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Orthogonal and biorthogonal polynomials in the theory of peakons

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Camassa–Holm peakons
Discrete string
Orthogonal polynomials

versus

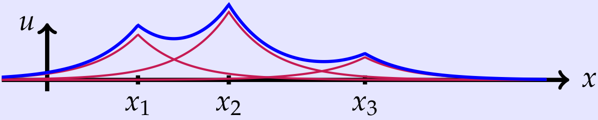
Degasperis–Procesi peakons
(& Novikov & Geng–Xue)
Discrete cubic string
Cauchy biorthogonal polynomials

The Camassa–Holm shallow water equation

$$\boxed{m_t + m_x u + 2 m u_x = 0 \qquad (m = u - u_{xx})}$$

admits "multipeakon" solutions of the form

$$\boxed{u(x,t) = \sum_{i=1}^{n} m_i(t)\, e^{-|x - x_i(t)|}}$$



$u = \sum m_i \, e^{-|x - x_i|}$ satisfies the PDE iff

$$\boxed{\dot{x}_k = u(x_k) \qquad \dot{m}_k = -m_k \langle u_x(m_k) \rangle}$$

which is shorthand notation for

$$\dot{x}_k = \sum_{i=1}^{n} m_i \, e^{-|x_k - x_i|}$$
$$\dot{m}_k = \sum_{i=1}^{n} m_k \, m_i \, \operatorname{sgn}(x_k - x_i) \, e^{-|x_k - x_i|}$$

(Geodesics for metric $g^{ij} = e^{-\left|x_i - x_j\right|}$.)

Explicit formulas for general n-peakon solution found by Beals, Sattinger & Szmigielski (2000).

Example. Formulas for positions when $n = 3$.

$$x_1(t) = \ln \frac{(\lambda_1 - \lambda_2)^2(\lambda_1 - \lambda_3)^2(\lambda_2 - \lambda_3)^2 b_1 b_2 b_3}{\sum_{j<k} \lambda_j^2 \lambda_k^2 (\lambda_j - \lambda_k)^2 b_j b_k}$$

$$x_2(t) = \ln \frac{\sum_{j<k} (\lambda_j - \lambda_k)^2 b_j b_k}{\lambda_1^2 b_1 + \lambda_2^2 b_2 + \lambda_3^2 b_3}$$

$$x_3(t) = \ln(b_1 + b_2 + b_3)$$

where $\lambda_k = \text{constant} \neq 0$, $b_k(t) = b_k(0)\, e^{t/\lambda_k} > 0$.

In general:

$$x_{n+1-k} = \ln \frac{\Delta_k^0}{\Delta_{k-1}^2} \qquad m_{n+1-k} = \frac{\Delta_k^0 \Delta_{k-1}^2}{\Delta_k^1 \Delta_{k-1}^1}$$

in terms of Hankel determinants of moments

$$\Delta_k^a = \begin{vmatrix} \beta_a & \beta_{a+1} & \beta_{a+2} & \cdots \\ \beta_{a+1} & \beta_{a+2} & & \\ \beta_{a+2} & & & \\ \vdots & & & \end{vmatrix} \qquad \text{(size } k \times k\text{)}$$

$$\beta_i = \int y^i \, d\mu(y) \quad \left(= \sum_{j=1}^{n} b_j \lambda_j^i \text{ since } \mu = \sum_{j=1}^{n} b_j \delta_{\lambda_j} \text{ here} \right)$$

Cf. orthogonal polynomials:

$$\begin{cases} \text{ON in } L^2(\mu): \quad \int P_r(y) P_s(y) d\mu(y) = \delta_{rs} \\ P_k \text{ has degree } k \text{ with positive leading coeff.} \end{cases} \implies$$

$$P_k(y) = \frac{1}{\sqrt{\Delta_k^0 \Delta_{k+1}^0}} \begin{vmatrix} \beta_a & \beta_{a+1} & \beta_{a+2} & \cdots & \\ \beta_{a+1} & \beta_{a+2} & & & \\ \beta_{a+2} & & & & \\ \vdots & & & & \\ 1 & y & y^2 & \cdots & y^k \end{vmatrix}$$

For example, $x_k = x_{k+1}$ iff $P_{n-k}(0) = 0$.
(Useful for study of peakon–antipeakon collisions.)

