The Canada Day Theorem

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The Canada Day Theorem:

$$X = \text{any symmetric } n \times n \text{ matrix}$$

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 1 & 0 & 0 & 0 & \\ 2 & 2 & 1 & 0 & 0 & \\ 2 & 2 & 2 & 1 & 0 & \\ 2 & 2 & 2 & 2 & 1 & \\ \vdots & & & & & \ddots \end{pmatrix} \qquad (\text{size } n \times n)$$

The sum of the **principal** $k \times k$ minors of TX

equals

the sum of **all** $k \times k$ minors of X

Example: $n = 2$

$$X = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

$$TX = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} a & b \\ 2a+b & 2b+c \end{pmatrix}$$

All 1×1 minors of X	$a+b+b+c$
Principal 1×1 minors of TX	$a+(2b+c)$
All 2×2 minors of X	$\det X$
Principal 2×2 minors of TX	$\det TX$

($\det TX = \det X$, since $\det T = 1$)

Example: $n = 3$

$$X = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}$$

$$TX = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix} = \begin{pmatrix} a & b & c \\ 2a+b & 2b+d & 2c+e \\ 2a+2b+c & 2b+2d+e & 2c+2e+f \end{pmatrix}$$

Easy cases: $\begin{cases} k = 1 \\ k = 3 \end{cases}$ (Like previous page.)

Less obvious case: $k = 2$

$$\begin{vmatrix} a & b \\ b & d \end{vmatrix} + \begin{vmatrix} a & c \\ c & f \end{vmatrix} + \begin{vmatrix} d & e \\ e & f \end{vmatrix} + 2\begin{vmatrix} a & c \\ b & e \end{vmatrix} + 2\begin{vmatrix} b & c \\ d & e \end{vmatrix} + 2\begin{vmatrix} b & c \\ e & f \end{vmatrix} =$$

$$\begin{vmatrix} a & b \\ 2a+b & 2b+d \end{vmatrix} + \begin{vmatrix} a & c \\ 2a+2b+c & 2c+2e+f \end{vmatrix} + \begin{vmatrix} 2b+d & 2c+e \\ 2b+2d+e & 2c+2e+f \end{vmatrix}$$

That was still quite easy to check.

But for $n \geq 4$ it gets more complicated!

(As we will see.)

Background: Why "Canada Day"?

The theorem appears in the paper

Explicit multipeakon solutions of Novikov's cubically nonlinear integrable Camassa–Holm type equation
by Andy Hone, Hans Lundmark, Jacek Szmigielski

There we study a certain $2n$-dimensional integrable Hamiltonian system

$$\dot{q}_k = \dots$$
$$\dot{p}_k = \dots$$

with constants of motion $H_1(q, p), \dots, H_n(q, p)$.

Example: the case $n = 3$

$$H_1 = p_1^2 + p_2^2 + p_3^2 + 2p_1p_2E_{12} + 2p_1p_3E_{13} + 2p_2p_3E_{23}$$

$$\begin{aligned} H_2 = {} & (1 - E_{12}^2)\, p_1^2\, p_2^2 + (1 - E_{13}^2)\, p_1^2\, p_3^2 + (1 - E_{23}^2)\, p_2^2\, p_3^2 \\ & + 2(E_{23} - E_{12}E_{13})\, p_1^2\, p_2\, p_3 + 2(E_{12} - E_{13}E_{23})\, p_1\, p_2\, p_3^2 \end{aligned}$$

$$H_3 = (1 - E_{12}^2)(1 - E_{23}^2)\, p_1^2\, p_2^2\, p_3^2$$

(Abbreviation: $E_{ij} = e^{-\left|q_i - q_j\right|}$)

Question: What's the pattern (for general n)?

Early conjecture (Hans):

$$X = \begin{pmatrix} p_1^2 & p_1 p_2 E_{12} & p_1 p_3 E_{13} \\ p_1 p_2 E_{12} & p_2^2 & p_2 p_3 E_{23} \\ p_1 p_3 E_{13} & p_2 p_3 E_{23} & p_3^2 \end{pmatrix}$$

$H_k =$ sum of all $k \times k$ minors of the $n \times n$ matrix X

whose entries are $X_{ij} = p_i p_j \, e^{-|q_i - q_j|}$

Later theorem (Jacek):

$H_k =$ sum of the principal $k \times k$ minors of TX

Does the theorem agree with the conjecture?

If so, does the particular form of X play any role?

Date: Sun, 29 Jun 2008 23:18:54 +0200 (MEST)
From: Hans Lundmark <halun@mai.liu.se>
To: Jacek Szmigielski <szmigiel@math.usask.ca>
Subject: Re: imaginary spectrum and other stuff

[...]

In fact, the following more general result seems to be true (I know it is true up to size 6x6 by brute force checking on the computer):

[...]

At first, I thought that it could be proved by just a routine application of Binet-Cauchy again, but it appears that it's more subtle than that.

