Explicit solutions of Novikov's peakon equation

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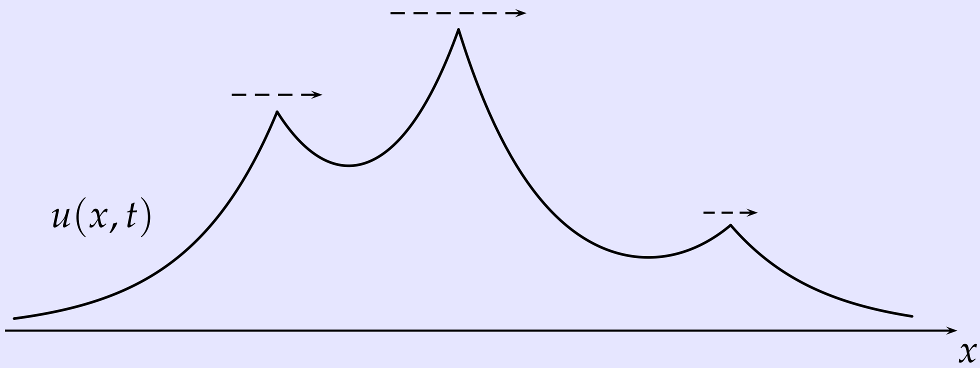
XXVIII Workshop on Geometric Methods in Physics

Białowieża 2009

Peakons = peaked solitons

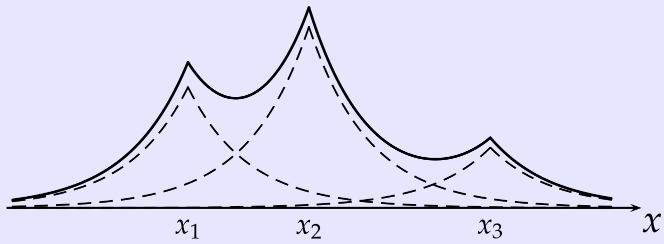
Some PDEs have wave solutions looking like this:



Main example: **Camassa–Holm** shallow water equation (1993)

$$m_t + m_x u + 2 m u_x = 0, \qquad \text{where} \quad m = u - u_{xx}$$

Simple, explicit structure: $\quad u(x,t) = \sum_{i=1}^{n} m_i(t)\, e^{-|x - x_i(t)|}$



Positions $\qquad x_1(t) < x_2(t) < \cdots < x_n(t)$

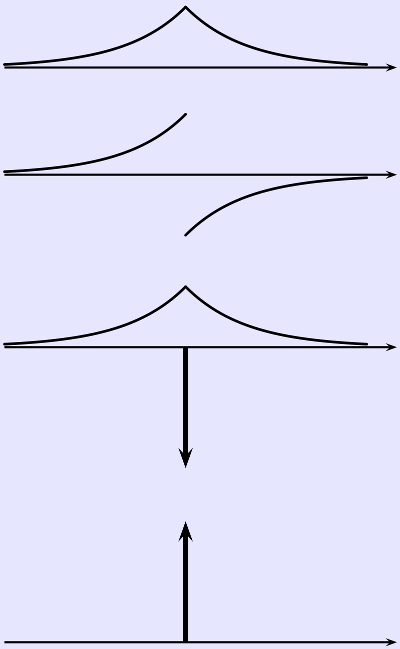
Amplitudes $\qquad m_1(t), m_2(t), \ldots, m_n(t)$

Dynamics given by a nonlinear system of ODEs:

$$\dot{x}_k = u(x_k) \qquad \dot{m}_k = -\, m_k \left\langle u_x(x_k) \right\rangle$$

$$\left(\Longleftrightarrow \text{geodesics for the metric } g^{ij} = e^{-|x_i - x_j|} \right)$$

Let's try to see if the peakons satisfy the PDE:



