and $\overline{\nabla} = (\partial_{q_1}, \partial_{q_2}, \partial_{p_1}, \partial_{p_2}, \partial_d)^T$ is the gradient operator on the extended phase space $\mathcal{M} = R^5$. The last column in the matrix Π equals the Hamiltonian vector field determined by the function $H(q, p, d) = d$, while the other entries are chosen so that $\{f, g\} = (\overline{\nabla} f)^T \Pi \overline{\nabla} g$ defines a Poisson bracket (in particular, so that the Jacobi identity is satisfied). The quadratic integral of motion E is a Casimir of Π, i.e., $\Pi \overline{\nabla} E = 0$.

The results for the system in example 1.1 gave rise to a general theory of two-dimensional QLN systems, developed in [2]. It was shown that they all admit a nonstandard Hamiltonian formulation similar to (1.7). In general, unlike in example 1.1, the parameter d which is used as an extra phase space variable is not present from the start, but has to be introduced by adding terms linear in d to the right-hand side of the original Newton system, which can be recovered as the restriction of the Hamiltonian system to the hyperplane $d = 0$.

Of special interest are the integrability properties of two-dimensional QLN systems with *two* functionally independent quadratic integrals of motion, say $E = \dot{q}^T A(q)\, \dot{q} + k(q)$ and $F = \dot{q}^T B(q)\, \dot{q} + l(q)$. It was shown that such a system can be embedded into a completely integrable bi-Hamiltonian system in extended phase space, in the sense that the trajectories of the extended system on the hyperplane $d = 0$ coincide with the trajectories of the original QLN system; however, they are in general traversed at a different speed. The reason for this extra complication is that some care has to be taken in order to ensure that both integrals of motion of the QLN system really give rise to corresponding integrals of motion of the extended system. The Poisson structures for the bi-Hamiltonian system are in general both non-canonical.

Both E and F can be used for generating the Newton system, which leads to the equality $\ddot{q} = -\frac{1}{2} A^{-1} \nabla k = -\frac{1}{2} B^{-1} \nabla l$. From $\nabla k = A B^{-1} \nabla l$ and the equality of mixed second order derivatives of k, one finds that l satisfies a certain second order linear PDE, whose coefficients depend on the entries of the matrices $A(q)$ and $B(q)$ (these entries are known to be quadratic polynomials of a certain form; see (3.1) below). Similarly one finds from $\nabla l = BA^{-1}\nabla k$ that k satisfies another PDE. A remarkable discovery made in [2] was that if one substitutes $k = K_1 \det A$ and $l = K_2 \det B$ in these equations, then K_1 and K_2 both satisfy *the same* second order linear PDE, which was named the *fundamental equation* associated with the matrices A and B. The coefficients in this equation are cubic polynomials in q, depending on the entries of A and B. It was shown that there is a one-to-one correspondence between fundamental equations and linear spans $\lambda A + \mu B$, which makes it possible to classify the types of systems that occur according to the polynomial degree of the matrices A and B. For example, when $B = I$ the fundamental equation reduces to the Bertrand–Darboux equation (1.2), which shows that this new class of system includes, but also significantly extends, the class of two-dimensional conservative systems with separable potentials. The fundamental equation was also used for constructing infinite families of integrable two-dimensional QLN system.

The aim of the present paper is to investigate what can be said in the n-dimensional case. In particular, we are interested in finding nonstandard Hamiltonian and bi-Hamiltonian formulations, similar to the ones in [2], which will allow us to show the integrability of (in general nonconservative) n-dimensional Newton systems with sufficiently many quadratic integrals of motions. The benefit of a Hamiltonian formulation is that only n integrals are needed, instead of $2n-1$ as in the general case. The rather unexpected result of our investigations is that even in n dimensions the existence of just *two* quadratic integrals of motion implies integrability, provided these integrals are of what we call *cofactor type*. Any Newton system with two such integrals of motion must in fact have n quadratic integrals of motion of a certain structure. Such systems are the principal objects of study in this paper, and we call them *cofactor pair systems*. We give a simple method of testing if a given Newton system is a cofactor pair system, and show how any cofactor pair system can be embedded in a bi-Hamiltonian system in $(2n+1)$-dimensional phase space. This bi-Hamiltonian system, whose Poisson structures are in general both non-canonical, is completely integrable under some mild non-degeneracy conditions, which explains in what sense cofactor pair systems can be considered integrable. We also find the analogue of the fundamental equation, which in this case is a system of $\binom{n}{2}$ second order linear PDEs, whose coefficients are cubic polynomials in q, and use this to recursively construct infinite families of cofactor pair systems.

This theory is richly illustrated by examples, and connects many different results obtained by other methods. In particular, we explain how n-dimensional separable potentials fit into this framework.

2 Quasi-Lagrangian Newton systems in n dimensions

In this section we review the basic facts about Newton systems with one or more quadratic integrals of motion. A characteristic feature of such a system is that it can easily be reconstructed from any of its quadratic integrals of motion $E = \dot{q}^T A(q)\, \dot{q} + k(q)$, either via the quasi-Lagrangian equations $\delta^+ E = 0$ or directly as $\ddot{q} = -\frac{1}{2} A^{-1} \nabla k$.

Theorem 2.1 (Newton systems with quadratic integrals)**.** *Let*

$$E(q, \dot{q}) = \dot{q}^T A(q)\, \dot{q} + k(q) = \sum_{i,j=1}^{n} A_{ij}(q)\, \dot{q}_i \dot{q}_j + k(q), \tag{2.1}$$

where $A^T = A$. Then E is an integral of motion for the Newton system $\ddot{q} = M(q)$ if and only if

$$\partial_i A_{jk} + \partial_j A_{ki} + \partial_k A_{ij} = 0, \quad \textit{for all } i,\ j,\ k = 1, \ldots, n, \tag{2.2}$$

and

$$2A(q)M(q) + \nabla k(q) = 0. \tag{2.3}$$

So if $\det A(q) \neq 0$, then the system can be reconstructed from its integral of motion E as

$$\ddot{q} = M(q) = -\frac{1}{2} A(q)^{-1} \nabla k(q), \tag{2.4}$$

which is equivalent to the system of quasi-Lagrangian equations $\delta^+ E = 0$ defined by (1.6)*.*

Proof. E is an integral of motion if and only if

$$\dot{E} = \sum_i (2(A\ddot{q})_i + \partial_i k)\dot{q}_i + \frac{1}{3} \sum_{i,j,k} (\partial_i A_{jk} + \partial_j A_{ki} + \partial_k A_{ij}) \dot{q}_i \dot{q}_j \dot{q}_k$$

vanishes identically, which proves the first statement. Moreover,

$$\delta_i^+ E = 2(A\ddot{q})_i + \partial_i k + \sum_{j,k} (\partial_i A_{jk} + \partial_j A_{ki} + \partial_k A_{ij}) \dot{q}_j \dot{q}_k,$$

so that $\delta^+ E = 2AM + \nabla k$ if (2.2) holds. □

Definition 2.2 (QLN system)**.** A Newton system of the form (2.4) in theorem 2.1, or, in other words, a Newton system with a quadratic integral of motion $E = \dot{q}^T A(q)\, \dot{q} + k(q)$ with $\det A \neq 0$, will be called a *quasi-Lagrangian Newton system*, or *QLN system*.

Definition 2.3 (Cyclic conditions)**.** The system (2.2) of linear first order PDEs $\partial_i A_{jk} + \partial_j A_{ki} + \partial_k A_{ij} = 0$ will be referred to as the *cyclic conditions* for the matrix $A(q)$.

Remark 2.4 (Killing tensor)**.** The cyclic conditions, with covariant instead of partial derivatives, are the equation for a second order Killing tensor on a Riemannian manifold (i.e., a tensor A_{ij} such that $A_{ij}\dot{q}^i\dot{q}^j$ is an integral of motion of the geodesic equations). Consequently, in our case we could speak of Killing tensors on R^n with the Euclidean metric. However, most of the time we will simply refer to "matrices satisfying the cyclic conditions."

Remark 2.5. That a Newton system $\ddot{q} = M(q)$ can be reconstructed from one of its integrals of motion was known already to Bertrand, whose method is not restricted to quadratic integrals [1, Sec. 151]. The quasi-Lagrangian formulation, however, was noticed only recently—it was first published in [3], and further theory was developed in [2]. It is at present unclear whether it has any geometric or similar significance, or if it is just an algebraic property. For example, unlike the ordinary Lagrange equations which admit arbitrary point transformations, the quasi-Lagrangian equations are only invariant under affine changes of variables. In any case, "QLN system" is a convenient designation for "velocity-independent Newton system, in general not conservative, with a nondegenerate quadratic integral of motion," and the notation $\delta^+ E = 0$ is also useful.

Remark 2.6. From theorem 2.1 it follows that if a Newton system $\ddot{q} = M(q)$ has two (or more) quadratic integrals of motion, say $E = \dot{q}^T A(q)\,\dot{q} + k(q)$ and $F = \dot{q}^T B(q)\,\dot{q} + l(q)$, then any of them can be used to reconstruct the system as long as the matrix is nonsingular. Thus,

$$M = -\frac{1}{2}A^{-1}\nabla k = -\frac{1}{2}B^{-1}\nabla l. \tag{2.5}$$

If B is singular but not A, then $B + \lambda A$ is nonsingular for some $\lambda \in R$, so we can replace F with $F + \lambda E$ to give

$$M = -\frac{1}{2}A^{-1}\nabla k = -\frac{1}{2}\big(B + \lambda A\big)^{-1}\nabla\big(l + \lambda k\big).$$

3 Matrices satisfying the cyclic conditions

A QLN system $\ddot{q} = -\frac{1}{2}A^{-1}\nabla k$ is completely determined by the matrix $A(q)$, which has to satisfy the cyclic conditions (2.2), and the arbitrary function $k(q)$, which plays the role of a "potential." In order to understand the class of QLN systems, it is essential to determine what a matrix A satisfying the cyclic conditions looks like. The general solution of the cyclic conditions is known (see for instance [4], which also gives results about Killing tensors on general manifolds). Since we will not use these results in full, we merely outline the general structure in proposition 3.1 below, which shows that the entries of A must be quadratic polynomials in q. The main purpose of this section is to introduce a special class of solutions that will be central in what follows: cofactor matrices of *elliptic coordinates matrices.*

Proposition 3.1. *If the symmetric matrix $A(q)$ satisfies the cyclic conditions* (2.2)*, then*

1. *For all i, j, k and l,*

$$\partial_{ij} A_{kl} = \partial_{kl} A_{ij} = const.$$

   In particular, each matrix entry $A_{ij}(q)$ is a polynomial of degree at most two.

2. *A_{ii} is independent of q_i for all i.*

3. *For $i \neq j$, A_{ij} contains no q_i^2 or q_j^2 terms.*

Proof. Taking $i = j = k$ the cyclic conditions read $3\partial_i A_{ii} = 0$, so A_{ii} does not depend on q_i. For $k = i \neq j$ we have $\partial_j A_{ii} + 2\,\partial_i A_{ij} = 0$, which shows that $\partial_i A_{ij}$ is independent of q_i. Thus, A_{ij} is linear in q_i and, by symmetry, in q_j. Finally, the stated relationship between the second derivatives follows from

$$\begin{aligned}
2(\partial_{kl} A_{ij} - \partial_{ij} A_{kl}) = &\,\partial_l(\partial_k A_{ij} + \partial_i A_{jk} + \partial_j A_{ki}) \\
&+ \partial_k(\partial_l A_{ij} + \partial_i A_{jl} + \partial_j A_{li}) \\
&- \partial_i(\partial_j A_{kl} + \partial_k A_{lj} + \partial_l A_{jk}) \\
&- \partial_j(\partial_i A_{kl} + \partial_k A_{li} + \partial_l A_{ik}) \\
= &\,0,
\end{aligned}$$

and they are constant since

$$\begin{aligned}
3\partial_{klm} A_{ij} &= \partial_{kl}\partial_m A_{ij} + \partial_{lm}\partial_k A_{ij} + \partial_{km}\partial_l A_{ij} \\
&= \partial_{kl}(-\partial_i A_{mj} - \partial_j A_{im}) + \partial_{lm}(-\partial_i A_{kj} - \partial_j A_{ik}) \\
&\quad + \partial_{km}(-\partial_i A_{lj} - \partial_j A_{il}) \\
&= -\frac{1}{2}\Big[\partial_{ik}(\partial_l A_{mj} + \partial_m A_{lj}) + \partial_{il}(\partial_k A_{mj} + \partial_m A_{kj}) \\
&\quad + \partial_{kj}(\partial_l A_{im} + \partial_m A_{il}) + \partial_{lj}(\partial_k A_{im} + \partial_m A_{ik}) \\
&\quad + \partial_{mi}(\partial_l A_{kj} + \partial_k A_{lj}) + \partial_{mj}(\partial_l A_{ik} + \partial_k A_{il})\Big] \\
&= \frac{1}{2}\Big[\partial_{ik}\partial_j A_{lm} + \partial_{il}\partial_j A_{km} + \partial_{kj}\partial_i A_{ml} \\
&\quad + \partial_{lj}\partial_i A_{km} + \partial_{mi}\partial_j A_{lk} + \partial_{mj}\partial_i A_{lk}\Big] \\
&= \partial_{ij}(\partial_k A_{lm} + \partial_l A_{mk} + \partial_m A_{kl}) \\
&= 0.
\end{aligned}$$

$\square$

With the help of these facts it is possible to find the general solution of (2.2) for any given n. For $n = 2$, it is

$$\begin{aligned} A &= \begin{pmatrix} \alpha q_2^2 + 2\beta_2 q_2 + \gamma_{22} & -(\alpha q_1 q_2 + \beta_1 q_2 + \beta_2 q_1 + \gamma_{12}) \\ -(\alpha q_1 q_2 + \beta_1 q_2 + \beta_2 q_1 + \gamma_{12}) & \alpha q_1^2 + 2\beta_1 q_1 + \gamma_{11} \end{pmatrix} \\ &= \operatorname{cof} \begin{pmatrix} \alpha q_1^2 + 2\beta_1 q_1 + \gamma_{11} & \alpha q_1 q_2 + \beta_1 q_2 + \beta_2 q_1 + \gamma_{12} \\ \alpha q_1 q_2 + \beta_1 q_2 + \beta_2 q_1 + \gamma_{12} & \alpha q_2^2 + 2\beta_2 q_2 + \gamma_{22} \end{pmatrix}, \end{aligned} \tag{3.1}$$

which depends on the six parameters α, β_1, β_2, γ_{11}, γ_{12} and γ_{22}. The choice of notation will be made clear below; see in particular remark 3.9. For $n = 3$ the general solution already involves 20 parameters, and in general the number of parameters is $n(n+1)^2(n+2)/12$ [4].