Date: Tue, 01 Jul 2008 01:09:22 -0600
From: Jacek Szmigielski <szmigiel@math.usask.ca>
To: Hans Lundmark <halun@mai.liu.se>
Subject: unfinished computation

Dear Hans,

I did not have enough time to finish the computation. I checked low (in j) dimensional cases, so I have a rough idea what to try. Tomorrow is Canada Day (July 1) so I will be in a festive mood and hopefully your conjecture will turn into a little theorem in honour of that day. I agree that this theorem will be valid for any symmetric matrix. I'll report later in the day.
Bye for now,

Jacek

Jacek in a festive mood

Date: Tue, 01 Jul 2008 23:08:06 +0200 (MEST)
From: Hans Lundmark <halun@mai.liu.se>
To: Jacek Szmigielski <szmigiel@math.usask.ca>
Subject: Re: unfinished computation

Well, July 1 is almost past over here, and I still haven't quite managed to prove the Canada Day Theorem, although I'm getting closer, so I hope your luck is better. The main thing that's missing for my idea to work is a proof of the following identity:

[...]

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Date: Tue, 01 Jul 2008 16:04:53 -0600
From: Jacek Szmigielski <szmigiel@math.usask.ca>
To: Hans Lundmark <halun@mai.liu.se>
Subject: Re: unfinished computation

Hi,
I am struggling here too. It is beautiful outside,
perhaps I should take a walk.
Jacek
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Eventually we did find a proof, which uses:

- The Cauchy–Binet formula
  (minors of a matrix product)
- Lindström's Lemma
  (for computing minors of T)
- Relations among minors of a symmetric matrix
  (the most technical part of the proof)

The Cauchy–Binet formula

For $n \times n$ matrices X and Y, and index sets I and J with k elements each:

$$\boxed{\det(XY)_{IJ} = \sum_M \det X_{IM} \det Y_{MJ}}$$

$X_{IJ} = (X_{i_r j_s})_{r,s=1}^k$ denotes the $k \times k$ submatrix of X formed from rows indexed by I and columns indexed by J.

$I = \{i_1 < i_2 < \cdots < i_k\} \subset \{1, 2, \ldots, n\}$

$J = \{j_1 < j_2 < \cdots < j_k\} \subset \{1, 2, \ldots, n\}$

In the sum, M runs over all index sets with k elements (so there are $\binom{n}{k}$ terms).

Sketch of proof:

$$
\begin{aligned}
\det(XY)_{IJ} &= \det\Big(\sum_{m=1}^{n} X_{i_r m} Y_{m j_s}\Big)_{r,s=1}^{k} \\
&= \sum_{m_1=1}^{n} \cdots \sum_{m_k=1}^{n} \det(X_{i_r m_r} Y_{m_r j_s})_{r,s=1}^{k} \\
&= \sum_{m_1=1}^{n} \cdots \sum_{m_k=1}^{n} (X_{i_1 m_1} \dots X_{i_k m_k}) \underbrace{\det(Y_{m_r j_s})_{r,s=1}^{k}}_{=0 \text{ or } \pm\det Y_{MK}} \\
&= \dots \\
&= \sum_{M} \det X_{IM} \det Y_{MJ}
\end{aligned}
$$

Remark: More abstractly, the formula says that $\bigwedge^k(X \circ Y) = (\bigwedge^k X) \circ (\bigwedge^k Y)$ for linear transformations X and Y on an n-dimensional vector space.

Lindström's Lemma

(Gessel–Viennot Theorem, Karlin–McGregor Theorem)

Planar network = planar acyclic directed graph with

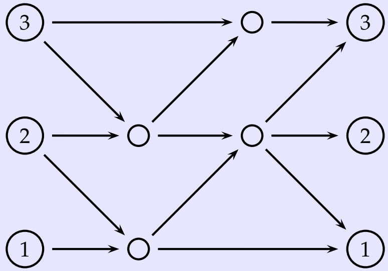
- source nodes numbered $1, 2, \ldots, n$ on the left,
- sink nodes numbered $1, 2, \ldots, n$ on the right,
- all arrows pointing towards the right.

Path matrix Ω:

Ω_{ij} = number of paths from source i to sink j

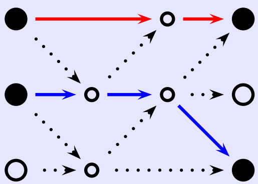
Then the minor $\det \Omega_{IJ}$ is equal to the number of **non-intersecting** (= **node-disjoint**) families of paths from the sources in the set I to the sinks in the set J.