$u = e^{-|x|}$

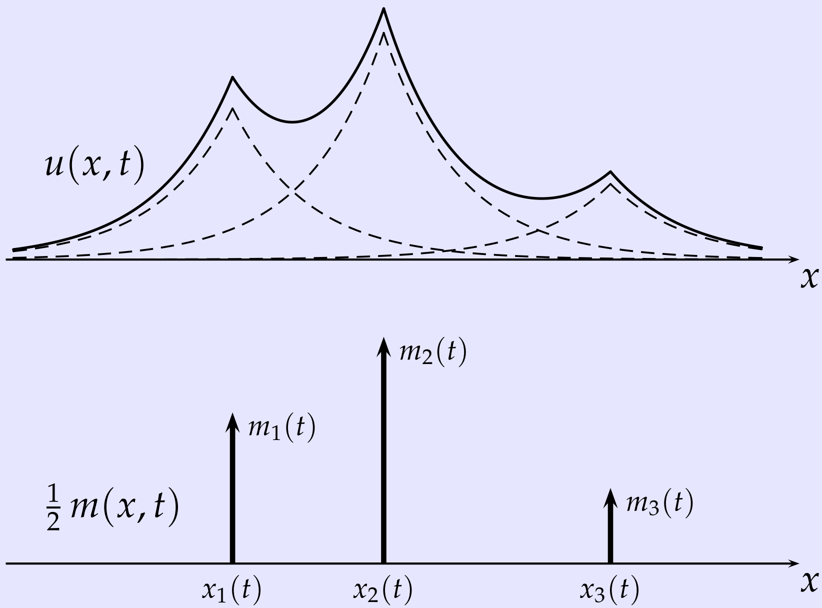
$u_x = -\operatorname{sgn}(x)\, e^{-|x|}$

$u_{xx} = e^{-|x|} - 2\,\delta_0$

(δ_0 is the Dirac delta at $x = 0$, the distributional derivative of the Heaviside step function)

Thus, $m = u - u_{xx} = 2\,\delta_0$

Likewise, $u = \sum_{i=1}^{n} m_i\, e^{-|x-x_i|}$ gives $m = 2\sum_{i=1}^{n} m_i\, \delta_{x_i}$



But inserting these u and m into the Camassa–Holm equation

$$m_t + m_x u + 2 m u_x = 0$$

means trouble at $x = x_k$: ***Dirac delta*** times ***jump discontinuity***!

- *Ad hoc* interpretation: use average $\langle u_x(x_k) \rangle$ in place of $u_x(x_k)$.
- Rigorous interpretation: rewrite the PDE as

$$(1 - \partial_x^2) u_t + (3 - \partial_x^2)\, \partial_x \left(\frac{1}{2}\, u^2 \right) + \partial_x \left(\frac{1}{2}\, u_x^2 \right) = 0$$

  and read it in the sense of distributions (equality in the space $\mathcal{D}'(\mathbf{R})$, for each fixed t).

In any case, one finds that the PDE is satisfied if the ODEs are.

Integrable peakon equations

Camassa–Holm (1993) $\quad m_t + m_x u + 2mu_x = 0$

$$\begin{cases} \dot{x}_k = u(x_k) \\ \dot{m}_k = -\, m_k \left\langle u_x(x_k) \right\rangle \end{cases}$$

Degasperis–Procesi (1998) $\quad m_t + m_x u + 3mu_x = 0$

$$\begin{cases} \dot{x}_k = u(x_k) \\ \dot{m}_k = -2\, m_k \left\langle u_x(x_k) \right\rangle \end{cases}$$

Vladimir Novikov (2008) $\quad m_t + (m_x u + 3mu_x)u = 0$

$$\begin{cases} \dot{x}_k = u(x_k)^2 \\ \dot{m}_k = -\, m_k \left\langle u_x(x_k) \right\rangle u(x_k) \end{cases}$$

Short interlude: Regularity of solutions

$$
\begin{array}{ll}
\text{CH} & (1-\partial_x^2)u_t + (3-\partial_x^2)\,\partial_x\left(\tfrac{1}{2}\,u^2\right) + \partial_x\left(\tfrac{1}{2}\,u_x^2\right) = 0 \\
\text{DP} & (1-\partial_x^2)u_t + (4-\partial_x^2)\,\partial_x\left(\tfrac{1}{2}\,u^2\right) = 0 \\
\text{Novikov} & (1-\partial_x^2)u_t + (4-\partial_x^2)\,\partial_x\left(\tfrac{1}{3}\,u^3\right) + \partial_x\left(\tfrac{3}{2}\,uu_x^2\right) + \tfrac{1}{2}u_x^3 = 0
\end{array}
$$

Locally integrable functions define distributions in $\mathcal{D}'(\mathbf{R})$.