How the spectral measure $\mu = \sum_{j=1}^{n} b_j \delta_{\lambda_j}$ arises:

$(\partial_x^2 - \frac{1}{4})\psi = -\frac{1}{2} z\, m\, \psi$ (from CH Lax pair) can be transformed to **string equation** $\boxed{\phi'' = -z\, g\, \phi}$ on the finite interval $(-1, 1)$.

For peakons, m & g are discrete measures.

$\phi(-1; z) = 0$, $\phi'(-1; z) = 1$ defines **Weyl function**

$$W(z) = \frac{\phi'(1; z)}{\phi(1; z)} = \frac{z}{2} \left(\frac{1}{z} + \sum_{j=1}^{n} \frac{b_j}{z - \lambda_j} \right)$$

Degasperis–Procesi equation (1998):

$$\boxed{m_t + m_x u + 3 m u_x = 0 \qquad (m = u - u_{xx})}$$

(Cf. Camassa–Holm $m_t + m_x u + 2 m u_x = 0$.)

Peakons: $\dot{x}_k = u(x_k) \qquad \dot{m}_k = -\mathbf{2}\, m_k \langle u_x(x_k) \rangle$

Explicit n-peakon solution involving less familiar-looking expressions than for CH.

(Lundmark & Szmigielski 2003, 2005)

Example. DP peakons, $n = 3$

$$x_1(t) = \ln \frac{U_3}{V_2} \quad m_1(t) = \frac{U_3 (V_2)^2}{V_3 W_2}$$
$$x_2(t) = \ln \frac{U_2}{V_1} \quad m_2(t) = \frac{(U_2)^2 (V_1)^2}{W_2 W_1}$$
$$x_3(t) = \ln U_1 \quad m_3(t) = \frac{(U_1)^2}{W_1}$$

with abbreviations explained on next page.
(Time evolution $b_k(t) = b_k(0)\, e^{t/\lambda_k}$ as before.)

$$U_1 = b_1 + b_2 + b_3 \qquad V_1 = \lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3$$

$$U_2 = \frac{(\lambda_1-\lambda_2)^2}{\lambda_1+\lambda_2} b_1 b_2 + \frac{(\lambda_1-\lambda_3)^2}{\lambda_1+\lambda_3} b_1 b_3 + \frac{(\lambda_2-\lambda_3)^2}{\lambda_2+\lambda_3} b_2 b_3$$

$$V_2 = \frac{(\lambda_1-\lambda_2)^2}{\lambda_1+\lambda_2} \lambda_1\lambda_2 b_1 b_2 + \frac{(\lambda_1-\lambda_3)^2}{\lambda_1+\lambda_3} \lambda_1\lambda_3 b_1 b_3 + \frac{(\lambda_2-\lambda_3)^2}{\lambda_2+\lambda_3} \lambda_2\lambda_3 b_2 b_3$$

$$U_3 = \frac{(\lambda_1-\lambda_2)^2(\lambda_1-\lambda_3)^2(\lambda_2-\lambda_3)^2}{(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)(\lambda_2+\lambda_3)} b_1 b_2 b_3$$

$$V_3 = \lambda_1\lambda_2\lambda_3 U_3$$

$$W_1 = U_1 V_1 - U_2 \qquad W_2 = U_2 V_2 - U_3 V_1$$

What's going on?

Explanatory framework:

Cauchy biorthogonal polynomials

$$\int p_i(x)\, q_j(y)\, \underbrace{\frac{1}{x+y}}_{\text{Cauchy kernel}}\, d\alpha(x)\, d\beta(y) = \delta_{ij}$$

(Bertola, Gekhtman & Szmigielski 2009, 2010)

Also relevant to other integrable peakon equations.

V. Novikov (2008):

$$m_t + (m_x u + 3 m u_x) u = 0 \qquad (m = u - u_{xx})$$

(Hone, Lundmark & Szmigielski 2009)

Geng–Xue (2009):

$$\begin{aligned} m_t + (m_x u + 3 m u_x) v = 0 \qquad & (m = u - u_{xx}) \\ n_t + (n_x v + 3 n v_x) u = 0 \qquad & (n = v - v_{xx}) \end{aligned}$$

(Lundmark & Szmigielski, work in progress)

$(\partial_x^3 - \partial_x)\psi = -z\, m\, \psi$ (from DP Lax pair) can be transformed to **cubic string equation** $\boxed{\phi''' = -z\, g\, \phi}$ on the finite interval $(-1, 1)$.