Example:

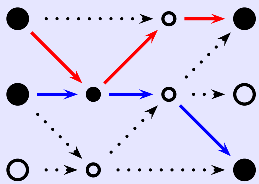


$$\Omega = \begin{pmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 1 & 1 & 3 \end{pmatrix}$$

Some ways of connecting sources $\{2, 3\}$ to sinks $\{1, 3\}$:



non-intersecting



intersecting

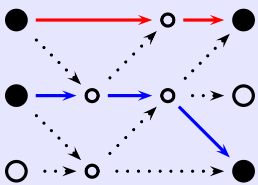
If Lindström was right, there should be

$$\det \Omega_{23,13} = \begin{vmatrix} \cdot & \cdot & \cdot \\ 3 & \cdot & 3 \\ 1 & \cdot & 3 \end{vmatrix} = 6$$

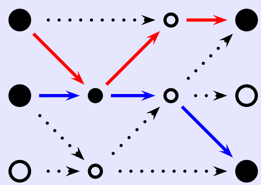
non-intersecting ways to connect them. Do you agree?

$$\det \Omega_{23,13} = \begin{vmatrix} \Omega_{21} & \Omega_{23} \\ \Omega_{31} & \Omega_{33} \end{vmatrix} = \begin{vmatrix} 3 & 3 \\ 1 & 3 \end{vmatrix} = 3 \cdot 3 - 1 \cdot 3 = 6$$

The term $\Omega_{21}\Omega_{33} = 3 \cdot 3$ counts **all** ways of connecting $2 \to 1$ and $3 \to 3$:



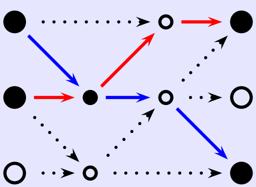
non-intersecting



intersecting

…and seven others.

The term $\Omega_{31}\Omega_{23} = 1 \cdot 3$ counts the ways to connect $3 \to 1$ and $2 \to 3$ (necessarily with intersections):



intersecting

…and two others. They cancel all intersecting contributions above in pairs. (Pairing: swap the colors to the left of the first intersection.) Only the non-intersecting ones remain!

The proof in the general case is similar:

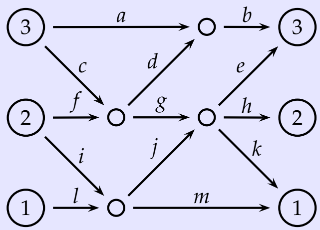
Each term in the minor counts some path families; think of it as $\pm(1+1+\cdots+1)$, where each 1 corresponds to a path family. Each path family from I to J appears exactly once in this expansion, contributing either $+1$ or -1 depending on the sign of the permutation of J naturally associated with it.

[Adjust the graph so that all nodes lie on *different vertical lines*.]

For each **intersecting** path family, find the *leftmost* intersection node, and from the paths that intersect there, pick the two with the lowest numbered source nodes, and swap the parts of those two paths that lie to the left of the intersection.

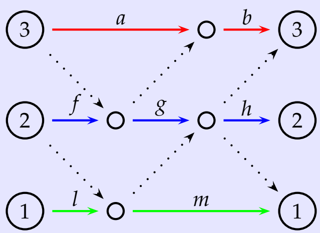
This defines a sign-reversing bijection on the set of intersecting path families, which causes all their contributions to cancel out in pairs. This leaves us with the contributions from the **non-intersecting** path families; each of them gives $+1$, since they are among the families counted by the main diagonal product.

Lindström's Lemma also works for networks with **weights**:



$$\Omega = \begin{pmatrix} lm + ljk & ljh & lje \\ fgk + ijk + im & fgh + ijh & fdb + fge + ije \\ cgk & cgh & ab + cdb + cge \end{pmatrix}$$

For example, when $I = J = \{1, 2, 3\}$ there is only one non-intersecting path family in this network, so in $\det \Omega$ all terms but one cancel out:



$$\det \Omega = abfghlm$$

Remark:

If the weights are nonnegative, then the weighted path matrix will be **totally nonnegative** (all minors ≥ 0), which implies, among other things, that all its eigenvalues are nonnegative.

In fact, the converse is also true: every totally nonnegative matrix is the weighted path matrix of some planar network.

Proof of the Canada Day Theorem

The statement to be proved is that

$$\sum_J \det(TX)_{JJ} = \sum_I \sum_J \det X_{IJ}$$

when X is symmetric, and $T_{ij} = 1 + \operatorname{sgn}(i - j)$.

The matrix size n and the minor size k are fixed.

Sums run over k-subsets of $\{1, 2, \ldots, n\}$ if nothing else is said.

Members of index sets will be numbered in increasing order:

$$I = \{i_1 < i_2 < \cdots < i_k\}$$

By Cauchy–Binet, the statement is the same as

$$\sum_I \sum_J \det T_{JI} \det X_{IJ} = \sum_I \sum_J \det X_{IJ}$$

If $\det T_{JI}$ were 1 for all I and J we would be done already. But that is false for all matrices (except the 1×1 matrix $T = (1)$ which is not very interesting).