For CH we need u^2 and u_x^2 locally integrable: $\quad u(\cdot,t) \in W^{1,2}_{\mathrm{loc}}(\mathbf{R})$

Similarly in the Novikov case for u^3 and u_x^3: $\quad u(\cdot,t) \in W^{1,3}_{\mathrm{loc}}(\mathbf{R})$

Sobolev $\Longrightarrow$ For CH and Novikov, $u(\cdot,t)$ must be continuous.

But DP can have discontinuous solutions!
(In particular “shockpeakons”.)

Back to our main track: Integrability

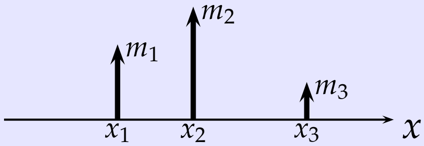
Lax pairs are known for all three equations.

CH	R. Camassa & D. Holm 1993
DP	A. Degasperis, D. Holm & A. Hone 2002
Novikov	A. Hone & J. P. Wang 2008

This makes it possible to use inverse spectral methods to find the *n*-peakon solution completely explicitly in terms of elementary functions.

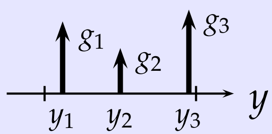
CH	R. Beals, D. Sattinger & J. Szmigielski 2000
DP	H. Lundmark & J. Szmigielski 2005
Novikov	A. Hone, H. Lundmark & J. Szmigielski 2009

CH integrability magic in a nutshell



Peakon measure $\frac{1}{2}m = \sum m_i \, \delta_{x_i}$ **evolving according to CH eqn**.

Bijection $\updownarrow$ $y_i = \tanh \frac{x_i}{2}$ $g_i = \frac{2m_i}{1-y_i^2}$



Associated distribution of point masses $g = \sum g_i \, \delta_{y_i}$ on the finite interval $y \in (-1, 1)$.

Spectral problem: $\downarrow$ forward $\uparrow$ inverse

$$\frac{W(z)}{z} = \frac{\phi'(1;z)}{z\,\phi(1;z)}$$
$$= \frac{1}{2z} + \sum_{k=1}^{n} \frac{b_k}{z-\lambda_k}$$

$$\dot{\lambda}_k = 0 \quad \dot{b}_k = b_k/\lambda_k$$

Linear evolution!

Weyl function of the **discrete string** with mass density g:
$$-\phi''(y;z) = z\,g(y)\,\phi(y;z)$$
$$\phi(-1;z) = 0 \qquad \phi'(-1;z) = 1$$

The discrete string

Wave equation for a vibrating string:

$$\underbrace{g(y)}_{\text{mass density}} F_{tt} = F_{yy} \qquad (F = 0 \text{ at } y = \pm 1)$$

Separate variables $F(y,t) = \phi(y)\,\psi(t)$ to find eigenmodes:

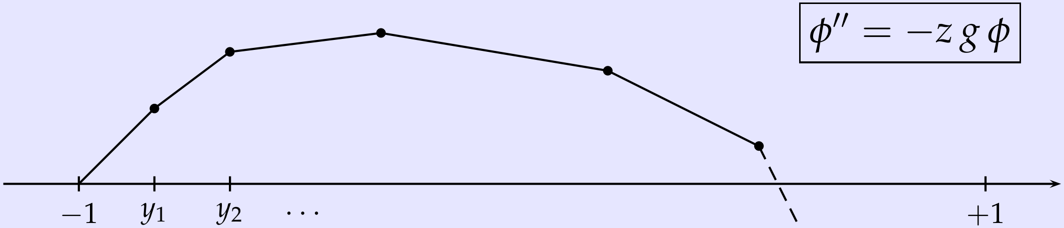
$$-\phi''(y;z) = z\,g(y)\,\phi(y;z) \qquad \phi(-1;z) = \phi(1;z) = 0$$

$$\Big[\ \text{Constant density} \quad \Longrightarrow \quad \text{sinusoidal eigenfunctions}\ \Big]$$

Discrete case:
n point masses
(Dirac deltas)

$$\Longrightarrow \begin{cases} n \text{ eigenvalues} \\ \text{piecewise linear eigenfunctions} \end{cases}$$

Shooting problem: $\phi(-1;z) = 0$ $\phi'(-1;z) = 1$



Point masses $g_1, \ldots, g_n$ at $y_1, \ldots, y_n$ (connected by weightless string).