Now we will focus on some special types of solutions of the cyclic conditions. First, we note that there is the following simple method of producing new solutions from a given one.

Proposition 3.2 (Change of variables)**.** *If $A(q)$ satisfies the cyclic conditions* (2.2)*, then so does $S^T A(Sq + v)S$, for any constant matrix $S \in R^{n\times n}$ and vector $v \in R^n$.*

Proof. This is easily verified directly using the chain rule. Alternatively, one can first verify that the quasi-Lagrangian equations are invariant under affine changes of variables $q = Sr + v$, which means that the Newton system $\delta^+ E(q, \dot{q}) = 0$ expressed in the new variables r is the Newton system generated by the integral of motion $E = \dot{q}^T A(q)\,\dot{q} + k(q)$ when expressed in r and $\dot{r}$:

$$E(r, \dot{r}) = (S\dot{r})^T A(Sr + v)\,(S\dot{r}) + k(Sr + v).$$

Thus, by theorem 2.1, $S^T A(Sr + v)S$ must satisfy the cyclic conditions (expressed in the r variables). $\square$

There is a class of matrices satisfying the cyclic conditions, that will be very important in what follows: cofactor matrices of *elliptic coordinates matrices.* Let us remind the reader that the cofactor (or adjoint) matrix $\operatorname{cof} X$ of a quadratic matrix X is the matrix whose (i, j) entry is the cofactor of X_{ji} in $\det X$, so that $X \operatorname{cof} X = (\det X)I$. Elliptic coordinates matrices and their cofactor matrices appear in a natural way when trying to find a Hamiltonian formulation for QLN systems, as will be seen in theorem 4.1 in the next section.

Definition 3.3 (Elliptic coordinates matrix)**.** A symmetric $n \times n$-matrix $G(q)$ whose entries are quadratic polynomials in q of the form

$$G_{ij}(q) = \alpha q_i q_j + \beta_i q_j + \beta_j q_i + \gamma_{ij} \tag{3.2}$$

will be called an *elliptic coordinates matrix*. Using matrix multiplication, $G(q)$ can be written

$$G(q) = \alpha q q^T + q\beta^T + \beta q^T + \gamma, \quad \text{where } \alpha \in R,\ \beta \in R^n,\ \gamma = \gamma^T \in R^{n\times n}.$$

(Let us emphasize, for clarity, that we consider elements in R^n as column vectors. Thus, qq^T is an $n \times n$-matrix, not to be confused with the scalar $q^T q = \sum q_i^2$.)

The reason for the terminology is that the eigenvalues $u_1(q), \dots, u_n(q)$ of an elliptic coordinates matrix (under some assumptions) determine a change of variables from Cartesian coordinates q to generalized elliptic coordinates u, which will be of interest when discussing separable potentials (see section 7). For the moment, we are only interested in the following remarkable property of such matrices:

Theorem 3.4 (Cofactor matrix)**.** *If $G(q)$ is an elliptic coordinates matrix, then $A(q) = \operatorname{cof} G(q)$ satisfies the cyclic conditions* (2.2)*.*

Proof. To begin with, we note that A is symmetric, since G is symmetric. Now let $N(q) = \alpha q + \beta$. Differentiating G we find, using the Kronecker delta notation,

$$\partial_k G_{ij} = \alpha(\delta_{ki} q_j + q_i \delta_{kj}) + \beta_i \delta_{kj} + \beta_j \delta_{ki} = \delta_{ki} N_j + \delta_{kj} N_i, \tag{3.3}$$

or, in matrix notation,

$$\partial_k G = (N e_k^T + e_k N^T), \tag{3.4}$$

where $e_k = (0, \dots, 1, \dots, 0)^T$ is the k'th standard basis vector of R^n.

Next, we show show that

$$\nabla \det G = 2AN, \tag{3.5}$$

a formula that also will be useful elsewhere in this article. For ease of notation, let us show the case $n = 3$:

$$\begin{aligned}
\partial_1 \begin{vmatrix} G_{11} & G_{12} & G_{13} \\ G_{12} & G_{22} & G_{23} \\ G_{13} & G_{23} & G_{33} \end{vmatrix} = \begin{vmatrix} 2N_1 & G_{12} & G_{13} \\ N_2 & G_{22} & G_{23} \\ N_3 & G_{23} & G_{33} \end{vmatrix} + \begin{vmatrix} G_{11} & N_2 & G_{13} \\ G_{12} & 0 & G_{23} \\ G_{13} & 0 & G_{33} \end{vmatrix} + \begin{vmatrix} G_{11} & G_{12} & N_3 \\ G_{12} & G_{22} & 0 \\ G_{13} & G_{23} & 0 \end{vmatrix} \\
&= (2N_1 A_{11} + N_2 A_{12} + N_3 A_{13}) + N_2 A_{12} + N_3 A_{13} \\
&= 2[AN]_1,
\end{aligned}$$

and similarly for the other ∂_k. The notation $[AN]_1$ means, of course, the first entry in the column vector AN. It is obvious that a similar calculation can be made for any n, which proves (3.5).

Thus, differentiating the identity $AG = (\det G)I$, we obtain

$$(\partial_k A)G + A(Ne_k^T + e_k N^T) = 2[AN]_k I.$$

After multiplying this by A from the right, we extract from the (i, j) entry

$$(\det G)\partial_k A_{ij} = 2[AN]_k A_{ij} - [AN]_i A_{kj} - [AN]_j A_{ik}. \tag{3.6}$$

Summing cyclically we obtain

$$(\det G)(\partial_i A_{jk} + \partial_j A_{ki} + \partial_k A_{ij}) = 0.$$

The left-hand side of this equation is a polynomial in q, whose coefficients are polynomials in the parameters α, β_r and γ_{rs}, while the right-hand side vanishes identically. Since $\det G$ is not identically zero as a function of these parameters, as can be seen by taking $\alpha = 0$, $\beta = 0$ and γ nonsingular, we conclude that the sum in parentheses must vanish identically. In other words, A satisfies the cyclic conditions for any values of α, β and γ (even such values that make $\det G = 0$). □

Remark 3.5. This theorem implies, by proposition 3.1, that the cofactors of $G(q)$ are polynomials in q of degree at most two. This is a rather surprising fact, since one could expect them to have degree 2^{n-1}, being determinants of $(n-1)\times(n-1)$-matrices of quadratic polynomials. What happens is that all the terms of degree higher than two cancel due to the special structure of G. Similarly, since $\det G$ is the cofactor of the lower right entry in an elliptic coordinates matrix of size $(n+1)\times(n+1)$, it must also be a quadratic polynomial. However, checking this by direct calculation is already for $n = 3$ a quite formidable task!

We can use theorem 3.4 to produce a "cofactor chain" of matrices satisfying the cyclic conditions. Such chains will be very useful later on.

Proposition 3.6. *Let* $G(q) = \alpha qq^T + q\beta^T + \beta q^T + \gamma$ *and* $\widetilde{G}(q) = \widetilde{\alpha} qq^T + q\widetilde{\beta}^T + \widetilde{\beta} q^T + \widetilde{\gamma}$ *be elliptic coordinates matrices. Then the matrices* $A^{(0)}, \ldots, A^{(n-1)}$ *defined by*

$$A_\mu = \operatorname{cof}(G + \mu\widetilde{G}) = \sum_{i=0}^{n-1} A^{(i)}\mu^i \tag{3.7}$$

all satisfy the cyclic conditions (2.2).

Proof. $G + \mu\widetilde{G}$ is an elliptic coordinates matrix, with $\alpha + \mu\widetilde{\alpha}$ instead of α and so on. By theorem 3.4, A_μ satisfies the cyclic conditions for all μ. These being linear equations, it follows that the coefficients at different powers of μ in A_μ each must satisfy the cyclic conditions. □

Remark 3.7. Note that $A^{(0)} = \operatorname{cof} G$ and $A^{(n-1)} = \operatorname{cof} \widetilde{G}$, but that the interjacent matrices $A^{(1)}, \dots, A^{(n-2)}$ in general are impossible to write as cofactor matrices of elliptic coordinates matrices.

Obviously, we can obtain even larger variation if we form linear combinations of more than two elliptic coordinates matrices. For example, in $\operatorname{cof}(G + \mu G' + \lambda G'')$ the coefficient at each different power $\mu^i \lambda^j$ will satisfy the cyclic conditions. However, we have not found any particular use for this. Combinations of two matrices, on the other hand, are absolutely fundamental for the construction of integrable Newton systems in section 5, as the following example indicates.

Example 3.8 (KdV stationary flow)**.** Define elliptic coordinates matrices G and $\widetilde{G}$ by

$$\alpha = 0,\ \beta = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},\gamma = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad \widetilde{\alpha} = 0,\ \widetilde{\beta} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},\widetilde{\gamma} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Then

$$\begin{aligned}
\operatorname{cof}(G + \mu\widetilde{G}) &= \operatorname{cof} \left[\begin{pmatrix} 0 & -1 & q_1 \\ -1 & 0 & q_2 \\ q_1 & q_2 & 2q_3 \end{pmatrix} + \mu \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right] \\
&= \begin{pmatrix} -q_2^2 & q_1 q_2 + 2q_3 & -q_2 \\ q_1 q_2 + 2q_3 & -q_1^2 & -q_1 \\ -q_2 & -q_1 & -1 \end{pmatrix} \\
&\quad + \mu \begin{pmatrix} 2q_3 & q_2 & -q_1 \\ q_2 & -2q_1 & -1 \\ -q_1 & -1 & 0 \end{pmatrix} + \mu^2 \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \\
&= A^{(0)} + \mu A^{(1)} + \mu^2 A^{(2)}.
\end{aligned}$$

According to proposition 3.6, the matrices $A^{(0)}$, $A^{(1)}$ and $A^{(2)}$ so defined all satisfy the cyclic conditions, and this is also easily verified directly. They occur in the integrals of motion of the Newton system

$$\begin{aligned}
\ddot{q}_1 &= -10q_1^2 + 4q_2, \\
\ddot{q}_2 &= -16q_1 q_2 + 10q_1^3 + 4q_3, \\
\ddot{q}_3 &= -20q_1 q_3 - 8q_2^2 + 30q_1^2 q_2 - 15q_1^4 + d,
\end{aligned} \tag{3.8}$$

which, under the substitution $q_1 = u/4$, is equivalent to the integrated form

$$\frac{1}{64}(u_{6x} + 14uu_{4x} + 28u_x u_{xxx} + 21u_{xx}^2 + 70uu_x^2 + 70u^2 u_{xx} + 35u^4) = d$$

of the seventh order stationary KdV flow [5]. Indeed, this system has three quadratic integrals of motion of the form $E^{(i)} = \dot{q}^T A^{(i)}\,\dot{q} + k^{(i)} - d\,D^{(i)}$, $i = 0, 1, 2$, where

$$\begin{aligned} k^{(0)} =& 24q_1^3q_2^2 - 8q_1q_2^3 - 10q_1^5q_2 - 16q_1q_3^2 - 10q_1^4q_3 \\ &+ 8q_1^2q_2q_3 - 8q_2^2q_3, \\ k^{(1)} =& 8q_1^2q_2^2 + 10q_1^4q_2 - 5q_1^6 - 8q_2^3 + 4q_3^2 - 24q_1q_2q_3, \\ k^{(2)} =& -20q_1^2q_3 + 8q_2q_3 - 16q_1q_2^2 + 20q_1^3q_2 - 6q_1^5, \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} D^{(0)} &= -2(q_1q_2 + q_3), \\ D^{(1)} &= -(q_1^2 + 2q_2), \\ D^{(2)} &= -2q_1. \end{aligned}$$

The system can be reconstructed from any one of these integrals as

$$\ddot{q} = -\frac{1}{2}[A^{(i)}]^{-1}\nabla(k^{(i)} - d\,D^{(i)}).$$

It was shown in [5] to be bi-Hamiltonian and completely integrable. The parameter d was used as an extra phase space variable in the bi-Hamiltonian formulation. Notice that the $D^{(i)}$ occur as coefficients in

$$\det(G + \mu\widetilde{G}) = -2(q_1q_2 + q_3) - (q_1^2 + 2q_2)\mu - 2q_1\mu^2 - \mu^3.$$

All this fits nicely into the general scheme to be developed in section 5, where we construct a large class of bi-Hamiltonian Newton systems containing this one as a special case. (In this particular example, the matrix $\widetilde{G}$ happens to be independent of q. This will not be the case in general.)