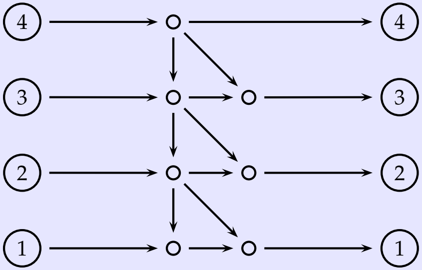
Can it be that $\det T_{JI}$ only assumes the values 0, 1, 2? Then maybe we could use the symmetry of X like this:

$$\det X_{IJ} + \det X_{JI} = 2 \det X_{IJ}.$$

Alas, it's not that simple either (unless $n \leq 3$)...

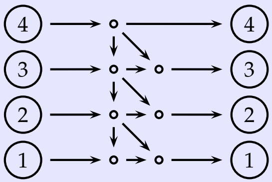
What are the minors $\det T_{JI}$ then?

We can compute them using Lindström's Lemma, since T is the path matrix of a network like this (illustrated for $n = 4$):



Number of paths from source a to sink b:

$$T_{ab} = \begin{cases} 2, & \text{if } a > b \\ 1, & \text{if } a = b \\ 0, & \text{if } a < b \end{cases}$$



The result is:

$$\det T_{JI} = \begin{cases} 2^{p(I,J)}, & \text{if } I \le J \\ 0, & \text{otherwise} \end{cases}$$

Notation:

$$I \le J \quad \Longleftrightarrow \quad i_1 \le j_1 \le \ldots \le i_k \le j_k \quad (\text{“interlacing”})$$

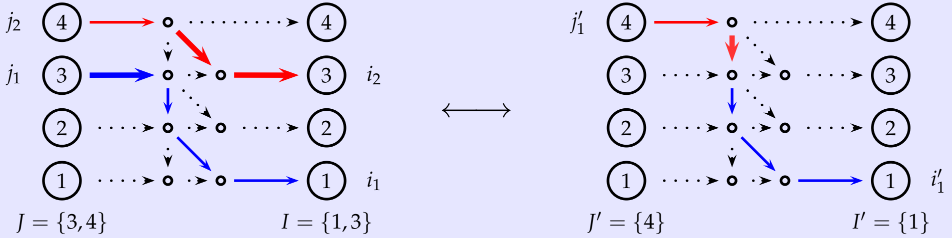
$$I < J \quad \Longleftrightarrow \quad i_1 < j_1 < \ldots < i_k < j_k \quad (\text{“strictly interlacing”})$$

$$p(I,J) = |I'| = |J'|, \quad \text{where } I' = I \setminus (I \cap J), \quad J' = J \setminus (I \cap J)$$

(“removal of shared elements”; note that $I \le J \iff I' < J'$)

$$\det T_{JI} = \begin{cases} 2^{p(I,J)}, & \text{if } I \leq J \\ 0, & \text{otherwise} \end{cases}$$

Proof: Obvious if I and J are disjoint, since then it is only a matter of whether they are strictly interlacing ($I < J$) or not. And if there are shared elements, then they can be removed (one at a time, in the way illustrated below) without affecting the number of non-intersecting path families, so that we can apply the result from the disjoint case to I' and J'.



Now what we have to prove is that

$$\sum_{I \leq J} 2^{p(I,J)} \det X_{IJ} = \sum_{A,B} \det X_{AB}$$

(If we want to actually compute the sum, we should stop here and just use the sum on the left-hand side. But if we want the "sum of all minors" statement, we have more work to do!)

To begin with, how many terms are there?

Right-hand side: $\binom{n}{k}^2$

Left-hand side: not as obvious, but it's $\sum_{p=0}^{k} \binom{n}{p}\binom{n-p}{2(k-p)}$

We'd better look at an example!

Example: $n = 5, k = 3$ $\quad$ (3×3 minors in a 5×5 matrix)

$$\sum_{A,B} \det X_{AB} \quad \text{has } \binom{5}{3}^2 = 100 \text{ terms.}$$

$$\sum_{I \leq J} 2^{p(I,J)} \det X_{IJ} \quad \text{turns out to have 45 terms:}$$

$$2^0 \Big(|X_{123,123}| + |X_{124,124}| + \cdots + |X_{345,345}| \Big)_{\text{10 terms}}$$
$$+ 2^1 \Big(|X_{123,124}| + |X_{123,125}| + |X_{124,125}| + \ldots$$
$$+ |X_{124,234}| + |X_{124,245}| + |X_{234,245}| + \ldots$$
$$+ |X_{145,245}| + |X_{145,345}| + |X_{245,345}| \Big)_{\text{30 terms}}$$
$$+ 2^2 \Big(|X_{124,135}|) + |X_{124,325}| + |X_{134,235}| + |X_{134,245}| + |X_{135,245}| \Big)_{\text{5 terms}}$$