(For peakons, *discrete* cubic string: $g = \sum_{k=1}^{n} g_k\, \delta_{y_k}$.)

Two Weyl functions:

$$W(z) = \frac{\phi'(1;z)}{\phi(1;z)} \qquad Z(z) = \frac{\phi''(1;z)}{\phi(1;z)}$$

where $\phi(-1;z) = \phi'(-1;z) = 0$, $\phi''(-1;z) = 1$.

$W(z)$ and $Z(z)$ have poles where $\phi(1;z) = 0$.

In other words: at the *eigenvalues* $z = \lambda_k$ of the discrete cubic string $\phi''' = -z\,g\,\phi$ with boundary conditions

$$\phi = \phi' = 0 \text{ at } y = -1, \quad \phi = 0 \text{ at } y = +1.$$

(When all $g_k > 0$, the eigenvalues λ_k are real and positive because of **total positivity**.)

$$\frac{W(z)}{z} = \frac{1}{z} + \sum_{k=1}^{n} \frac{b_k}{z - \lambda_k} \qquad \frac{Z(z)}{z} = \frac{1}{2z} + \sum_{k=1}^{n} \frac{c_k}{z - \lambda_k}$$

The even part of $Z(z) = \frac{\phi''(1)}{\phi(1)}$ is determined by $W(z) = \frac{\phi'(1)}{\phi(1)}$.

(For $g \geq 0$ this means that W determines Z completely.)

Proof. Let $\eta(y;z) = \phi(y;-z)$.
Then $\phi''' = -zg\phi$ and $\eta''' = +zg\eta$ so that

$$0 = \eta\phi''' + \eta'''\phi = (\eta\phi'' - \eta'\phi' + \eta''\phi)'.$$

Integration over $-1 \leq y \leq 1$ gives

$$0 = \eta(1)\,\phi''(1) - \eta'(1)\,\phi'(1) + \eta''(1)\,\phi(1)$$

since the boundary conditions kill contributions from $y = -1$.

Division by $\eta(1)\,\phi(1)$ gives

$$Z(z) - W(-z)\,W(z) + Z(-z) = 0. \qquad \square$$

With $\frac{W(z)}{z} = \frac{1}{z} + \sum \frac{b_k}{z-\lambda_k}$ and $\frac{Z(z)}{z} = \frac{1}{2z} + \sum \frac{c_k}{z-\lambda_k}$ we see that Z is completely determined by W (unless some $\lambda_i + \lambda_j = 0$):

$$Z(z) - W(-z)\,W(z) + Z(-z) = 0$$

$$\operatorname*{res}_{z=\lambda_k} \left(\frac{Z(z)}{z} - W(-z)\,\frac{W(z)}{z} + \frac{Z(-z)}{z} \right) = 0$$

$$c_k - W(-\lambda_k)\,b_k + 0 = 0$$

$$c_k = \left(1 + \sum_{j=1}^{n} \underbrace{\frac{\lambda_k b_j}{\lambda_k + \lambda_j}}_{\text{Cauchy kernel appears here!}} \right) b_k$$

Inverse spectral problem for the discrete cubic string: determine $\{y_k, g_k\}_{k=1}^n$ from $\{\lambda_k, b_k\}_{k=1}^n$.

Explicit solution in terms of determinants of **bi-moments** (with respect to the Cauchy kernel) of the spectral measure $\mu = \sum_{k=1}^n b_k \, \delta_{\lambda_k}$:

$$I_{ab} = \iint \frac{x^a y^b}{x+y} \, d\mu(x) \, d\mu(y) = \sum_{i,j=1}^{n} \frac{\lambda_i^a \lambda_j^b b_i b_j}{\lambda_i + \lambda_j}$$

A curious approximation problem plays a key role:

For $1 \leq k \leq n$, seek polynomials $Q(z)$, $P(z)$, $\widehat{P}(z)$ of degree $k-1$ such that

- $W = \dfrac{P}{Q} + \mathcal{O}(z^{-(k-1)}), \quad Z = \dfrac{\widehat{P}}{Q} + \mathcal{O}(z^{-(k-1)})$
- $Z^*Q + W^*P + \widehat{P} = \mathcal{O}(z^{-k}) \qquad \begin{pmatrix} W^*(z) := -W(-z) \\ Z^*(z) := Z(-z) \end{pmatrix}$
- $P(0) = 1, \quad \widehat{P}(0) = 0$

(Similar to Hermite–Padé, but the degrees are too low, and the functions W & Z to be approximated are not independent.)