Remark 3.9. For $n = 2$ every matrix satisfying the cyclic conditions is the cofactor matrix of an elliptic coordinates matrix, as equation (3.1) shows. For $n > 2$ this is not the case, as we have already noticed in remark 3.7. As another example, a matrix with the block structure

$$A = \begin{pmatrix} \operatorname{cof} G_1(q_1, \dots, q_r) & 0 \\ 0 & \operatorname{cof} G_2(q_{r+1}, \dots, q_n) \end{pmatrix},$$

where G_1 and G_2 are elliptic coordinate matrices of smaller dimensions, satisfies the cyclic conditions but cannot in general be written as the cofactor matrix of a single elliptic coordinates matrix. Applying proposition 3.2 we can obtain matrices for which the same is true, but without the blocks of zeros immediately revealing them as "decomposable."

An interesting open problem is how to detect whether the reverse process is possible, i.e., if a given solution A of the cyclic conditions can be transformed, by changing variables according to proposition 3.2, into such a decomposable form with "cofactor blocks" along the diagonal and zeros elsewhere. If in that case $k(q) = k_1(q_1, \dots, q_r) + k_2(q_{r+1}, \dots, q_n)$ in the new variables, then the QLN system $0 = \delta^+(\dot{q}^T A(q)\, \dot{q} + k(q))$ splits into the direct sum of two smaller QLN systems, one for $q_1, \dots, q_r$ and one for $q_{r+1}, \dots, q_n$, to which the theory that we will develop for "cofactor systems" can be applied separately.

4 Hamiltonian formulation and cofactor systems

Now we turn to the question of integrability of QLN systems. The notion of complete integrability concerns *Hamiltonian systems*. If one has a Hamiltonian formulation for some system under study, then the task of showing the system's integrability is just a matter of finding sufficiently many Poisson commuting integrals of motion. In this section, we present a (nonstandard) Hamiltonian formulation for a certain class of QLN systems, the *cofactor systems*.

Recall that a *Poisson manifold* is a manifold endowed with *Poisson bracket*, i.e., a bilinear antisymmetric mapping $\{\cdot,\cdot\} : C^\infty(\mathcal{M}) \times C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ which satisfies the Leibniz rule and the Jacobi identity. In coordinates $(x_1, \dots, x_n)$ the bracket takes the form

$$\{f, g\}(x) = (\overline{\nabla} f(x))^T \Pi(x) \overline{\nabla} g(x)$$

for some antisymmetric *Poisson matrix* $\Pi(x)$, where $\overline{\nabla} = (\partial_{x_1}, \dots, \partial_{x_n})^T$. A *Hamiltonian system* on $\mathcal{M}$ is a dynamical system of the form $\dot{x}_i = \{x_i, H\}$, or $\dot{x} = \Pi\overline{\nabla}H$, for some function $H(x)$.

Conservative Newton systems $\ddot{q} = -\nabla V(q)$ on R^n admit the standard Hamiltonian formulation

$$\frac{d}{dt}\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \overline{\nabla} H, \quad \text{where } H(q,p) = \frac{1}{2} p^T p + V(q),$$

on the phase space $\mathcal{M} = R^{2n}$ with coordinates (q, p). In this section, the manifold $\mathcal{M}$ will be R^{2n+1} with coordinates (q, p, d) and we will investigate the possibility of finding a nonstandard Hamiltonian formulation for some nonconservative Newton systems $\ddot{q} = M(q)$. The idea is that several known nonstandard Hamiltonian formulations of integrable Newton systems derived from soliton equations [5, 6] or bi-Hamiltonian formulations for systems with separable potentials [3, 7], involve Poisson matrices on R^{2n+1} with a certain block structure (see (4.1) below). We investigate what the most general form of a Poisson matrix with this structure is. The answer leads us to define *cofactor systems*, which are just the systems which admit this type of Hamiltonian formulation. The previously known systems are special instances of this class. Our results generalize the ones found using similar methods in [2, 3] for the cases $n = 2$ or $M = -\nabla V$.

Theorem 4.1 (Poisson matrix). *Let $\mathcal{M}$ denote the space R^{n+n+1} with coordinates (q,p,d). Let Π be an antisymmetric $(n+n+1)\times(n+n+1)$-matrix with the block structure*

$$\Pi = \begin{pmatrix} 0 & \frac{\lambda}{2}G(q) & p \\ * & \frac{\lambda}{2}F(q,p) & \widehat{M}(q,d) \\ * & * & 0 \end{pmatrix}, \tag{4.1}$$

where F and G are $n\times n$-matrices, p and $\widehat{M}$ column n-vectors, λ a nonzero real parameter (introduced for later convenience) and stars denote elements determined by antisymmetry. Then Π is a Poisson matrix if and only if:

1. *G is an elliptic coordinates matrix, i.e.,*

$$G(q) = \alpha qq^T + q\beta^T + \beta q^T + \gamma \tag{4.2}$$

   for some $\alpha\in R$, $\beta\in R^n$ and $\gamma=\gamma^T\in R^{n\times n}$.

2. *F is given by*

$$F(q,p) = Np^T - pN^T, \quad \text{where} \quad N(q) = \alpha q + \beta. \tag{4.3}$$

3. *$\widehat{M}$ has the structure*

$$\widehat{M}(q,d) = M(q) + \lambda d\, N(q), \tag{4.4}$$

   where $M(q)$ satisfies the equations

$$0 = P_{ij} - P_{ji}, \quad \text{where } P_{ij} = 3N_iM_j + \sum_{k=1}^{n} G_{ki}\partial_k M_j, \tag{4.5}$$

   for all $i,j=1,\dots,n$. If $\det G(q)\neq 0$, this is equivalent to

$$M = -\frac{1}{2}A^{-1}\nabla k, \quad \text{for some function } k(q), \tag{4.6}$$

   where $A(q)=\operatorname{cof} G(q)$. In other words, $\ddot{q}=M(q)$ is the QLN system generated by $E=\dot{q}^T A(q)\,\dot{q}+k(q)=\dot{q}^T\operatorname{cof} G(q)\,\dot{q}+k(q)$.

Moreover, then the function

$$\widehat{E}(q,p,d) = p^T\operatorname{cof} G(q)\,p + k(q) - \lambda d\det G(q) \tag{4.7}$$

is a Casimir of Π, i.e.,

$$\Pi\,\overline{\nabla}E = 0,$$

where $\overline{\nabla} = (\partial_{q_1},\dots,\partial_{q_n},\partial_{p_1},\dots,\partial_{p_n},\partial_d)^T$.

Proof. First of all, F must be antisymmetric in order for Π to be so. Then we must determine what form F, G and $\widehat{M}$ must take in order for the Jacobi identity to be satisfied for all combinations of the coordinates q, p, d. Let us use the abbreviation $J(f,g,h) = \{\{f,g\},h\} + \{\{h,f\},g\} + \{\{g,h\},f\}$. We find that $J(q_i,q_j,q_k)$ and $J(q_i,q_j,p_k)$ are both identically zero for all i, j, k, while $J(q_i,q_j,d) = \frac{\lambda}{2}(G_{ij} - G_{ji})$, which implies that the matrix $G(q)$ must be symmetric. Further, $J(q_i,p_j,d) = \frac{\lambda}{2}(p^T\nabla G_{ij} - F_{ij}) - p_i\partial_d\widehat{M}_j$, which shows that $\partial_d\widehat{M}(q,d)$ is independent of d, and thus $\widehat{M}(q,d) = M(q) + \lambda d N(q)$ for some $M(q)$ and $N(q)$. ($\nabla = (\partial_{q_1},\dots,\partial_{q_n})^T$, as usual, and λ is introduced here for convenience.) With this expression for $\widehat{M}$ we obtain $0 = p^T\nabla G_{ij} - F_{ij} - 2p_iN_j$. Adding and subtracting this expression and the corresponding one with i and j interchanged, and using $G = G^T$, $F = -F^T$, we obtain

$$F_{ij} = N_ip_j - N_jp_i,$$

which is (4.3) except that we do not know the form of N yet, and

$$p^T\nabla G_{ij} = N_ip_j + N_jp_i.$$

Let us now go back to writing ∂_k instead of ∂_{q_k}, since only derivatives with respect to the q variables remain. Taking $i = j$ we see that $N_i = \frac{1}{2}\partial_iG_{ii}$ and that G_{ii} and N_i must depend on q_i only. For i, j and k different, we obtain $\partial_iG_{ij} = N_j(q_j)$, $\partial_jG_{ij} = N_i(q_i)$, and $\partial_kG_{ij} = 0$. Since mixed derivatives are equal, this gives $\partial_iN_i(q_i) = \partial_jN_j(q_j)$ for all i, j, and so $\partial_1N_1 = \dots = \partial_nN_n = \alpha$ for some constant α. This shows that $N_i = \alpha q_i + \beta_i$, from which it follows that $G_{ij} = \alpha q_iq_j + \beta_iq_j + \beta_jq_i + \gamma_{ij}$. We have now established (4.2) and (4.3). With F, G and N given by these formulas it is easy to check that $J(q_i,p_j,p_k)$ and $J(p_i,p_j,p_k)$ vanish identically. For the only remaining condition, we obtain $J(p_i,p_j,d) = \frac{\lambda}{2}(P_{ij} - P_{ji})$, from which (4.5) follows. When $\det G \neq 0$, the equations $0 = P_{ij} - P_{ji}$ are equivalent, through the forming of suitable linear combinations, to the equations $0 = \partial_i[AM]_j - \partial_j[AM]_i$, where $A(q) = \operatorname{cof} G(q)$ (the proof of this is slightly technical and has therefore been relegated to the appendix). It follows that there is a function $k(q)$ such that $AM = -\frac{1}{2}\nabla k$. We have now completely determined the structure of the Poisson matrix Π. Recall from theorem 3.4 that $A = \operatorname{cof} G$ satisfies the cyclic conditions, so that $M = -\frac{1}{2}A^{-1}\nabla k$ really is a QLN system.

It remains to verify that the function $\widehat{E}$ given by (4.7) is a Casimir of Π. One needs to use the facts that $GA = (\det G)I$ and $G\nabla(p^TAp) = 2FAp$. The latter equality is established as follows. By theorem 3.4, A satisfies the cyclic conditions, so that $\partial_kA_{ij} = -\partial_iA_{kj} - \partial_jA_{ki}$. Thus, using (3.3) and

(3.5), we obtain

$$
\begin{aligned}
[G\nabla(p^TAp)]_a =& \sum_k G_{ak}\,\partial_k\left(\sum_{i,j} A_{ij}p_ip_j\right) = -2\sum_{i,j,k} G_{ak}(\partial_i A_{kj})p_ip_j \\
=& -2\sum_{i,j,k}\bigl(\partial_i(G_{ak}A_{kj}) - (\delta_{ia}N_k + \delta_{ik}N_a)A_{kj}\bigr)p_ip_j \\
=& -2\left(\sum_{i,j}\partial_i(\delta_{aj}\det G)p_ip_j - \sum_{j,k} N_kA_{jk}p_ap_j - N_a\sum_{i,j}A_{ij}p_ip_j\right) \\
=& 2\sum_{k,m}(N_ap_k - N_kp_a)A_{km}p_m = 2[FAp]_a.
\end{aligned}
$$

Knowing this, the result $\Pi\overline{\nabla}\widehat{E} = 0$ follows from a relatively straightforward calculation which we omit here. □

Remark 4.2. If we assume from the outset that $M(q) = -\nabla V(q)$, as was done in [3], then (4.5) takes the form

$$
0 = \sum_{r=1}^{n}\bigl(G_{ir}\,\partial_{rj}V - G_{jr}\,\partial_{ri}V\bigr) + 3\bigl(N_i\partial_jV - N_j\partial_iV\bigr). \tag{4.8}
$$

As pointed out in [3], this system of equations has been found before as a criterion for the separability of the potential V. We will return to this in section 7.

We need a name for the type of QLN systems occurring in theorem 4.1.

Definition 4.3 (Cofactor system)**.** A QLN system $\delta^+E = 0$ generated by $E = \dot{q}^TA\,\dot{q} + k$, where A is the cofactor matrix of a nonsingular elliptic coordinates matrix, i.e.,

$$
A(q) = \operatorname{cof} G(q), \quad G(q) = \alpha qq^T + q\beta^T + \beta q^T + \gamma, \quad \det G(q) \neq 0,
$$

will be called a *cofactor system*, and E an integral of motion of *cofactor type.*

In two dimensions any QLN system is a cofactor system, by remark 3.9. Theorem 4.1 leads immediately to a Hamiltonian formulation for cofactor systems:

Theorem 4.4 (Hamiltonian formulation)**.** *Let $\ddot{q} = M(q)$ be a cofactor system, generated by $E = \dot{q}^TA(q)\,\dot{q} + k(q)$, with $A = \operatorname{cof} G$. Then, using the notation of theorem 4.1, there is on the extended phase space $\mathcal{M} = R^{2n+1}$ with coordinates (q, p, d) a related Hamiltonian system*

$$
\frac{d}{dt}\begin{pmatrix} q \\ p \\ d \end{pmatrix} = \begin{pmatrix} p \\ M(q) + \lambda d\,N(q) \\ 0 \end{pmatrix} = \Pi\,\overline{\nabla}d, \tag{4.9}
$$

whose motion on the hyperplane $d = 0$ coincides with the motion of the original system $\ddot{q} = M(q)$ in $(q, \dot{q} = p)$-space.