Idea: Each term $2^{p(I,J)} \det X_{IJ}$ (with $I \leq J$) corresponds to its own **group of terms** in $\sum_{A,B} \det X_{AB}$, namely those with the same **shared elements** ($I \cap J = A \cap B$), and also the same set of non-shared elements but in any order ($I \cup J = A \cup B$, but $A \not\leq B$ unless $A = I$ and $B = J$). For example:

$$2^1 \left| X_{123,124} \right| = \left| X_{123,124} \right| + \left| X_{124,123} \right|$$

$$\begin{aligned} 2^2 \left| X_{124,135} \right| &= \left| X_{123,145} \right| + \left| X_{124,135} \right| + \left| X_{125,134} \right| \\ &\quad + \left| X_{134,125} \right| + \left| X_{135,124} \right| + \left| X_{145,123} \right| \end{aligned}$$

First equation: clearly true, since $X_{AB} = X_{BA}$ when $X = X^t$.

Second equation: not obvious (but in fact true; see below).

The grouping of terms indicated on the previous page indeed works in general:

Technical Lemma. Assume that X is symmetric. For given interlacing k-sets $I \leq J$ the identity

$$2^{p(I,J)} \det X_{IJ} = \sum_{A,B} \det X_{AB}$$

holds, where the sum runs over all k-sets A and B such that $A \cap B = I \cap J$ and $A \cup B = I \cup J$.

Proving this will finish the proof of the Canada Day Theorem. To get a feeling for what needs to be done, let us write out the second equation from the previous page in all its gory details…

$$I = \{1,2,4\} \quad J = \{1,3,5\} \quad \text{(interlacing)}$$

$$I' = \{2,4\} \quad J' = \{3,5\} \quad \text{(strictly interlacing)}$$

$$I \cap J = \{1\} \quad I \cup J = \{1,2,3,4,5\} \quad p(I,J) = 2$$

$$
\begin{aligned}
& |X_{123,145}| = X_{11}X_{24}X_{35} + \boldsymbol{X_{15}X_{21}X_{34}} + \boldsymbol{X_{14}X_{25}X_{31}} - \boldsymbol{X_{11}X_{25}X_{34}} - X_{14}X_{21}X_{35} - X_{15}X_{24}X_{31} \\
& |X_{124,135}| = \boldsymbol{X_{11}X_{23}X_{45}} + \boldsymbol{X_{15}X_{21}X_{43}} + \boldsymbol{X_{13}X_{25}X_{41}} - \boldsymbol{X_{11}X_{25}X_{43}} - \boldsymbol{X_{13}X_{21}X_{45}} - \boldsymbol{X_{15}X_{23}X_{41}} \\
& |X_{125,134}| = \boldsymbol{X_{11}X_{23}X_{54}} + X_{14}X_{21}X_{53} + X_{13}X_{24}X_{51} - X_{11}X_{24}X_{53} - \boldsymbol{X_{13}X_{21}X_{54}} - \boldsymbol{X_{14}X_{23}X_{51}} \\
& |X_{134,125}| = \boldsymbol{X_{11}X_{32}X_{45}} + X_{15}X_{31}X_{42} + X_{12}X_{35}X_{41} - X_{11}X_{35}X_{42} - \boldsymbol{X_{12}X_{31}X_{45}} - \boldsymbol{X_{15}X_{32}X_{41}} \\
& |X_{135,124}| = \boldsymbol{X_{11}X_{32}X_{54}} + \boldsymbol{X_{14}X_{31}X_{52}} + \boldsymbol{X_{12}X_{34}X_{51}} - \boldsymbol{X_{11}X_{34}X_{52}} - \boldsymbol{X_{12}X_{31}X_{54}} - \boldsymbol{X_{14}X_{32}X_{51}} \\
+ \quad & |X_{145,123}| = X_{11}X_{42}X_{53} + \boldsymbol{X_{13}X_{41}X_{52}} + \boldsymbol{X_{12}X_{43}X_{51}} - \boldsymbol{X_{11}X_{43}X_{52}} - X_{12}X_{41}X_{53} - X_{13}X_{42}X_{51} \\
\hline
& \sum_{\substack{A\cap B=\{1\} \\ A\cup B=\{1,2,3,4,5\}}} \det X_{AB} = \qquad 4 \det X_{124,135} = 2^{p(I,J)} \det X_{IJ}
\end{aligned}
$$

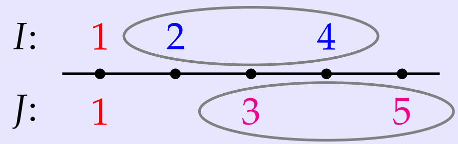
Some terms cancel in pairs (as indicated by the colors).

(Mechanism, explained below: Terms are of two kinds – some terms contain a “hostile pair of indices”, some don’t. Sign-reversing bijection on terms of the first kind: “flip the lowest hostile pair”.)

The remaining ones (boldface) add up to the desired result.

Here's how it works:

The elements in I' hate each other!



The elements in J' also hate each other!