All this fits into the general theory of Cauchy biorthogonal polynomials.

And we'll finish with a quick glimpse of that theory.

Let α and β be measures supported on the positive real axis, with finite moments

$$\alpha_k = \int x^k \, d\alpha(x), \qquad \beta_k = \int y^k \, d\beta(y)$$

and finite Cauchy bimoments

$$I_{ab} = \iint \frac{x^a y^b}{x+y} \, d\alpha(x) \, d\beta(y).$$

Then

$$\langle f \mid g \rangle = \iint \frac{f(x)\, g(y)}{x+y} \, d\alpha(x) \, d\beta(y)$$

is a bilinear form on the space of polynomials (not symmetric in general).

There are unique **monic** polynomials $\{\widetilde{p}_k, \widetilde{q}_k\}_{k=0}^{\infty}$, with degree equal to the subscript, such that

$$\langle \, \widetilde{p}_i \, | \, \widetilde{q}_j \, \rangle = h_i \delta_{ij}.$$

They can be written in terms of the bimoments:

$$\widetilde{p}_k(x) = \frac{1}{D_k} \begin{vmatrix} I_{00} & I_{01} & \cdots & 1 \\ I_{10} & I_{11} & \cdots & x \\ \vdots & \vdots & & \vdots \\ I_{k0} & I_{k1} & \cdots & x^k \end{vmatrix} \quad \widetilde{q}_k(y) = \frac{1}{D_k} \begin{vmatrix} I_{00} & I_{01} & \cdots & I_{0k} \\ I_{10} & I_{11} & \cdots & I_{1k} \\ \vdots & \vdots & & \vdots \\ 1 & y & \cdots & y^k \end{vmatrix}$$

where D_k is the $k \times k$ bimoment determinant which starts with I_{00} in the upper left corner.

Normalized biorthogonal polynomials $\{p_k, q_k\}_{k=0}^{\infty}$ such that

$$\langle\, p_i \,|\, q_j \,\rangle = \delta_{ij}$$

are given by

$$p_k(x) = \frac{1}{\sqrt{D_k D_{k+1}}} \begin{vmatrix} I_{00} & I_{01} & \cdots & 1 \\ I_{10} & I_{11} & \cdots & x \\ \vdots & \vdots & & \vdots \\ I_{k0} & I_{k1} & \cdots & x^k \end{vmatrix} = \sqrt{\frac{D_k}{D_{k+1}}}\, x^k + \cdots$$

$$q_k(y) = \frac{1}{\sqrt{D_k D_{k+1}}} \begin{vmatrix} I_{00} & I_{01} & \cdots & I_{0k} \\ I_{10} & I_{11} & \cdots & I_{1k} \\ \vdots & \vdots & & \vdots \\ 1 & y & \cdots & y^k \end{vmatrix} = \sqrt{\frac{D_k}{D_{k+1}}}\, y^k + \cdots$$

A basic property of the Cauchy kernel is that

$$\begin{aligned} I_{a+1,b} + I_{a,b+1} &= \iint \frac{x^{a+1}y^b + x^a y^{b+1}}{x+y}\, d\alpha(x)\, d\beta(y) \\ &= \iint x^a y^b \, d\alpha(x)\, d\beta(y) = \alpha_a \beta_b. \end{aligned}$$

For the Hessenberg matrices X and Y given by

$$x \begin{pmatrix} p_0(x) \\ p_1(x) \\ \vdots \end{pmatrix} = X \begin{pmatrix} p_0(x) \\ p_1(x) \\ \vdots \end{pmatrix}, \quad y \begin{pmatrix} q_0(y) \\ q_1(y) \\ \vdots \end{pmatrix} = Y \begin{pmatrix} q_0(y) \\ q_1(y) \\ \vdots \end{pmatrix},$$

this implies that $X + Y^t$ has rank 1, which leads to **four-term recurrence relations** for p_k & q_k.

In addition:

- interlacing of zeros
- Christoffel–Darboux-type identities
- Hermite–Padé-like approximation
- Riemann–Hilbert problems
- random matrix models

and so on. But that's another talk!

THE END