Proof. Since $M = -\frac{1}{2}A^{-1}\nabla k$ by theorem 2.1, all the conditions of theorem 4.1 are satisfied. Thus, Π is a Poisson matrix and the system is Hamiltonian. Trajectories with initial values in the hyperplane $d = 0$ remain there, since $\dot{d} = 0$. The motion

$$\frac{d}{dt}\begin{pmatrix} q \\ p \\ d \end{pmatrix} = \begin{pmatrix} p \\ M(q) \\ 0 \end{pmatrix}$$

in that hyperplane is clearly equivalent to $\ddot{q} = M(q)$. □

Remark 4.5. The restriction of the extended system (4.9) to any hyperplane of constant d (not necessarily $d = 0$) is equivalent to the Newton system $\ddot{q} = M(q) + \lambda d\, N(q)$, which is just the QLN system generated by $\widehat{E}(q, \dot{q}, d) = \dot{q}^T A\, \dot{q} + k(q) - \lambda d \det G(q)$, since $-\frac{1}{2}A^{-1}\nabla\big(k(q) - \lambda d \det G(q)\big) = M + \lambda d\, N$, by (3.5). Here we can view d just as a parameter in $\widehat{E}$, which is indeed how it first turns up in integrable Newton systems derived from soliton theory. In that context, d is typically an integration constant appearing when integrating the stationary flow of some soliton PDE. See for instance [5, 6] and example 3.8.

Remark 4.6. We have shown that one integral of motion of cofactor type is enough for a Newton system $\ddot{q} = M(q)$ to admit a certain type of Hamiltonian formulation, but it is of course not enough to guarantee integrability of any kind. If the extended system (4.9) admits $n-1$ functionally independent Poisson commuting extra integrals of motion in addition to the Casimir $\widehat{E}$ and the Hamiltonian d, then it is completely integrable. Indeed, the restriction of the system to any level surface of $\widehat{E}$ is a Hamiltonian system [8, Prop. 6.19], which is symplectic, since Π obviously has rank $2n$ if $\det G \neq 0$, and has n commuting integrals of motion. Since the original Newton system $\ddot{q} = M(q)$ is obtained by restriction to the hyperplane $d = 0$, it can in this case be considered as completely integrable too. For instance, the system in example 3.8 falls into this category; setting $p = \dot{q}$ it is actually the "extended system," while what we have called here the "original system" corresponds to the case $d = 0$.

This, however, does not mean that any cofactor system with n integrals $E(q, \dot{q})$, $F_2(q, \dot{q})$, ..., $F_n(q, \dot{q})$ must be integrable in this sense, because it may not be possible to incorporate d-dependence into the F_i to even make them integrals $\widehat{F}_i(q, p, d)$ of the extended system, not to mention that the $\widehat{F}_i$ have to Poisson commute. In the next section, we will see how it is possible to overcome this difficulty for the class of *cofactor pair* systems, i.e., systems with *two* integrals of motion of cofactor type, by using a slightly different (bi-Hamiltonian) extended system. The system in example 3.8 is in fact a cofactor pair system, but of a rather special kind ($\widetilde{G}$ is constant), which is why already the theory in this section is sufficient for proving its integrability. (Actually, one can get by with even less. That system has a Lagrangian with indefinite kinetic energy $\dot{q}_1\dot{q}_3 + \frac{1}{2}\dot{q}_2^2$, so when introducing momenta $s_1 = \dot{q}_3$, $s_2 = \dot{q}_2$, $s_3 = \dot{q}_1$ as was done in [5], one obtains a canonical Hamiltonian formulation.)

5 Bi-Hamiltonian formulation and cofactor pair systems

This section forms the central part of the paper. We show that *cofactor pair systems*, i.e., QLN systems with *two* independent integrals of motion of cofactor type, automatically must have n quadratic integrals of motion, and that they under some non-degeneracy assumptions can be considered as completely integrable via embedding into bi-Hamiltonian completely integrable systems in $(2n+1)$-dimensional phase space. The rest of the article is then devoted to the explicit construction of cofactor pair systems in large numbers, and to showing that many known integrable Newton systems from the literature, in particular conservative systems with separable potentials, fit into this framework as special cases. However, the main part of the class of cofactor pair systems seems not to have been considered before.

We now show how the results from the previous section lead naturally to the concept of a cofactor pair system. The matrix Π in theorem 4.1 depends linearly on the parameters α, β, γ in the G, N and F blocks. In order to construct a pencil of compatible Poisson matrices, let these parameters in turn depend linearly on a variable μ:

$$\begin{aligned} \alpha_\mu &= \alpha + \mu\widetilde{\alpha}, \\ \beta_\mu &= \beta + \mu\widetilde{\beta}, \\ \gamma_\mu &= \gamma + \mu\widetilde{\gamma}, \end{aligned} \tag{5.1}$$

where α, β, γ and $\widetilde{\alpha}$, $\widetilde{\beta}$, $\widetilde{\gamma}$ are two separate sets of parameters. Then the corresponding G_μ, N_μ and F_μ also depend linearly on μ:

$$\begin{aligned} G_\mu &= \alpha_\mu qq^T + q\beta_\mu^T + \beta_\mu q^T + \gamma_\mu &&= G + \mu\widetilde{G}, \\ N_\mu &= \alpha_\mu q + \beta_\mu &&= N + \mu\widetilde{N}, \\ F_\mu &= N_\mu p^T - pN_\mu^T &&= F + \mu\widetilde{F}, \end{aligned} \tag{5.2}$$

where, for instance, $N = \alpha q + \beta$ and $\widetilde{N} = \widetilde{\alpha}q + \widetilde{\beta}$, and similarly for G and F. On the other hand, $A_\mu = \operatorname{cof} G_\mu$ is a polynomial in μ of degree $n-1$:

$$A_\mu = \operatorname{cof}(G + \mu\widetilde{G}) = \sum_{i=0}^{n-1} A^{(i)}\mu^i. \tag{5.3}$$

Note that $A^{(0)} = \operatorname{cof} G$ and $A^{(n-1)} = \operatorname{cof} \widetilde{G}$, that G and $\widetilde{G}$ are both elliptic coordinates matrices, and that the matrices $A^{(i)}$ so defined all satisfy the cyclic conditions (2.2) (cf. proposition 3.6 and remark 3.7).

If we now require the M block in the matrix Π *not* to depend on μ, we obtain the following result.

Theorem 5.1 (μ-dependent Poisson matrix)**.** *Let G_μ, N_μ and F_μ be given by* (5.2)*, and suppose that G and $\widetilde{G}$ are nonsingular and linearly independent. Then the matrix*

$$\Pi_\mu = \begin{pmatrix} 0 & \frac{\lambda}{2} G_\mu(q) & p \\ * & \frac{\lambda}{2} F_\mu(q,p) & M(q) + \lambda d\, N_\mu(q) \\ * & * & 0 \end{pmatrix} \tag{5.4}$$

is a Poisson matrix (for all μ) if and only if the Newton system $\ddot{q} = M(q)$ has n quadratic integrals of motion

$$E^{(i)} = \dot{q}^T A^{(i)}(q)\, \dot{q} + k^{(i)}(q), \quad i = 0, \dots, n-1, \tag{5.5}$$

where the matrices $A^{(i)}$ are defined by (5.3)*.*

Moreover, then the function

$$\widehat{E}_\mu(q,p,d) = p^T A_\mu(q)\, p + k_\mu(q) - \lambda d \det G_\mu(q), \tag{5.6}$$

where

$$k_\mu(q) = \sum_{i=0}^{n-1} k^{(i)}(q) \mu^i, \tag{5.7}$$

is a Casimir of Π_μ.

Proof. Nearly all of the requirements of theorem 4.1 are automatically fulfilled. What remains is that we must have $A_\mu M = -\frac{1}{2} \nabla k_\mu$ for some function $k_\mu(q)$ in order for Π_μ to be Poisson. If this is to be an identity in μ, k_μ must have the form $k_\mu = \sum_{i=0}^{n-1} k^{(i)} \mu^i$, and $A^{(i)} M = -\frac{1}{2} \nabla k^{(i)}$ must hold for all i. The latter condition is, by theorem 2.1, equivalent to $E^{(i)} = \dot{q}^T A^{(i)}\, \dot{q} + k^{(i)}$ being an integral of motion of $\ddot{q} = M(q)$, which proves the first part of the theorem. The Casimir follows immediately from theorem 4.1. $\square$

Remark 5.2 (Poisson pencil)**.** The matrix Π_μ splits in the following way:

$$\begin{aligned} \Pi_\mu &= \Pi + \mu \Pi_0 \\ &= \begin{pmatrix} 0 & \frac{\lambda}{2} G(q) & p \\ * & \frac{\lambda}{2} F(q,p) & M(q) + \lambda d\, N(q) \\ * & * & 0 \end{pmatrix} + \mu \begin{pmatrix} 0 & \frac{\lambda}{2} \widetilde{G}(q) & 0 \\ * & \frac{\lambda}{2} \widetilde{F}(q,p) & \lambda d\, \widetilde{N}(q) \\ * & * & 0 \end{pmatrix}, \end{aligned} \tag{5.8}$$

where Π is Poisson by theorem 4.1, and likewise Π_0, by a similar (but simpler) calculation. (Alternatively, we could infer this from $\Pi_0 = \lim_{\mu\to\infty} \Pi_\mu/\mu$.) Thus, Π_μ is a *Poisson pencil* of compatible Poisson matrices Π and Π_0.

We have already, in example 3.8 (with $d = 0$), seen a Newton system of the type required in theorem 5.1. In the remainder of this article, we will show that such systems exist in large numbers, including for example systems with separable potentials, and we will also show that they are completely integrable (in a slightly generalized sense). To begin with, we have the following theorem, which says that the existence of n integrals of motion of the special form required in theorem 5.1 is guaranteed by the existence of just *two* integrals of motion of cofactor type. This is clearly a feature which cannot be seen until one considers more than two dimensions, and thus it has no counterpart in the two-dimensional theory [2].

Theorem 5.3 ("2 implies n")**.** *In the notation of theorem 5.1, if the Newton system $\ddot{q} = M(q)$ has integrals of motion $E^{(0)}$ and $E^{(n-1)}$ of cofactor type, then it also has integrals of motion of the form $E^{(2)}, \dots, E^{(n-2)}$.*

Proof. The question is whether each vector field $A^{(i)}M$ has a potential $-\frac{1}{2}k^{(i)}$, given that $A^{(0)}M$ and $A^{(n-1)}M$ do, where the matrices $A^{(i)}$ are defined by (5.3). We will show this in a rather indirect way. By theorem 4.1, applied first with $(\operatorname{cof} G)M = -\frac{1}{2}\nabla k^{(0)}$ and then with $(\operatorname{cof} \widetilde{G})M = -\frac{1}{2}\nabla k^{(n-1)}$, the matrices

$$\Pi' = \begin{pmatrix} 0 & \frac{\lambda}{2}G(q) & p \\ * & \frac{\lambda}{2}F(q,p) & M(q) + \lambda d\, N(q) \\ * & * & 0 \end{pmatrix}$$

and

$$\Pi'' = \begin{pmatrix} 0 & \frac{\lambda}{2}\widetilde{G}(q) & p \\ * & \frac{\lambda}{2}\widetilde{F}(q,p) & M(q) + \lambda d\, \widetilde{N}(q) \\ * & * & 0 \end{pmatrix}$$

are both Poisson. At the same time,

$$\Pi' - \Pi'' = \begin{pmatrix} 0 & \frac{\lambda}{2}(G - \widetilde{G}) & 0 \\ * & \frac{\lambda}{2}(F - \widetilde{F}) & \lambda d\,(N - \widetilde{N}) \\ * & * & 0 \end{pmatrix}$$

has the form of Π_0 in (5.8), so it is also Poisson. This implies [8, Lemma

7.20] that

$$\Pi' + \mu\Pi'' = \begin{pmatrix} 0 & \frac{\lambda}{2}G_\mu & (1+\mu)p \\ * & \frac{\lambda}{2}F_\mu & (1+\mu)M + \lambda d\, N_\mu \\ * & * & 0 \end{pmatrix}$$

is Poisson for all μ, and also for all λ, since λ is just an arbitrary numerical parameter. Replacing λ with $\lambda(1+\mu)$, and dividing the matrix by $(1+\mu)$, we obtain precisely the matrix Π_μ in (5.4), which we thus have shown to be Poisson for all μ. Theorem 5.1 now implies that $\ddot{q} = M(q)$ has n integrals of motion $E^{(i)}$, as claimed. □

Remark 5.4. In [9] Newton systems on R^{2m} are constructed which have m quadratic and m quartic integrals of motion. This shows that the existence of two quadratic integrals of motion which are not of cofactor type is not sufficient for n quadratic integrals to exist.

Theorem 5.3 motivates the following definition:

Definition 5.5 (Cofactor pair system)**.** An n-dimensional QLN system with two independent quadratic integrals of motion $E = \dot{q}^T A\,\dot{q} + k$ and $\widetilde{E} = \dot{q}^T \widetilde{A}\,\dot{q} + \widetilde{k}$, where A and $\widetilde{A}$ both are cofactor matrices of linearly independent nonsingular elliptic coordinates matrices, i.e.,

$$\begin{aligned} A(q) &= \operatorname{cof} G(q), \quad G(q) = \alpha q q^T + q\beta^T + \beta q^T + \gamma, \quad \det G(q) \neq 0, \\ \widetilde{A}(q) &= \operatorname{cof} \widetilde{G}(q), \quad \widetilde{G}(q) = \widetilde{\alpha} q q^T + q\widetilde{\beta}^T + \widetilde{\beta} q^T + \widetilde{\gamma}, \quad \det \widetilde{G}(q) \neq 0, \end{aligned}$$

will be called a *cofactor pair system*.