The slope ϕ' changes by $-z\,g_k\,\phi(y_k;z)$ at each point mass.

For some values of z, we hit $\phi = 0$ at $y = +1$.

These are the **eigenvalues** $z = \lambda_1, \ldots, \lambda_n$ of the string.
(Real & simple.)

The **Weyl function** $W(z) = \dfrac{\phi'(1;z)}{\phi(1;z)}$ is rational. Poles at $z = \lambda_k$.

Inverse spectral problem: recover $g(y)$ from $W(z)$

Spectral data $\{\lambda_k, b_k\}_{k=1}^n \quad \xrightarrow{?} \quad$ String data $\{y_k, g_k\}_{k=1}^n$

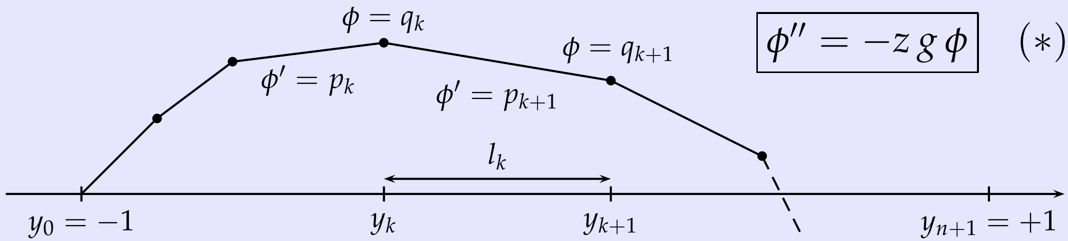
Solved by M. G. Krein (1951) using Stieltjes continued fractions. (See next page.)

Hence, there are connections to

- Padé approximation
- orthogonal polynomials
- the classical moment problem
- Riemann–Hilbert problems
- random matrix models

and much more.

Here's how it works:



$$\boxed{\phi'' = -z\,g\,\phi} \qquad (*)$$

$$\frac{W(z)}{z} = \frac{\phi'(1;z)}{z\,\phi(1;z)} = \boxed{\frac{p_{n+1}}{z\,q_{n+1}}} = \frac{p_{n+1}}{z(q_n + l_n p_{n+1})} = \cfrac{1}{z l_n + \cfrac{z\,q_n}{p_{n+1}}}$$

$$\stackrel{(*)}{=} \cfrac{1}{z l_n + \cfrac{z\,q_n}{p_n - z\,g_n\,q_n}} = \cfrac{1}{z l_n + \cfrac{1}{-g_n + \boxed{\cfrac{p_n}{z\,q_n}}}} = \ldots$$

Thus the quantities we seek are the coefficients in the **Stieltjes continued fraction expansion** of the Weyl function:

$$\frac{W(z)}{z} = \frac{\phi'(1;z)}{z\,\phi(1;z)} = \cfrac{1}{zl_n + \cfrac{1}{-g_n + \cfrac{1}{zl_{n-1} + \cfrac{1}{\ddots + \cfrac{1}{-g_2 + \cfrac{1}{zl_1 + \cfrac{1}{-g_1 + \cfrac{1}{zl_0}}}}}}}}$$

And the formulas to compute such an expansion for a given function $W(z)$ were derived by T. J. Stieltjes in 1895. This solves the inverse problem in the discrete case.

Example: General three-peakon solution of CH.

Initial conditions determine eigenvalues λ_k and residues $b_k(0)$.

Time evolution: $b_k(t) = b_k(0)\, e^{t/\lambda_k}$.

$$x_1(t) = \log \frac{(\lambda_1-\lambda_2)^2(\lambda_1-\lambda_3)^2(\lambda_2-\lambda_3)^2 b_1 b_2 b_3}{\sum_{j<k} \lambda_j^2 \lambda_k^2 (\lambda_j-\lambda_k)^2 b_j b_k}$$

$$x_2(t) = \log \frac{\sum_{j<k} (\lambda_j-\lambda_k)^2 b_j b_k}{\lambda_1^2 b_1 + \lambda_2^2 b_2 + \lambda_3^2 b_3}$$