The first of the determinants $\det X_{AB}$ from the previous page:

$$\det X_{\underset{A}{123},\underset{B}{145}} = X_{11}X_{24}X_{35} + \cdots - X_{11}X_{25}X_{34} - X_{14}X_{21}X_{35} - \ldots$$

Hostile pairs Friendly pairs Hostile pairs

Shared elements act as links: $X_{21}X_{14}$ means that 2 is linked with 4, forming a hostile pair.

More generally: $a \in A \setminus (\boldsymbol{I} \cap \boldsymbol{J})$ is said to be linked with $b \in B \setminus (\boldsymbol{I} \cap \boldsymbol{J})$ in a term if that term contains either the factor X_{ab} or a chain of factors $X_{a\boldsymbol{m_1}}X_{\boldsymbol{m_1 m_2}} \ldots X_{\boldsymbol{m_{s-1}m_s}}X_{\boldsymbol{m_s}b}$ with $\boldsymbol{m_1}, \ldots, \boldsymbol{m_s} \in \boldsymbol{I} \cap \boldsymbol{J}$.

Flipping a linked pair (a,b) in a term means swapping the indices in all factors involved in the linking:

$$X_{ab} \to X_{ba} \qquad \text{or} \qquad X_{a\boldsymbol{m_1}}X_{\boldsymbol{m_1m_2}}\ldots X_{\boldsymbol{m_{s-1}m_s}}X_{\boldsymbol{m_s}b} \to X_{b\boldsymbol{m_s}}X_{\boldsymbol{m_sm_{s-1}}}\ldots X_{\boldsymbol{m_2m_1}}X_{\boldsymbol{m_1}a}$$

Since X is symmetric, this gives a new term with the same numerical value, but appearing somewhere else (and possibly with a different sign) in the expansion of $\sum_{A,B} \det X_{AB}$.

Facts (to be proved shortly):

Flipping a **friendly** pair gives another term appearing with the **same** sign in the expansion of $\sum_{A,B} \det X_{AB}$.

Flipping a **hostile** pair gives another term appearing with the **opposite** sign in the expansion of $\sum_{A,B} \det X_{AB}$.

Given this, we can prove the Technical Lemma:

"*Flip the hostile pair that contains the smallest number*" is a sign-reversing involution on the set of terms in the expansion

$$\sum_{A,B} \det X_{AB} = \sum_{A,B} \sum_{\sigma} (-1)^{\sigma} X_{a_1 b_{\sigma(1)}} X_{a_2 b_{\sigma(2)}} \ldots$$

that contain at least one hostile pair. Thus those terms will cancel in pairs.

The remaining terms contain only friendly pairs. There are $p(I, J)$ pairs in each term, and each can be flipped either way without changing the term's value or sign. Exactly one way of flipping the pairs will give a term appearing in the expansion of $\det X_{IJ}$, so the friendly terms add up to $2^{p(I,J)} \det X_{IJ}$.

Now it remains to prove the sign rules for flipping…

Again, the ideas are best seen in an example, but our previous example is too small.

Here is a bigger example ($n \geq 8$, $k = 6$, four elements shared):

$$\begin{array}{lcccccccc} I: & 1 & 2 & & 4 & 5 & 6 & & 8 \\ & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ J: & & 2 & 3 & 4 & 5 & & 7 & 8 \end{array}$$

The sum $\sum_{A,B} \det X_{AB}$ contains $\binom{4}{2} = 6$ terms, each of which expands into $k! = 720$ terms. Let's focus on a single term in the expansion of one particular $\det X_{AB}$:

$$\det X_{234568,124578} = \cdots - X_{28} X_{34} X_{42} X_{55} X_{61} X_{87} + \ldots$$

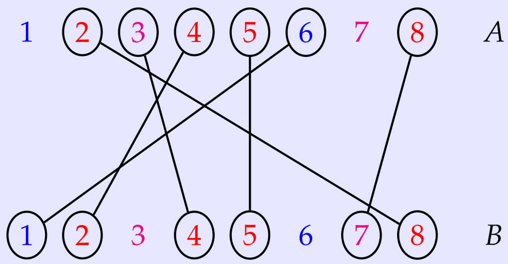
The term chosen as an example here contains two hostile pairs:
$(6, 1)$ because of the factor X_{61}
$(3, 7)$ because of the factors $X_{34} X_{42} X_{28} X_{87}$

Let's see what happens when we flip the pair $(3,7)$!

Term before flipping:

$X_{28}X_{34}X_{42}X_{55}X_{61}X_{87}$

Appears with **negative** sign in $\det X_{234568,124578}$, since the crossing number is 9 (**odd**).