Note that A and $\widetilde{A}$ are the same as $A^{(0)}$ and $A^{(n-1)}$ in theorems 5.1 and 5.3, and similarly for k, $\widetilde{k}$ and E, $\widetilde{E}$. By theorem 5.3, a cofactor pair system always has n quadratic integrals of motion $E^{(i)} = \dot{q}^T A^{(i)}\,\dot{q} + k^{(i)}$ which can be found by solving the equations $-2A^{(i)}M = \nabla k^{(i)}$ for $k^{(i)}$. Theorem 5.1 leads immediately to the following theorem, which is the key to explaining in what sense cofactor pair systems can be considered to be integrable.

Theorem 5.6 (Bi-Hamiltonian formulation)**.** *Let $\ddot{q} = M(q)$ be a cofactor pair system. Then there is on the extended phase space $\mathcal{M} = R^{2n+1}$ with coordinates (q, p, d) a related bi-Hamiltonian system*

$$\frac{d}{dt}\begin{pmatrix} q \\ p \\ d \end{pmatrix} = \Pi\,\overline{\nabla}(\lambda d \det \widetilde{G}) = \Pi_0\,\overline{\nabla}(\widetilde{E} - \lambda d D^{(n-1)}), \tag{5.9}$$

where Π and Π_0 are given by (5.8)*, and $D^{(n-1)}$ is defined by*

$$D_\mu = \det G_\mu = \det(G + \mu\widetilde{G}) = \sum_{i=0}^{n} D^{(i)}\mu^i. \tag{5.10}$$

The trajectories of this system on the hyperplane $d = 0$ coincide with the trajectories of the original system in $(q, \dot{q} = p)$-space, but are traversed with $\lambda \det \widetilde{G}(q)$ times the velocity at each point.

Proof. From theorem 5.1 we know that

$$\widehat{E}_\mu(q, p, d) = p^T A_\mu(q)\, p + k_\mu(q) - \lambda d D_\mu(q) = \sum_{i=0}^{n} \widehat{E}^{(i)} \mu^i,$$

is a Casimir of the Poisson pencil $\Pi_\mu = \Pi + \mu \Pi_0$. Collecting powers of μ, we obtain the following bi-Hamiltonian chain:

$$\begin{aligned} 0 =& \Pi_\mu \overline{\nabla} \widehat{E}_\mu = (\Pi + \mu \Pi_0) \overline{\nabla} \left(\sum_{i=0}^{n} \widehat{E}^{(i)} \mu^i \right) \\ =& \Pi \overline{\nabla} \widehat{E}^{(0)} + \mu \left[\Pi \overline{\nabla} \widehat{E}^{(1)} + \Pi_0 \overline{\nabla} \widehat{E}^{(0)} \right] \\ & + \cdots + \mu^n \left[\Pi \overline{\nabla} \widehat{E}^{(n)} + \Pi_0 \overline{\nabla} \widehat{E}^{(n-1)} \right] \\ & + \mu^{n+1} \Pi_0 \overline{\nabla} \widehat{E}^{(n)}. \end{aligned} \tag{5.11}$$

Since $\widehat{E}^{(n)} = -\lambda d D^{(n)} = -\lambda d \det \widetilde{G}$ and $\widehat{E}^{(n-1)} = E^{(n-1)} - \lambda d D^{(n-1)} = \widetilde{E} - \lambda d D^{(n-1)}$, we identify at μ^n the bi-Hamiltonian system (5.9). Computing the right-hand side of the system explicitly yields

$$\frac{d}{dt} \begin{pmatrix} q \\ p \\ d \end{pmatrix} = \Pi\, \overline{\nabla} (\lambda d \det \widetilde{G}) = \lambda \begin{pmatrix} (\det \widetilde{G}) p \\ -\frac{\lambda}{2} d G (\nabla \det \widetilde{G}) + (\det \widetilde{G})(M + \lambda d\, N) \\ -d p^T \nabla \det \widetilde{G} \end{pmatrix},$$

which for $d = 0$ reduces to

$$\frac{d}{dt} \begin{pmatrix} q \\ p \\ d \end{pmatrix} = \lambda \det \widetilde{G} \begin{pmatrix} p \\ M \\ 0 \end{pmatrix}.$$

The last claim follows. □

Corollary 5.7. *If the functions $\widehat{E}^{(i)}(q, p, d)$, $i = 0, \ldots, n$, are functionally independent, then the bi-Hamiltonian system* (5.9) *is completely integrable.*

Proof. This follows by similar reasoning as in remark 4.6, since the functions $\widehat{E}^{(i)}(q, p, d)$ Poisson commute with respect to Π and Π_0 by Magri's theorem [10]. □

Remark 5.8. Since a completely integrable Hamiltonian system can, in principle, be solved by quadrature, the same is true for cofactor pair systems satisfying the assumptions of corollary 5.7. The final step, from the solution of the extended bi-Hamiltonian system back to the original Newton system, is just a matter of re-parameterizing the trajectories to obtain the correct velocity at each point. This can be done with one further quadrature.

Remark 5.9. The assumption about functional independence of the functions $\widehat{E}^{(i)}$ seems to be fulfilled for most cofactor pair systems, like for instance in example 3.8. As an example of a degenerate case when it is not, consider $G = qq^T$ and $\widetilde{G} = I$ for $n = 3$. The system $\ddot{q} = -q$ is a cofactor pair system with these matrices. It is just a harmonic oscillator, so it is integrable. The integrals of motion in the cofactor chain (for the extended system) are $\widehat{E}^{(0)} = 0$, $\widehat{E}^{(1)} = l_{12}^2 + l_{13}^2 + l_{23}^2$, where $l_{ij} = q_i p_j - q_j p_i$, $\widehat{E}^{(2)} = p^T p + q^T q + \lambda d q^T q$, and $\widehat{E}^{(3)} = \lambda d$. Since $A^{(0)} = \operatorname{cof} G = 0$, the cofactor chain does not provide us with all the integrals of motion of this system. (G is singular in this example, which simplifies the formulas a little, but it could be replaced with the nonsingular matrix $G = qq^T + I$ with essentially the same results; $\widehat{E}^{(0)}$ would not be zero, but the functions $\widehat{E}^{(i)}$ would be dependent.)

The harmonic oscillator above is integrable, so there are integrals of motion which do not appear in the degenerate cofactor chain. It might also be possible that there exist non-integrable cofactor pair systems, with dependent $\widehat{E}^{(i)}$ and no other integrals of motion.

The standard test for functional independence is the following: the functions $\widehat{E}^{(i)}(q, p, d)$ are functionally dependent in an open set U if and only if their gradients $\overline{\nabla} \widehat{E}^{(i)}(q, p, d)$ are linearly dependent everywhere in U [8]. This shows that a sufficient conditions for the functions $E^{(i)}$ to be functionally independent is that the vectors $A^{(i)} p$, $i = 0, \dots, n-1$ are linearly independent. It would be nice to have some simple criterion, expressed directly in terms of G and $\widetilde{G}$, which would guarantee this, but we have not been able to find any such. As the above example shows, it is not enough that G and $\widetilde{G}$ are nonsingular.

Remark 5.10. Theorems 4.1 and 5.6 generalize the corresponding results obtained in [2] for $n = 2$. When $\widetilde{G} = I$ they reproduce, respectively, theorem 4.1 and corollary 4.2 of [3] (bi-Hamiltonian formulation for separable potentials; see also section 7).

6 The fundamental equations and recursive construction of cofactor pair systems

Considering the results of the previous section, which show that cofactor pair systems can be considered as completely integrable, it is natural to ask how large the class of such systems is, and how to find or identify them in practice. In this section, we show that cofactor pair systems are closely related to a system of $\binom{n}{2}$ second order linear PDEs, which we call the *fundamental equations*. This yields an extremely simple method of constructing infinite families of cofactor pair systems.

Definition 6.1 (Fundamental equations)**.** Let $G(q) = \alpha qq^T + q\beta^T + \beta q^T + \gamma$

and $\widetilde{G}(q) = \widetilde{\alpha} q q^T + q\widetilde{\beta}^T + \widetilde{\beta} q^T + \widetilde{\gamma}$ be elliptic coordinates matrices. Let, as usual, $N = \alpha q + \beta$ and $\widetilde{N} = \widetilde{\alpha} q + \widetilde{\beta}$. The *fundamental equations* associated with the pair $(G, \widetilde{G})$ are, for $i, j = 1, \dots, n$,

$$\begin{aligned} 0 &= \sum_{r,s=1}^{n} \big(G_{ir}\widetilde{G}_{js} - G_{jr}\widetilde{G}_{is}\big)\partial_{rs}K \\ &\quad + 3\sum_{r=1}^{n}\big(G_{ir}\widetilde{N}_j + \widetilde{G}_{jr}N_i - G_{jr}\widetilde{N}_i - \widetilde{G}_{ir}N_j\big)\partial_r K \\ &\quad + 6\big(N_i\widetilde{N}_j - N_j\widetilde{N}_i\big)K. \end{aligned} \tag{6.1}$$

The number of independent equations is (at most) $\binom{n}{2}$ since the equations are antisymmetric in i and j.

The coefficients in this system are polynomials in q. The highest powers of q cancel in each coefficient, so that the coefficient at $\partial_{rs}K$ is in general of degree three, at $\partial_r K$ of degree two, and at K of degree one.

The fundamental equations are antisymmetric not only with respect to i and j, but also under swapping of corresponding parameters with and without tilde. This means that the fundamental equations associated with the pair $(G, \widetilde{G})$ are the same as the fundamental equations for $(\widetilde{G}, G)$, or even for any linear combination $(\lambda_1 G + \lambda_2 \widetilde{G}, \mu_1 G + \mu_2 \widetilde{G})$. Consequently, we might say that the fundamental equations are associated with the linear span of the matrices G and $\widetilde{G}$. The following theorem shows the intimate connection between cofactor pair systems and the corresponding fundamental equations.

Theorem 6.2 (Fundamental equations)**.** *For a cofactor pair system with integrals of motion $E = \dot{q}^T A\,\dot{q} + k$ and $\widetilde{E} = \dot{q}^T \widetilde{A}\,\dot{q} + \widetilde{k}$, where $A = \operatorname{cof} G$ and $\widetilde{A} = \operatorname{cof} \widetilde{G}$, the functions*

$$K'(q) = \frac{k(q)}{\det G(q)} \quad \text{and} \quad K''(q) = \frac{\widetilde{k}(q)}{\det \widetilde{G}(q)},$$

although in general different, both satisfy the fundamental equations associated with the pair of matrices $(G, \widetilde{G})$.

Conversely, for each solution K of the fundamental equations the two different QLN systems

$$\begin{aligned} 0 &= \delta^+ \widetilde{E}, \quad \text{where} \quad \widetilde{E} = \dot{q}^T \widetilde{A}\,\dot{q} + \widetilde{k}, \quad \widetilde{k} = K \det \widetilde{G}, \\ 0 &= \delta^+ F, \quad \text{where} \quad F = \dot{q}^T A\,\dot{q} + l, \quad l = K \det G \end{aligned}$$

are both cofactor pair systems. Explicitly, there exist extra integrals of motion

$$E = \dot{q}^T A\,\dot{q} + k \quad \text{and} \quad \widetilde{F} = \dot{q}^T \widetilde{A}\,\dot{q} + \widetilde{l}$$

for the first and second system respectively.

Proof. The cofactor pair system can be written $\ddot{q} = -\frac{1}{2}A^{-1}\nabla k = -\frac{1}{2}\widetilde{A}^{-1}\nabla\widetilde{k} = M$. This means that the vector field $AM = -\frac{1}{2}\nabla k$ satisfies the integrability conditions

$$\partial_a[AM]_b - \partial_b[AM]_a = 0$$

for all a, b. As shown in the proof of theorem 4.1, these conditions are equivalent to the equations

$$0 = P_{ij} - P_{ji}, \text{where } P_{ij} = 3N_iM_j + \sum_{k=1}^{n} G_{ik}\partial_k M_j. \tag{6.2}$$

Expressing $M = -\frac{1}{2}\widetilde{A}^{-1}\nabla\widetilde{k}$ in terms of $K = K'' = \widetilde{k}/\det\widetilde{G}$ yields

$$-2M = \widetilde{A}^{-1}\nabla\widetilde{k} = \widetilde{A}^{-1}(\nabla K \det\widetilde{G} + K\nabla\det\widetilde{G}) = \widetilde{G}\nabla K + 2K\,\widetilde{N},$$

where we have used equation (3.5) (with tildes attached) and the relation $\widetilde{G}\widetilde{A} = (\det\widetilde{G})I$. Substituting this into (6.2) we obtain after a short calculation the fundamental equations (6.1), which thus are satisfied by $K = K''$. Exchanging the roles of E and $\widetilde{E}$, we find that $K = K' = k/\det G$ satisfies the corresponding equations with coefficients with and without tilde interchanged. But this is in fact the same system, since (6.1) is completely antisymmetric under that operation.

The second part of the theorem follows easily by doing the same calculations backwards. Indeed, if K satisfies the fundamental equations and we let $\widetilde{k} = K\det\widetilde{G}$, then the vector field $-2AM = A(\widetilde{G}\nabla K + 2K\,\widetilde{N}) = A\widetilde{A}^{-1}\nabla\widetilde{k}$ satisfies the integrability conditions, so there exists a function k such that $\nabla k = A\widetilde{A}^{-1}\nabla\widetilde{k}$. Thus, $\ddot{q} = \frac{1}{2}A^{-1}\nabla k = \frac{1}{2}\widetilde{A}^{-1}\nabla\widetilde{k}$ is the cofactor pair system $\delta^+E = 0 = \delta^+\widetilde{E}$. Similarly, if we let $l = K\det G$, then, because of the antisymmetry, the fundamental equations are also equivalent to the integrability conditions for the vector field $\widetilde{A}A^{-1}\nabla l$, so that we obtain the cofactor pair system $\delta^+\widetilde{F} = 0 = \delta^+F$. $\square$

Corollary 6.3. *$K(q) = 1/\det G(q)$ and $K(q) = 1/\det\widetilde{G}(q)$ are solutions of the fundamental equations* (6.1).