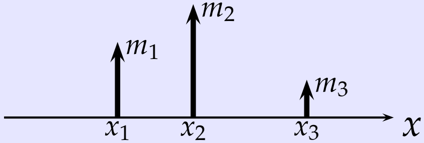
$$x_3(t) = \log(b_1 + b_2 + b_3)$$

$$m_1(t) = \frac{\sum_{j<k} \lambda_j^2 \lambda_k^2 (\lambda_j-\lambda_k)^2 b_j b_k}{\lambda_1\lambda_2\lambda_3 \sum_{j<k} \lambda_j \lambda_k (\lambda_j-\lambda_k)^2 b_j b_k}$$

$$m_2(t) = \frac{\left(\lambda_1^2 b_1 + \lambda_2^2 b_2 + \lambda_3^2 b_3\right) \sum_{j<k} (\lambda_j-\lambda_k)^2 b_j b_k}{(\lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3) \sum_{j<k} \lambda_j \lambda_k (\lambda_j-\lambda_k)^2 b_j b_k}$$

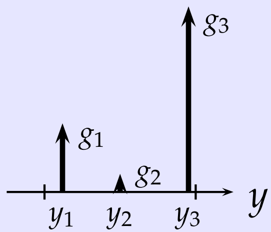
$$m_3(t) = \frac{b_1 + b_2 + b_3}{\lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3}$$

DP integrability magic in a nutshell



Peakon measure $\frac{1}{2}m = \sum m_i \delta_{x_i}$ **evolving according to DP eqn**.

Bijection $\updownarrow$ $y_i = \tanh \frac{x_i}{2}$ $g_i = \frac{8m_i}{(1-y_i^2)^2}$



Associated distribution of point masses $g = \sum g_i \delta_{y_i}$ on the finite interval $y \in (-1, 1)$.

Spectral problem: $\downarrow$ forward $\uparrow$ inverse

Weyl function of the **discrete <u>cubic</u> string** with mass density g:

$$-\phi'''(y;z) = z\, g(y)\, \phi(y;z)$$

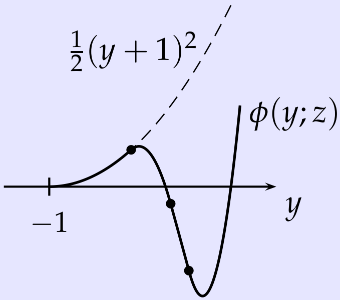
$$\phi(-1;z) = \phi'(-1;z) = 0 \quad \phi''(-1;z) = 1$$

$$\frac{W(z)}{z} = \frac{\phi'(1;z)}{z\,\phi(1;z)}$$
$$= \frac{1}{z} + \sum_{k=1}^{n} \frac{b_k}{z-\lambda_k}$$

$$\dot{\lambda}_k = 0 \quad \dot{b}_k = b_k/\lambda_k$$

Linear evolution!

Forward spectral problem for the cubic string



Shooting problem:

$$-\phi''' = zg\phi$$

$$\phi(-1;z) = \phi'(-1;z) = 0$$
$$\phi''(-1;z) = 1$$

The **third** derivative ϕ''' vanishes away from the point masses.

The eigenfunctions are quadratic splines.

The second derivative ϕ'' changes by $-z\, g_k\, \phi(y_k;z)$ at y_k.

Eigenvalues: those z for which we hit $\phi = 0$ at $y = +1$.

Non-selfadjoint problem, but the eigenvalues are still real and simple for positive mass distributions (Gantmacher–Krein theory of oscillatory kernels).

Inverse spectral problem for the discrete cubic string

Determinants of bimoments

$$I_{ab} = \iint \frac{\lambda^a \kappa^b}{\lambda + \kappa}\, d\mu(\lambda)\, d\mu(\kappa)$$

of the spectral measure $\mu = \sum_{k=1}^{n} b_k\, \delta_{\lambda_k}$ with respect to the Cauchy kernel $K(x,y) = (x+y)^{-1}$.

Curious simultaneous approximation of Weyl functions

$$W(z) = \frac{\phi'(1;z)}{\phi(1;z)} \qquad \text{and} \qquad Z(z) = \frac{\phi''(1;z)}{\phi(1;z)}$$

by rational functions with a common denominator.