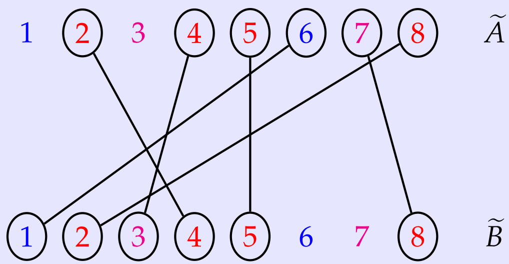


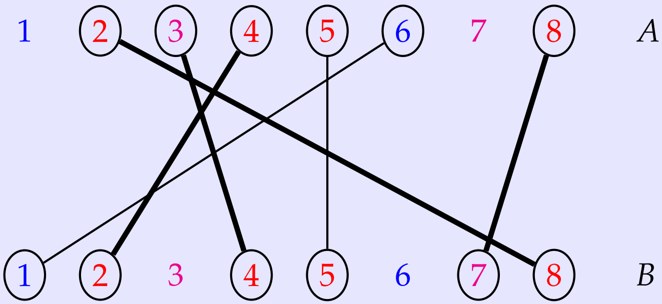
Flip: $X_{34}X_{42}X_{28}X_{87} \longrightarrow X_{78}X_{82}X_{24}X_{43}$

Term after flipping:

$X_{24}X_{43}X_{55}X_{61}X_{78}X_{82}$

Appears with **positive** sign in $\det X_{245678,123458}$, since the crossing number is 8 (**even**).



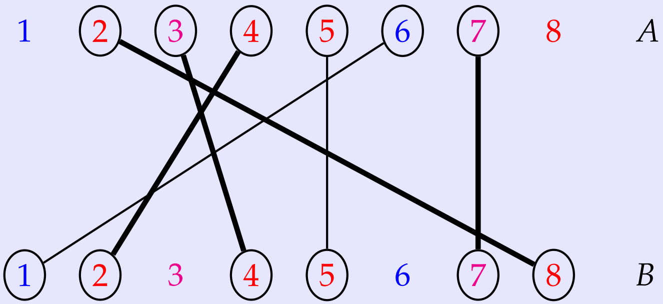


Crossings: 9

Before flip

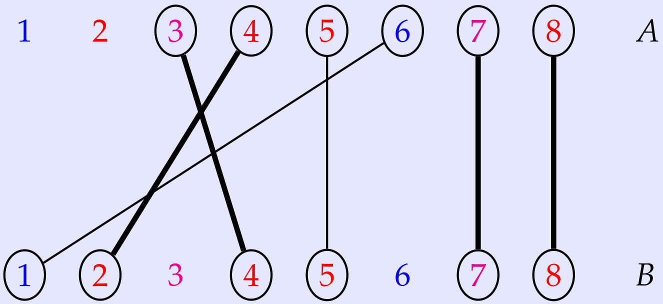
Flipping the pair $(3, 7)$ involves **reflecting** the **thick edges** in the figure ($\searrow$ to $\swarrow$ and vice versa).

By breaking this process down into small steps, we can keep track of how the crossing number changes.