Proof. The Newton system $\ddot{q} = 0$ is trivially a cofactor pair systems for any pair $(G, \widetilde{G})$ and any constant k and $\widetilde{k}$, so we just take $k = \widetilde{k} = 1$ in the preceding theorem. $\square$

Remark 6.4. Since $A^{-1}\nabla k = -2\ddot{q} = \widetilde{A}^{-1}\nabla\widetilde{k}$, the first part of the theorem can be expressed by saying that the equation

$$A^{-1}\nabla(K'\det G) = \widetilde{A}^{-1}\nabla(K''\det\widetilde{G})$$

is an auto-Bäcklund transformation between solutions K' and K'' of the fundamental equations. For example, when $G = \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$ and $\widetilde{G} = \left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)$ this reproduces the Cauchy–Riemann equations, a well-known auto-Bäcklund transformation for the Laplace equation, which is the fundamental equation in this case (when $n = 2$, there is just one fundamental equation).

Theorem 6.2 opens up the possibility of recursively constructing families of solutions to the fundamental equations, or, equivalently, constructing families of cofactor pair systems whose integrals of motion all have the same "kinetic" parts $\dot{q}^T A^{(i)} \dot{q}$, determined by $\operatorname{cof}(G + \mu\widetilde{G}) = \sum_0^{n-1} A^{(i)} \mu^i$, but different "potential" parts $k^{(i)}$. We can combine the two statements of the theorem as the following diagram illustrates:

$$\boxed{\begin{array}{c}\text{Cofactor pair}\\ \text{system } \delta^+\widetilde{E} = 0.\end{array}} \xrightarrow{K=\frac{\widetilde{k}}{\det \widetilde{G}}} \boxed{\text{Solution } K} \xrightarrow{l=K \det G} \boxed{\begin{array}{c}\text{Cofactor pair}\\ \text{system } \delta^+F = 0.\end{array}}$$

I.e., starting with a cofactor pair system $\delta^+E = 0 = \delta^+\widetilde{E}$, we obtain another cofactor pair system $\delta^+F = 0 = \delta^+\widetilde{F}$ by defining $l = K \det G = (\widetilde{k}/\det \widetilde{G}) \det G$ and determining $\widetilde{l}$ from

$$\nabla\widetilde{l} = \widetilde{A}A^{-1}\nabla l. \tag{6.3}$$

That this integration is possible is precisely what theorem 6.2 says. Then we can repeat the procedure to find yet another cofactor pair system $\delta^+G = 0 = \delta^+\widetilde{G}$, and so on. We can also go to the left, thereby producing a bi-infinite sequence of cofactor pair systems

$$\cdots \longleftrightarrow \delta^+E = 0 \longleftrightarrow \delta^+F = 0 \longleftrightarrow \delta^+G = 0 \longleftrightarrow \cdots$$

The next theorem shows that there is a purely algebraic relation between the integrals of motion of adjacent systems in this sequence. This means that we can get from one system to the next without having to integrate (6.3), but instead we need to keep track of all n integrals of motion of each system.

Theorem 6.5 (Recursion formula)**.** *Let* $0 = \delta^+E$ *and* $\delta^+F = 0$ *be cofactor pair systems related as in the second part of theorem 6.2. Let, as usual,* $E^{(i)} = \dot{q}^T A^{(i)} \dot{q} + k^{(i)}$ *and* $F^{(i)} = \dot{q}^T A^{(i)} \dot{q} + l^{(i)}$, $i = 0, \ldots, n-1$ *denote their integrals of motion, where* $\widetilde{k} = k^{(n-1)}$ *and* $l = l^{(0)}$, *and let* $k_\mu = \sum_{i=0}^{n-1} k^{(i)} \mu^i$ *and* $l_\mu = \sum_{i=0}^{n-1} l^{(i)} \mu^i$. *Then, up to an arbitrary additive constant in each* $l^{(i)}$,

$$l_\mu = \frac{\det(G + \mu\widetilde{G})}{\det \widetilde{G}} \widetilde{k} - \mu k_\mu, \tag{6.4}$$

with the inverse relationship

$$k_\mu = \frac{1}{\mu} \left(\frac{\det(G + \mu\widetilde{G})}{\det G} l - l_\mu \right). \tag{6.5}$$

Proof. This proof is quite technical and has therefore been put in the appendix. $\square$

Remark 6.6. Setting

$$\frac{\det(G+\mu\widetilde{G})}{\det\widetilde{G}} = \sum_{i=0}^{n} X^{(i)}\mu^i$$

we can write (6.4) as

$$\begin{pmatrix} l^{(0)} \\ l^{(1)} \\ \vdots \\ l^{(n-2)} \\ l^{(n-1)} \end{pmatrix} = \begin{pmatrix} 0 & & & & X^{(0)} \\ -1 & 0 & & & X^{(1)} \\ & & \ddots & & \vdots \\ & & -1 & 0 & X^{(n-2)} \\ & & & -1 & X^{(n-1)} \end{pmatrix} \begin{pmatrix} k^{(0)} \\ k^{(1)} \\ \vdots \\ k^{(n-2)} \\ k^{(n-1)} \end{pmatrix}, \tag{6.6}$$

which is sometimes convenient. We note that the matrix is (minus) what is known as the companion matrix of the polynomial $\mu^n + X^{(n-1)}\mu^{n-1} + \cdots + X^{(0)}$.

Remark 6.7 (Families of cofactor pair systems)**.** With the help of the recursion theorem we can easily construct a bi-infinite family of cofactor pair systems for any given pair $(G,\widetilde{G})$. Namely, we observe that any k_μ which is independent of q gives rise to the trivial cofactor pair system $\ddot{q}=0$, which can be used as a starting point for the recursion. For example, we can take $k_\mu = \mu^{n-1}$ and iterate (6.4) to obtain the "upwards" part of the family, or start with $l_\mu = 1$ and iterate (6.5) to obtain the "downwards" part. (Starting with other choices of constant k_μ or l_μ will only lead to systems which are linear combinations of the systems in this family.) For systems $\ddot{q} = M(q)$ obtained in this way, $M_1(q), \ldots, M_n(q)$ will always be a rational functions. However, if we find some cofactor pair system which does not depend rationally on q, then we can use the recursion formula in both directions to obtain another bi-infinite family, associated with this system, whose members will all be non-rational. This is illustrated in example 7.7.

Example 6.8. To illustrate the procedure in the case $n=3$, define elliptic coordinates matrices G and $\widetilde{G}$ by

$$\alpha = 1,\ \beta = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}; \quad \widetilde{\alpha} = 0,\ \widetilde{\beta} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \widetilde{\gamma} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$
\begin{aligned}
&\operatorname{cof}(G+\mu\widetilde{G}) = \\
&= \operatorname{cof}\left[\begin{pmatrix} q_1^2+1 & q_1q_2 & q_1q_3 \\ q_1q_2 & q_2^2+2 & q_2q_3 \\ q_1q_3 & q_2q_3 & q_3^2+3 \end{pmatrix} + \mu \begin{pmatrix} 1 & 0 & q_1 \\ 0 & 1 & q_2 \\ q_1 & q_2 & 2q_3 \end{pmatrix}\right] \\
&= \begin{pmatrix} 3q_2^2+2q_3^2+6 & -3q_1q_2 & -2q_1q_3 \\ -3q_1q_2 & 3q_1^2+q_3^2+3 & -q_2q_3 \\ -2q_1q_3 & -q_2q_3 & 2q_1^2+q_2^2+2 \end{pmatrix} \\
&+ \mu \begin{pmatrix} q_3^2+4q_3+3 & 0 & -q_1q_3-2q_1 \\ 0 & q_3^2+2q_3+3 & -q_2q_3-q_2 \\ -q_1q_3-2q_1 & -q_2q_3-q_2 & q_1^2+q_2^2+3 \end{pmatrix} \\
&+ \mu^2 \begin{pmatrix} -q_2^2+2q_3 & q_1q_2 & -q_1 \\ q_1q_2 & -q_1^2+2q_3 & -q_2 \\ -q_1 & -q_2 & 1 \end{pmatrix} \\
&= A^{(0)} + \mu A^{(1)} + \mu^2 A^{(2)}
\end{aligned}
$$

and

$$
\begin{aligned}
\det(G+\mu\widetilde{G}) = {} & (6q_1^2+3q_2^2+2q_3^2+6) + (3q_1^2+3q_2^2+3q_3^2+4q_3+9)\mu \\
& + (-2q_1^2-q_2^2+q_3^2+6q_3+3)\mu^2 + (-q_1^2-q_2^2+2q_3)\mu^3.
\end{aligned}
$$

An application of the "upwards" recursion formula (6.4) with $k_\mu = \mu$ gives $l_\mu = l^{(0)} + l^{(1)}\mu + l^{(2)}\mu^2$, where

$$
\begin{aligned}
l^{(0)} &= \frac{6q_1^2+3q_2^2+2q_3^2+6}{-q_1^2-q_2^2+2q_3}, \\
l^{(1)} &= \frac{3q_1^2+3q_2^2+3q_3^2+4q_3+9}{-q_1^2-q_2^2+2q_3}, \\
l^{(2)} &= \frac{-2q_1^2-q_2^2+q_3^2+6q_3+3}{-q_1^2-q_2^2+2q_3}.
\end{aligned}
$$

This corresponds to the nontrivial Newton system

$$
\ddot{q} = -\frac{1}{2}[A^{(i)}]^{-1}\nabla l^{(i)} = \frac{-1}{(-q_1^2-q_2^2+2q_3)^2}\begin{pmatrix} q_1q_3+q_1 \\ q_2q_3+2q_2 \\ q_3^2-3 \end{pmatrix} \tag{6.7}
$$

with integrals of motion $\dot{q}^T A^{(i)}\dot{q} + l^{(i)}$ $(i = 0, 1, 2)$.

Applying the "downwards" recursion formula (6.5) with $l_\mu = 1$ gives

$k_\mu = k^{(0)} + k^{(1)}\mu + k^{(2)}\mu^2$, where

$$
\begin{aligned}
k^{(0)} &= \frac{3q_1^2 + 3q_2^2 + 3q_3^2 + 4q_3 + 9}{6q_1^2 + 3q_2^2 + 2q_3^2 + 6}, \\
k^{(1)} &= \frac{-2q_1^2 - q_2^2 + q_3^2 + 6q_3 + 3}{6q_1^2 + 3q_2^2 + 2q_3^2 + 6}, \\
k^{(2)} &= \frac{-q_1^2 - q_2^2 + 2q_3}{6q_1^2 + 3q_2^2 + 2q_3^2 + 6},
\end{aligned}
$$

corresponding to the Newton system

$$
\ddot{q} = -\frac{1}{2}[A^{(i)}]^{-1}\nabla k^{(i)} = \frac{1}{(6q_1^2 + 3q_2^2 + 2q_3^2 + 6)^2}\begin{pmatrix} 2q_1q_3 + 6q_1 \\ 2q_2q_3 + 3q_2 \\ 2q_3^2 - 6 \end{pmatrix} \tag{6.8}
$$

with integrals of motion $\dot{q}^T A^{(i)}\dot{q} + k^{(i)}$ $(i = 0, 1, 2)$.

The systems (6.7) and (6.8) are integrable in the sense described in the previous section, but it is not known if they admit, for example, any kind of variable separation. Further systems in the recursive sequence are easily computed with the help of symbolic algebra software, but the expressions quickly become rather long.

7 Identifying cofactor pair system

There is a straightforward way of testing if a given Newton system $\ddot{q} = M(q)$ is a cofactor pair system.

Theorem 7.1. *The Newton system $\ddot{q} = M(q)$ admits an integral of motion $E = \dot{q}^T \operatorname{cof} G(q)\, \dot{q} + k(q)$ of cofactor type if and only if the equations*

$$
0 = P_{ij} - P_{ji}, \quad \textit{where } P_{ij} = 3N_iM_j + \sum_{k=1}^{n} G_{ki}\partial_k M_j, \tag{7.1}
$$

viewed as a linear system for the parameters α, β, γ in $G = \alpha qq^T + q\beta^T + \beta q^T + \gamma$ and $N = \alpha q + \beta$, has a nontrivial solution with G nonsingular. It is a cofactor pair system if and only if there is a two-parameter family of solutions $G = sG' + tG''$, from which it is possible to choose $G = s_1G' + t_1G''$ and $\widetilde{G} = s_2G' + t_2G''$ nonsingular and linearly independent.

Proof. These equations occurred previously as equations (4.5). The claim follows immediately from the statement in theorem 4.1 connecting equations (4.5) and (4.6). □

Example 7.2 (Harry Dym stationary flow)**.** As a simple example, let us apply this test to the system (1.4) from example 1.1. Inserting $M(q)$ from

(1.4) into (7.1) (with $i = 1$, $j = 2$) yields $0 = -\alpha\, q_1^{-5} q_2^2 - 5\beta_1\, q_1^{-6} q_2^2 + \gamma_{22}\, q_1^{-5} - 5\gamma_{12}\, q_1^{-6} q_2 +$ (polynomial terms). Since different powers are linearly independent we must have $\alpha = \beta_1 = \gamma_{12} = \gamma_{22} = 0$, and with these values the polynomial terms cancel as well, leaving β_2 and γ_{11} free to attain any values. Thus

$$G = s \begin{pmatrix} 0 & q_1 \\ q_1 & 2q_2 \end{pmatrix} + t \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

is the general solution of (7.1) in this case. The matrices at s and t correspond to the two known quadratic integrals of motion E and F from example 1.1. If we want both G and $\widetilde{G}$ to be nonsingular, we can take for example $\left(\begin{smallmatrix} 0 & q_1 \\ q_1 & 2q_2 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & q_1 \\ q_1 & 2q_2 \end{smallmatrix}\right)$.