Biorthogonal polynomials, four-term recurrence, Riemann–Hilbert problems, random matrix models.
(M. Bertola, M. Gekhtman & J. Szmigielski)

Example: General three-peakon solution of DP.

$$x_1(t) = \log \frac{U_3}{V_2} \qquad x_2(t) = \log \frac{U_2}{V_1} \qquad x_3(t) = \log U_1$$

$$m_1(t) = \frac{U_3 (V_2)^2}{V_3 W_2} \quad m_2(t) = \frac{(U_2)^2 (V_1)^2}{W_2 W_1} \quad m_3(t) = \frac{(U_1)^2}{W_1}$$

with time evolution $b_k(t) = b_k(0)\, e^{t/\lambda_k}$ and abbreviations

$$U_1 = b_1 + b_2 + b_3 \qquad V_1 = \lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3$$

$$U_2 = \frac{(\lambda_1 - \lambda_2)^2}{\lambda_1 + \lambda_2} b_1 b_2 + \frac{(\lambda_1 - \lambda_3)^2}{\lambda_1 + \lambda_3} b_1 b_3 + \frac{(\lambda_2 - \lambda_3)^2}{\lambda_2 + \lambda_3} b_2 b_3$$

$$V_2 = \frac{(\lambda_1 - \lambda_2)^2}{\lambda_1 + \lambda_2} \lambda_1 \lambda_2 b_1 b_2 + \frac{(\lambda_1 - \lambda_3)^2}{\lambda_1 + \lambda_3} \lambda_1 \lambda_3 b_1 b_3 + \frac{(\lambda_2 - \lambda_3)^2}{\lambda_2 + \lambda_3} \lambda_2 \lambda_3 b_2 b_3$$

$$U_3 = \frac{(\lambda_1 - \lambda_2)^2 (\lambda_1 - \lambda_3)^2 (\lambda_2 - \lambda_3)^2}{(\lambda_1 + \lambda_2)(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3)} b_1 b_2 b_3 \qquad V_3 = \lambda_1 \lambda_2 \lambda_3 U_3$$

$$W_1 = U_1 V_1 - U_2 = \lambda_1 b_1^2 + \lambda_2 b_2^2 + \lambda_3 b_3^2 + \frac{4 \lambda_1 \lambda_2}{\lambda_1 + \lambda_2} b_1 b_2 + \frac{4 \lambda_1 \lambda_3}{\lambda_1 + \lambda_3} b_1 b_3 + \frac{4 \lambda_2 \lambda_3}{\lambda_2 + \lambda_3} b_2 b_3$$

$$W_2 = U_2 V_2 - U_3 V_1 = \frac{(\lambda_1 - \lambda_2)^4}{(\lambda_1 + \lambda_2)^2} \lambda_1 \lambda_2 (b_1 b_2)^2 + \cdots + \frac{4 \lambda_1 \lambda_2 \lambda_3 (\lambda_1 - \lambda_2)^2 (\lambda_1 - \lambda_3)^2 b_1^2 b_2 b_3}{(\lambda_1 + \lambda_2)(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3)} + \ldots$$

New stuff: Novikov peakons

Summary of our results (with A. Hone & J. Szmigielski):

- Rigorous verification that the Lax pair for the Novikov equation really is valid in the context of peakons. (There are some subtle interpretation problems.)
- Transformation of the spatial Lax equation to the **dual cubic string** on the interval $y \in (-1, 1)$. With the help of this, explicit peakon solution formulas for arbitrary n.
- A curious combinatorial result related to the structure of the constants of motion H_k. (For $1 \leq k \leq n$, H_k equals the sum of all $k \times k$ minors – principal and nonprincipal – of a certain symmetric $n \times n$ matrix.)

The Lax pair found by Hone & Wang is

$$\frac{\partial}{\partial x}\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\end{pmatrix} = \begin{pmatrix}0 & zm & 1\\ 0 & 0 & zm\\ 1 & 0 & 0\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\end{pmatrix}$$

$$\frac{\partial}{\partial t}\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\end{pmatrix} = \begin{pmatrix}-uu_x & u_x z^{-1} - u^2 mz & u_x^2\\ uz^{-1} & -z^{-2} & -u_x z^{-1} - u^2 mz\\ -u^2 & uz^{-1} & uu_x\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\end{pmatrix}$$

For peakons, $m = 2\sum m_i \delta_{x_i}$ as before.