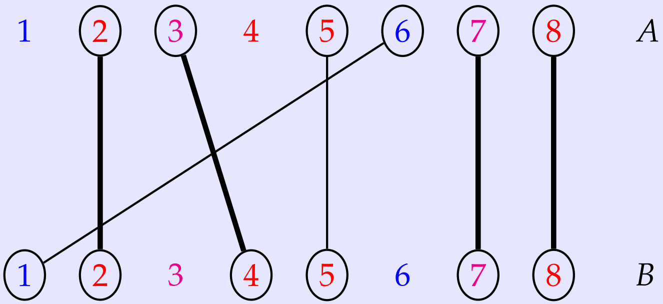
Crossings: 9

Stage I, step 1



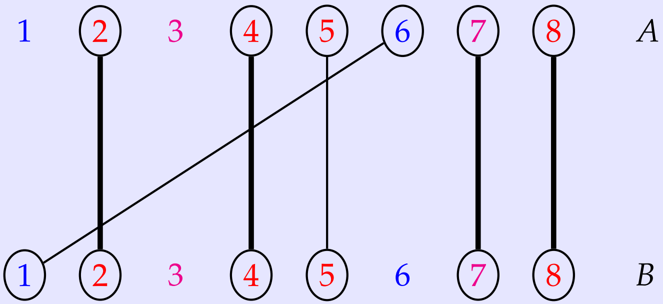
Crossings: 4

Stage I, step 2



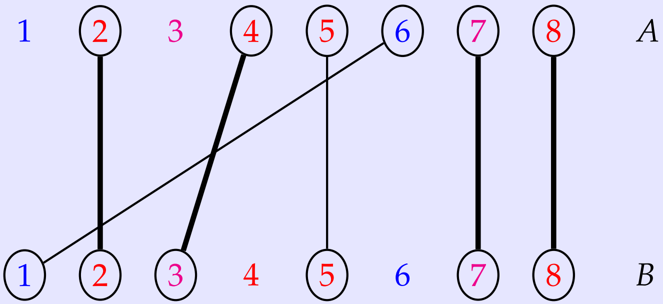
Crossings: 3

Stage I, step 3



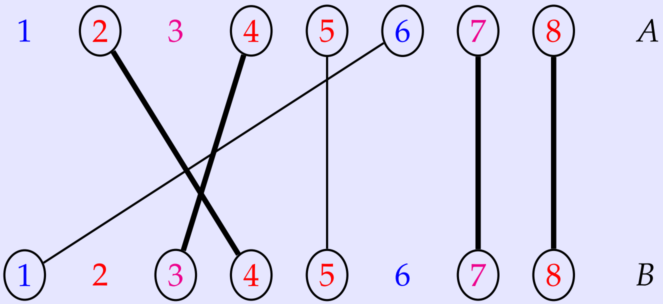
Crossings: 3

Stage I, done



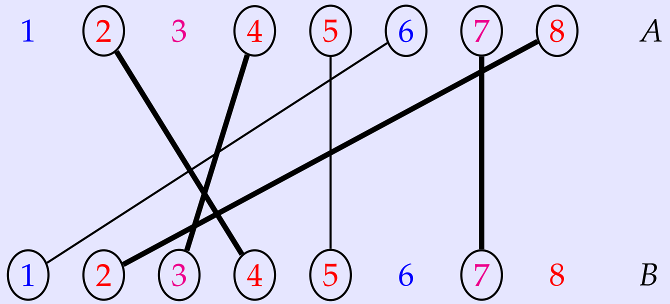
Crossings: 3

Stage II, step 1



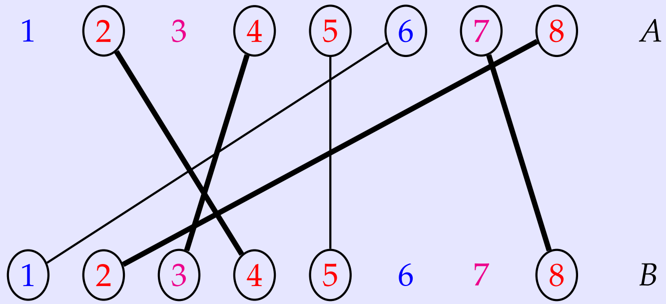
Crossings: 4

Stage II, step 2



Crossings: 8

Stage II, step 3



Crossings: 8

Stage II, done

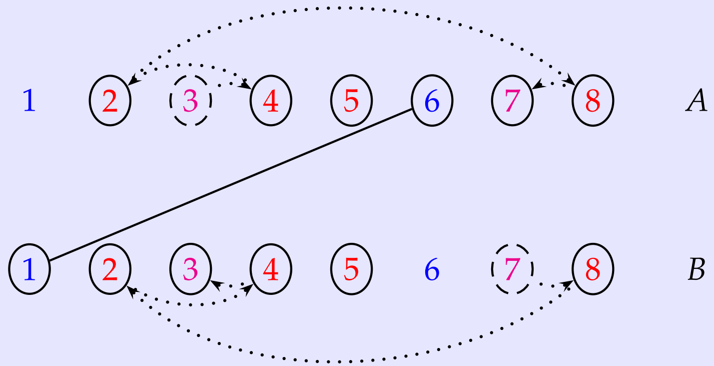
The flip is completed. How did the crossing number change?

The crossing number changes parity whenever we move the end of an edge past some node that has an edge attached to it (= past a circled node).

The **thick edges** have the same intersections among themselves before and after the flip, so we need not count how many times we pass their nodes.

Other nodes with **shared elements** are passed the same number of times in Stage I as in Stage II, so they don't cause any net change in parity. Therefore we need not count how many times we pass those nodes either.

With irrelevant edges removed, here are the moves that we made:



We see that we really only passed one node that made a change to the parity of the crossing number (node 6, during Stage I).

In general, we count the number of times we pass a node lying in $A \setminus (\boldsymbol{I} \cap \boldsymbol{J})$ (during Stage I) or in $B \setminus (\boldsymbol{I} \cap \boldsymbol{J})$ (during Stage II), except that we don't count passing the two nodes that are being flipped.

[Again: We count the number of times we pass a node lying in $A \setminus (\boldsymbol{I} \cap \boldsymbol{J})$ (during Stage I) or in $B \setminus (\boldsymbol{I} \cap \boldsymbol{J})$ (during Stage II), except that we don't count passing the two nodes that are being flipped.]

Such nodes will be passed an **odd** number of times if they lie **between** the two being flipped, and an **even** number of times otherwise.

So **the parity of the crossing number changes** iff an **odd** number of nodes in the set

$$(A \cup B) \setminus (\boldsymbol{I} \cap \boldsymbol{J}) = (I \cup J) \setminus (\boldsymbol{I} \cap \boldsymbol{J})$$

lie between the two nodes that are being flipped.

And since I' and J' are strictly interlacing, this is exactly the case when the two nodes being flipped both belong to I' or both to J'; in other words, when they form a **hostile** pair.

QED (phew!)