Example 7.3 (KdV stationary flow)**.** For a three-dimensional example, consider the Newton system (3.8) from example 3.8. If we apply theorem 7.1 to this system, we obtain first $0 = P_{12} - P_{21} = 60\alpha\, q_1^4 + 90\beta_1\, q_1^3 +$ (lower order terms), from which it follows that $\alpha = \beta_1 = 0$. This simplifies the expressions considerably. What remains is $0 = P_{12} - P_{21} = (30\gamma_{11} + 34\beta_2)q_1^2 + 4(\beta_3 + \gamma_{12})q_1 - (20\beta_2 + 16\gamma_{11})q_2 + 4(\gamma_{13} - \gamma_{22})$, which forces $\beta_2 = \gamma_{11} = 0$, $\beta_3 = -\gamma_{12}$ and $\gamma_{13} = \gamma_{22}$. Taking this into account, we find $P_{13} - P_{31} = -4\gamma_{23}$ and $P_{23} - P_{32} = -4\gamma_{23}\, q_1 - 4\gamma_{33}$, which gives $\gamma_{23} = \gamma_{33} = 0$. Consequently, the most general matrix G for which the system has an integral of motion of the form $\dot{q}^T(\operatorname{cof} G)\dot{q} + k(q)$ is

$$G = s \begin{pmatrix} 0 & -1 & q_1 \\ -1 & 0 & q_2 \\ q_1 & q_2 & 2q_3 \end{pmatrix} + t \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

In this way we recover the matrices G and $\widetilde{G}$ from example 3.8.

The system (3.8) has an indefinite Lagrangian $\dot{q}_1\dot{q}_3 + \dot{q}_2^2/2 - V(q) + d\, q_1$, where $-2V(q) = k^{(2)}$ using our notation from (3.9). This gives a canonical Hamiltonian formulation via the Legendre transformation to momenta $s_1 = \dot{q}_3$, $s_2 = \dot{q}_2$, $s_3 = \dot{q}_1$, and there is also a second, non-canonical, Hamiltonian formulation given in [5]. Except for naming the momenta in the reverse order, this bi-Hamiltonian formulation is just a special case of the one in theorem 5.6. The system was shown in [11] to be separable in the Hamilton–Jacobi sense, using results about so-called quasi-bi-Hamiltonian systems [12]. The same can be shown to hold for any cofactor pair system where one of the matrices G or $\widetilde{G}$ (say $\widetilde{G}$, as in this case) is independent of q. Briefly, when changing to momenta $s = \widetilde{G}^{-1}p$ instead of p, our bi-Hamiltonian formulation of theorem 5.6 takes the form required for the methods used in [11] to apply. However, it is not known if general cofactor pair system, with both G and $\widetilde{G}$ depending on q, can be solved through separation of variables. A separation procedure not using the Hamilton–Jacobi equation was given in [2] for a special class of two-dimensional cofactor pair systems, the so-called *driven systems*. Similar results have been found also for $n > 2$ and will be published in a separate paper.

Finally, the fundamental equations (6.1) for $K = k^{(2)}/\det\widetilde{G} = -k^{(2)} = 2V$ associated with the pair $(G, \widetilde{G})$ reduce to precisely the system (4.20) for V in [5], found there as the conditions for the Jacobi identity of the non-canonical Poisson matrix to be fulfilled. The authors note that any V satisfying these equations gives rise to a completely integrable bi-Hamiltonian system, but do not address the question of finding such V. Our recursion formula (6.4), which in this case can be written

$$\begin{pmatrix} k^{(0)} \\ k^{(1)} \\ k^{(2)} \end{pmatrix} \longmapsto \begin{pmatrix} 0 & 0 & 2q_1q_2 + 2q_3 \\ -1 & 0 & q_1^2 + 2q_2 \\ 0 & -1 & 2q_1 \end{pmatrix} \begin{pmatrix} k^{(0)} \\ k^{(1)} \\ k^{(2)} \end{pmatrix},$$

immediately provides us with an infinite family of solutions, one of which corresponds to the Newton system (3.8). In fact, starting with $k^{(0)} = k^{(1)} = 0$, $k^{(2)} = -1$, and iterating, we obtain the $k^{(i)}$ of (3.9) after five steps.

Separable potentials

There is an interesting special case of theorem 7.1 that deserves mentioning. The Newton system $\ddot{q} = M(q)$ is conservative when $G = I$ (identity matrix) is a solution of (7.1). A two-parameter solution $sG + tI$ with G non-constant indicates a special kind of cofactor pair system, namely a conservative system with separable potential (in the Hamilton–Jacobi sense). Indeed, I being a solution implies that $M = -\nabla V$ for some potential V, and inserting this into (7.1) shows that G and V satisfy the equations (cf. remark 4.2)

$$\begin{aligned} 0 &= \sum_{r=1}^{n} \big(G_{ir}\, \partial_{rj} V - G_{jr}\, \partial_{ri} V\big) + 3\big(N_i \partial_j V - N_j \partial_i V\big) \\ &= \sum_{r=1}^{n} \Big((\alpha q_i q_r + \beta_i q_r + \beta_r q_i + \gamma_{ir})\, \partial_{rj} V - (\alpha q_j q_r + \beta_j q_r + \beta_r q_j + \gamma_{jr})\, \partial_{ri} V \Big) \\ &\quad + 3\big((\alpha q_i + \beta_i)\partial_j V - (\alpha q_j + \beta_j)\partial_i V\big), \end{aligned} \tag{7.2}$$

which have been found before in various forms [13, 14, 15, 16, 3] as a criterion for the potential V to be separable in generalized elliptic coordinates or some degeneration thereof. The matrix G determines in which coordinates the separation takes place, in a way which we will now describe briefly. The proofs of the following three propositions, which finally justify the terminology "elliptic coordinates matrix," can be found in the appendix.

Proposition 7.4 (Standard form). *Let $G(q) = \alpha q q^T + q\beta^T + \beta q^T + \gamma$ be an elliptic coordinates matrix with α and β not both zero. Any (G, I) cofactor* *pair system can be transformed by an orthogonal change of reference frame $q \to Sq + v$, $S \in SO(n)$, $v \in R^n$, to an equivalent system where G has the standard form*

$$G(q) = -qq^T + \mathrm{diag}(\lambda_1, \ldots, \lambda_n), \tag{7.3}$$

if $\alpha \neq 0$, or

$$G(q) = e_n q^T + q e_n^T + \mathrm{diag}(\lambda_1, \ldots, \lambda_{n-1}, 0), \tag{7.4}$$

where $e_n = (0, \ldots, 0, 1)^T$, if $\alpha = 0$, $\beta \neq 0$.

Proposition 7.5 (Elliptic coordinates)**.** *If*

$$G(q) = -qq^T + \mathrm{diag}(\lambda_1, \ldots, \lambda_n),$$

then the eigenvalues $u_1(q), \ldots, u_n(q)$ of G satisfy

$$\prod_{i=1}^{n}(z - u_i) \bigg/ \prod_{j=1}^{n}(z - \lambda_j) = 1 + \sum_{m=1}^{n} \frac{q_m^2}{z - \lambda_m}, \tag{7.5}$$

which, when all λ_i are distinct, is the defining equation for generalized elliptic coordinates u with parameters $(\lambda_1, \ldots, \lambda_n)$.

Proposition 7.6 (Parabolic coordinates)**.** *Let $e_n = (0, \ldots, 0, 1)^T$. If*

$$G(q) = e_n q^T + q e_n^T + \mathrm{diag}(\lambda_1, \ldots, \lambda_{n-1}, 0),$$

then the eigenvalues $u_1(q), \ldots, u_n(q)$ of G satisfy

$$-\prod_{i=1}^{n}(z - u_i) \bigg/ \prod_{j=1}^{n-1}(z - \lambda_j) = \sum_{m=1}^{n-1} \frac{q_m^2}{z - \lambda_m} + (2q_n - z), \tag{7.6}$$

which, when all λ_i are distinct, is the defining equation for generalized parabolic coordinates u with parameters $(\lambda_1, \ldots, \lambda_{n-1})$.

Now, to find the separation coordinates, first change Euclidean reference frame so as to transform G to standard form. Then change to the elliptic or parabolic coordinates defined by the eigenvalues of G, and the Hamilton–Jacobi equation separates. See for example [16] for a nice summary of the theory. (There are some technicalities concerning degenerate cases; see [13].)

The equations (7.2) are precisely the fundamental equations (6.1) for the special case $\widetilde{G} = I$ (and with V instead of K). Thus, the recursion theorem 6.5 provides an easy way of producing separable potentials for any elliptic coordinates matrix G. In fact, when G takes one of the standard forms above, the recursion formula reduces to known recursion formulas [15] for elliptic and parabolic separable potentials respectively.

Example 7.7. When $n = 2$ and $\widetilde{G} = I$, the matrix form (6.6) of the recursion formula reduces to

$$\begin{pmatrix} l \\ \widetilde{l} \end{pmatrix} = \begin{pmatrix} 0 & \det G \\ -1 & \operatorname{tr} G \end{pmatrix} \begin{pmatrix} k \\ \widetilde{k} \end{pmatrix},$$

with $\widetilde{k}$ corresponding to the potential V and k occurring in the second quadratic integral of motion. We can solve the recursion explicitly by computing the powers of the matrix. For example, with

$$G = \begin{pmatrix} 0 & q_1 \\ q_1 & 2q_2 \end{pmatrix} \tag{7.7}$$

it is not hard to show that we recover the combinatorial potentials

$$V_m = \sum_{k=0}^{\lfloor m/2 \rfloor} \binom{m-k}{k} q_1^{2k} (2q_2)^{m-2k}, \quad m \geq 0, \tag{7.8}$$

found in [17], as well as the accompanying downwards family

$$V_{-m} = \frac{(-1)^m}{q_1^{2m}} V_{m-1}, \quad m \geq 1. \tag{7.9}$$

These potentials are all separable in the parabolic coordinates $u_{1,2} = q_2 \pm \sqrt{q_1^2 + q_2^2}$ defined by the eigenvalues of G, and in fact constitute the two-dimensional case of a more general family of parabolic separable potentials in n dimensions [15].

The two-dimensional Kepler potential

$$W_0(q) = -(q_1^2 + q_2^2)^{-1/2} \tag{7.10}$$

(which is not rational in q) is also separable in these same parabolic coordinates, with second integral $F_0 = \dot{q}^T (\operatorname{cof} G) \dot{q} + 2q_2 W_0(q) = 2q_2 \dot{q}_1^2 - 2q_1 \dot{q}_1 \dot{q}_2 + 2q_2 W_0(q)$. Starting the recursion with $k_\mu = (q_2 + \mu) W_0(q)$, we obtain what might be called the Kepler family of parabolic separable potentials:

$$W_m = (-q_2 V_{m-1} + V_m) W_0, \quad m \geq 0, \tag{7.11}$$

and

$$W_{-m} = (q_2 V_{-m} - V_{-(m-1)}) W_0, , \quad m \geq 1, \tag{7.12}$$

where the V_i are given by (7.8) and (7.9).

8 Conclusions

We have introduced the class of cofactor pair Newton systems in n dimensions, and explained their integrability properties through embedding into bi-Hamiltonian systems in extended phase space. As well as providing many new integrable systems, this gives a framework into which several previously known systems fit, such as separable potentials and some integrable Newton systems derived from soliton theory. Perhaps the most remarkable feature of cofactor pair systems is the algebraic structure of their integrals of motion; namely, that a Newton system with two integrals of motion of cofactor type must have an entire "cofactor chain" consisting of n quadratic integrals of motion. We have shown how to construct infinite families of cofactor pair systems, and how to determine if a given Newton system is a cofactor pair system. Whether all cofactor pair systems can be integrated through some kind of variable separation is an interesting open question, but only partial results are known yet.

9 Acknowledgments

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10 Note added in proof

Several works have already appeared which elaborate further on the subject of cofactor pair systems. In my PhD thesis [18] it is shown how the theory developed in this paper can be used to separate variables for a class of time-dependent potentials, and also how to obtain new cofactor pair systems using a "multiplication" formula (of which the recursion formula is a special case). The geometric properties of cofactor pair systems have been much clarified through the coordinate-free description given by Crampin and Sarlet [19, 20, 21], who define cofactor systems on (pseudo-)Riemannian manifolds. In that setting Topalov [22] constructs hierarchies of metrics admitting cofactor systems. Marciniak and Błaszak [23] have investigated the separability of cofactor pair systems with G and $\widetilde{G}$ both non-constant.

A Appendix

This appendix contains the missing part of the proof of the Poisson matrix theorem 4.1 as well as the proofs of the recursion theorem 6.5 and propositions 7.4, 7.5, and 7.6.

Proof of theorem 4.1. It remains to show that the $\binom{n}{2}$ equations ($1 \le i < j \le n$)

$$0 = P_{ij} - P_{ji}, \quad \text{where } P_{ij} = 3N_iM_j + \sum_{k=1}^{n} G_{ik}\partial_k M_j, \tag{A.1}$$

are equivalent to the $\binom{n}{2}$ equations ($1 \le a < b \le n$)

$$0 = \partial_a[AM]_b - \partial_b[AM]_a, \quad \text{where } A(q) = \operatorname{cof} G(q),\ \det G(q) \ne 0. \tag{A.2}$$

Consider the following linear combination of equations (A.2):

$$\begin{aligned}
&\sum_{a<b}(G_{ia}G_{jb} - G_{ja}G_{ib})(\partial_a[AM]_b - \partial_b[AM]_a) \\
&= \frac{1}{2}\sum_{a,b}(G_{ia}G_{jb} - G_{ja}G_{ib})(\partial_a[AM]_b - \partial_b[AM]_a) \\
&= \sum_{a,b}(G_{ia}G_{jb} - G_{ja}G_{ib})\partial_a[AM]_b \\
&= Q_{ij} - Q_{ji},
\end{aligned}$$

where

$$\begin{aligned}
Q_{ij} =& \sum_{a,b} G_{ia}G_{jb}\partial_a[AM]_b \\
=& \sum_{a,b} \partial_a(G_{ia}G_{jb}[AM]_b) - \sum_{a,b} \partial_a(G_{ia}G_{jb})[AM]_b \\
=& \sum_{a} \partial_a(G_{ia}[GAM]_j) - \sum_{a}(\partial_a G_{ia})[GAM]_j \\
&- \sum_{a,b} G_{ia}(\delta_{aj}N_b + \delta_{ab}N_j)[AM]_b \\
=& \sum_{a} G_{ia}\partial_a(M_j \det G) - G_{ij}N^TAM - N_j[GAM]_i \\
=&(\det G)\left(\sum_{a} G_{ia}\partial_a M_j - N_jM_i\right) + M_j[G\,2AN]_i - G_{ij}N^TAM \\
=&(\det G)\left(\sum_{a} G_{ia}\partial_a M_j - N_jM_i + 2M_jN_i\right) - G_{ij}N^TAM.
\end{aligned}$$

Here we have made use of (3.3) and (3.5), as well as the fact $GA = (\det G)I$. It follows that

$$Q_{ij} - Q_{ji} = (\det G)(P_{ij} - P_{ji})$$

so that (A.2) implies (A.1). The opposite implication follows from the fact that equations (A.1) can be linearly combined to yield (A.2). Explicitly, the inverse transformation of the linear combination above is obtained by multiplying (A.1) by the algebraic complement of $G_{ia}G_{jb} - G_{ja}G_{ib}$ in $\det G$ and summing over $i < j$. This completes the proof. (It can be noted, for completeness, that the implication (A.1) $\implies$ (A.2) does not require $\det G \neq 0$.) $\square$

Before turning to the proof of theorem 6.5 we need some preliminaries. From (3.5) we know that $\nabla \det G = 2AN$, or, since $2N = \nabla \operatorname{tr} G$ and $GA = (\det G)I$,

$$G\nabla \det G = (\det G)\nabla \operatorname{tr} G. \tag{A.3}$$

This can be generalized in the following way.

Lemma A.1. *If $X = \widetilde{G}^{-1}G$, where G and $\widetilde{G}$ are elliptic coordinates matrices, then*

$$X\nabla \det X = (\det X)\nabla \operatorname{tr} X. \tag{A.4}$$

Moreover,

$$\frac{X\nabla \det X}{\det X} = \frac{(X + \mu I)\nabla \det(X + \mu I)}{\det(X + \mu I)}. \tag{A.5}$$

Proof. Multiplying (A.4) by $\widetilde{G}$, we see that it is equivalent to

$$G\nabla \left(\frac{\det G}{\det \widetilde{G}} \right) = \frac{\det G}{\det \widetilde{G}}\, \widetilde{G}\, \nabla \operatorname{tr}(\widetilde{G}^{-1}G).$$

Using (A.3) we find that the left-hand side equals

$$G \left(\frac{\nabla \det G}{\det \widetilde{G}} - \det G \frac{\nabla \det \widetilde{G}}{(\det \widetilde{G})^2} \right) = \frac{\det G}{\det \widetilde{G}} \left(\nabla \operatorname{tr} G - G\widetilde{G}^{-1}\nabla \operatorname{tr} \widetilde{G} \right).$$

We are done if we can show that the expression in parentheses equals $\widetilde{G}\, \nabla \operatorname{tr}(\widetilde{G}^{-1}G)$. Let us temporarily use the notation $\widetilde{H} = \widetilde{G}^{-1}$. Then the general formula for the derivative of an inverse matrix, together with (3.3), yields

$$\partial_k \widetilde{H}_{rs} = -[\widetilde{H}(\partial_k \widetilde{G})\widetilde{H}]_{rs} = -\widetilde{H}_{kr}[\widetilde{H}\widetilde{N}]_s - \widetilde{H}_{ks}[\widetilde{H}\widetilde{N}]_r.$$

Now we can compute

$$\begin{aligned}
[\widetilde{G}\nabla \operatorname{tr}(\widetilde{G}^{-1}G)]_m =& \sum_{k,r,s} \widetilde{G}_{mk}\partial_k(\widetilde{H}_{rs}G_{sr}) \\
=& \sum_{k,r,s} \widetilde{G}_{mk}\big(-\widetilde{H}_{kr}[\widetilde{H}\widetilde{N}]_s - \widetilde{H}_{ks}[\widetilde{H}\widetilde{N}]_r\big)G_{sr} \\
&+ \sum_{k,r,s} \widetilde{G}_{mk}\widetilde{H}_{rs}(\delta_{ks}N_r + \delta_{kr}N_s) \\
=& -2\sum_{r,s}[\widetilde{G}\widetilde{H}]_{mr}[\widetilde{H}\widetilde{N}]_s G_{sr} + 2[\widetilde{G}\widetilde{H}N]_m \\
=& -2[G\widetilde{H}\widetilde{N}]_m + 2N_m \\
=& [\nabla \operatorname{tr} G - G\widetilde{G}^{-1}\nabla \operatorname{tr} \widetilde{G}]_m.
\end{aligned}$$

This establishes (A.4).

To prove (A.5), observe that (A.4) can be applied with $X+\mu I = \widetilde{G}^{-1}(G+\mu\widetilde{G})$ instead of X, since $G+\mu\widetilde{G}$ is an elliptic coordinates matrix. This shows that (A.5) is just a restatement of the identity $\nabla \operatorname{tr}(X+\mu I) = \nabla \operatorname{tr} X$. $\square$

Proof of theorem 6.5. We remind the reader that $l = l^{(0)}$, $\widetilde{k} = k^{(n-1)}$, $A = A^{(0)}$, and $\widetilde{A} = A^{(n-1)}$. If $\ddot{q} = M(q)$ is the cofactor pair system $\delta^+ E = 0 = \delta^+ \widetilde{E}$, then we know that it is generated by any of its integrals of motions, so that $-2A_\mu M = \nabla k_\mu$. In particular, $M = -\frac{1}{2}\widetilde{A}^{-1}\nabla\widetilde{k}$, which shows that

$$\nabla k_\mu = A_\mu \widetilde{A}^{-1}\nabla\widetilde{k}. \tag{A.6}$$

Similarly, l_μ is determined up to integration constant by

$$\nabla l_\mu = A_\mu A^{-1}\nabla l. \tag{A.7}$$

The relationship between $\widetilde{k}$ and l is by construction given by $K = \widetilde{k}/\det\widetilde{G} = l/\det G$, where K is some solution of the fundamental equations. This is in agreement with the recursion formula (6.4) that we are trying to prove. What needs to be verified is consequently that the expression (6.4) for l_μ as a function of k_μ satisfies (A.7), given that k_μ satisfies (A.6). Rewriting this in terms of $X = \widetilde{G}^{-1}G$, we have to verify that

$$\nabla\left[\det(X+\mu I)\widetilde{k} - \mu k_\mu\right] = \frac{\det(X+\mu I)}{\det X}(X+\mu I)^{-1}X\,\nabla\left[(\det X)\widetilde{k}\right] \tag{A.8}$$

when

$$\nabla k_\mu = \det(X+\mu I)(X+\mu I)^{-1}\nabla\widetilde{k}.$$

With the help of lemma A.1, we find

$$\begin{aligned}
(X+\mu I)\times[\text{RHS of (A.8)}] &= \frac{\det(X+\mu I)}{\det X}X\,\nabla\left[(\det X)\widetilde{k}\right] \\
&= \det(X+\mu I)\frac{X\nabla\det X}{\det X}\,\widetilde{k} + \det(X+\mu I)(X+\mu I-\mu I)\nabla\widetilde{k} \\
&= (X+\mu I)\nabla\det(X+\mu I)\widetilde{k} + (X+\mu I)\det(X+\mu I)\nabla\widetilde{k} - \mu(X+\mu I)\nabla k_\mu \\
&= (X+\mu I)\times[\text{LHS of (A.8)}].
\end{aligned}$$

This completes the proof of (6.4). The inverse formula (6.5) follows immediately, since $k^{(n-1)} = l^{(0)}\det\widetilde{G}/\det G$. $\square$

Proof of proposition 7.4. We need to study how the velocity-dependent parts of the integrals of motion transform under the stated change of variables. Clearly, $\dot{q}^T I\,\dot{q}$ in $\widetilde{E}$ does not change, while $\dot{q}^T(\operatorname{cof}G(q))\,\dot{q}$ in E goes to $(S\dot{q})^T(\operatorname{cof}G(Sq+v))(S\dot{q}) = \dot{q}^T\operatorname{cof}(S^T G(Sq+v)\,S)\dot{q}$, since $S^T = \operatorname{cof}S$ if $S\in SO(n)$. Thus, we must show that we can choose S and v such that $S^T G(Sq+v)\,S$ takes the stated standard form.

Consider first the case $\alpha \neq 0$. Dividing G by $-\alpha$ and adjusting the cofactor chain accordingly, we can assume $\alpha = -1$ without loss of generality. If we then take $v = \beta$ and choose S so as to diagonalize the symmetric matrix $\gamma + \beta\beta^T$, i.e., $S^T(\gamma + \beta\beta^T)S = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$, it is easily verified that $S^T G(Sq+v)\, S = -qq^T + \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$.

Similarly, in the case $\alpha = 0$ we can assume that the vector β is normalized. Direct calculation shows that $S^T G(Sq+v)\, S = (S^T\beta)q^T + q(S^T\beta)^T + S^T(\gamma + \beta v^T + v\beta^T)S$, which, if the last column in the orthogonal matrix S equals β, equals $e_n q^T + q e_n^T + S^T\gamma S + e_n v^T + v e_n^T$. Now, to choose the remaining columns of S, let R be any orthogonal matrix with last column β, and let P be an orthogonal $(n-1)\times(n-1)$ matrix which diagonalizes the upper left $(n-1)\times(n-1)$ block Q in $R^T\gamma R$, i.e., $P^T Q P = \operatorname{diag}(\lambda_1, \ldots, \lambda_{n-1})$. Setting

$$S = R \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix},$$

we find $S^T\gamma S = \operatorname{diag}(\lambda_1, \ldots, \lambda_{n-1}, 0) + e_n c^T + c e_n^T$ for some vector c. Finally we complete the proof by taking $v = -c$, which gives $S^T G(Sq+v)\, S = e_n q^T + q e_n^T + \operatorname{diag}(\lambda_1, \ldots, \lambda_{n-1}, 0)$. □

Proof of proposition 7.5. We have $\prod_{i=1}^n (z-u_i) = \det(zI - G) = \det(qq^T + \operatorname{diag}(z-\lambda_1, \ldots, z-\lambda_n))$, so the statement follows from taking $\mu_i = z - \lambda_i$ in the identity

$$\begin{aligned} \det(qq^T + \operatorname{diag}(\mu_1, \ldots, \mu_n)) &= \prod_{i=1}^n \mu_i + \sum_{m=1}^n q_m^2 \left(\prod_{\substack{i=1 \\ i\neq m}}^n \mu_i \right) \\ &= \left(1 + \sum_{m=1}^n \frac{q_m^2}{\mu_m} \right) \prod_{i=1}^n \mu_i, \end{aligned} \tag{A.9}$$

which can be proved by induction on the dimension n, as follows. It is obviously true for $n = 1$. Let $A(q) = \operatorname{cof}(qq^T + \operatorname{diag}(\mu_1, \ldots, \mu_n))$. The diagonal entries A_{aa} are determinants of the same form as the one we are computing, so they are $A_{aa} = \prod_{i\neq a} \mu_i + \sum_{m\neq a} q_m^2 (\prod_{i\neq m,a} \mu_i)$ by the induction hypothesis. From them the off-diagonal entries are found, using the cyclic conditions $\partial_a A_{ab} = -\frac{1}{2}\partial_b A_{aa}$ (theorem 3.4), to be $A_{ab} = -q_a q_b \prod_{i\neq a,b} \mu_i$ (there can be no constant term since all entries from row a in G that occur in the determinant A_{ab} contain the factor q_a). A cofactor expansion along any row or column now yields (A.9). □

Proof of proposition 7.6. This is similar to the elliptic case, but easier.

The proposition follows quickly once we prove

$$\begin{aligned}
&\det(e_n q^T + q e_n^T + \operatorname{diag}(\mu_1, \ldots, \mu_n)) \\
&\quad = \left(\prod_{i=1}^{n-1} \mu_i\right)(2q_n + \mu_n) - \sum_{m=1}^{n-1} q_m^2 \left(\prod_{\substack{i=1 \\ i\neq m}}^{n-1} \mu_i\right) \\
&\quad = \left(\prod_{i=1}^{n-1} \mu_i\right)\left(2q_n + \mu_n - \sum_{m=1}^{n-1} \frac{q_m^2}{\mu_m}\right).
\end{aligned} \tag{A.10}$$

The elements in the cofactor matrix which correspond to nonzero elements in the first column are A_{11}, which by induction is given by (A.10) with sum and product indices starting from 2 instead of 1, and $A_{n1} = -q_1 \prod_{i=2}^{n-1} \mu_i$. Cofactor expansion along the first column completes the proof. □

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