The equation $\partial_x \psi_2 = zm\psi_3$ forces ψ_2 to have jumps at the points x_i, but ψ_2 is multiplied by m in the equation for $\partial_x \psi_1$. The way out of this problem is to use the average $\langle \psi_2(x_i) \rangle$ to define what $\psi_2(x)\,\delta_{x_i}$ means, and this can be rigorously justified.

Eliminating ψ_1 gives $(\partial_x^2 - 1)\psi_3 = zm\psi_2$. The CH Lax pair contains the operator $\partial_x^2 - \frac{1}{4}$, and the mapping to a finite interval used there was precisely devised to get rid of the constant term in such an operator. So we use a similar transformation here too. After some trial and error, it turns out that

$$\begin{aligned} y &= \tanh x \\ \phi_1(y) &= \psi_1(x)\cosh x - \psi_3(x)\sinh x \\ \phi_2(y) &= z\,\psi_2(x) \\ \phi_3(y) &= z^2\,\psi_3(x)/\cosh x \\ g(y) &= m(x)\,\cosh^3 x \\ \lambda &= -z^2 \end{aligned}$$

transforms the Lax equation for $\partial_x \Psi$ into

$$\frac{\partial}{\partial y}\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\end{pmatrix} = \begin{pmatrix} 0 & g(y) & 0 \\ 0 & 0 & g(y) \\ -\lambda & 0 & 0 \end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\end{pmatrix}$$

(Notice that the 1 in the upper right corner is gone!)

From the previous page:

$$\frac{\partial}{\partial y}\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\end{pmatrix} = \begin{pmatrix}0 & g(y) & 0\\ 0 & 0 & g(y)\\ -\lambda & 0 & 0\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\end{pmatrix}$$

We call this the **dual cubic string**.

The ordinary cubic string $\partial_y^3\phi = -\lambda g \phi$ can be written as

$$\frac{\partial}{\partial y}\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\end{pmatrix} = \begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -\lambda g(y) & 0 & 0\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\end{pmatrix}$$

by letting $(\phi_1, \phi_2, \phi_3) = (\phi, \phi_y, \phi_{yy})$.

One maps to the other under the transformation $\dfrac{d\tilde{y}}{dy} = g(y) = 1/\tilde{g}(\tilde{y})$.

(In the discrete case: interchange of masses g_k and distances $l_k = y_{k+1} - y_k$.)

Because of the duality we can reuse results from the DP case to derive n-peakon solution formulas for Novikov's equation.

The Novikov three-peakon solution doesn't fit on one page!
But here's the two-peakon solution:

$$x_1 = \frac{1}{2}\log \frac{\dfrac{(\lambda_1-\lambda_2)^4}{(\lambda_1+\lambda_2)^2\lambda_1\lambda_2} b_1^2 b_2^2}{\lambda_1 b_1^2 + \lambda_2 b_2^2 + \dfrac{4\lambda_1\lambda_2}{\lambda_1+\lambda_2} b_1 b_2}$$

$$x_2 = \frac{1}{2}\log\left(\frac{b_1^2}{\lambda_1} + \frac{b_2^2}{\lambda_2} + \frac{4}{\lambda_1+\lambda_2} b_1 b_2\right)$$

$$m_1 = \frac{\left(\lambda_1 b_1^2 + \lambda_2 b_2^2 + \dfrac{4\lambda_1\lambda_2}{\lambda_1+\lambda_2} b_1 b_2\right)^{1/2}}{\sqrt{\lambda_1\lambda_2}(b_1+b_2)}$$

$$m_2 = \frac{\left(\dfrac{b_1^2}{\lambda_1} + \dfrac{b_2^2}{\lambda_2} + \dfrac{4}{\lambda_1+\lambda_2} b_1 b_2\right)^{1/2}}{b_1+b_2}$$

Constants of motion

Eliminate ψ_1 from the spatial Lax equation: $\partial_x \psi_2 = zm\psi_3$ and $(\partial_x^2 - 1)\,\psi_3 = zm\psi_2$. Assuming vanishing boundary conditions at infinity, write this as integral equations instead:

$$\psi_2(x) = z\int_{-\infty}^{x} \psi_3(y)\,dm(y) \qquad \psi_3(x) = -z\int_{-\infty}^{\infty} \frac{1}{2}\,e^{-|x-y|}\psi_2(y)\,dm(y)$$

Evaluation at the points $x = x_k$ gives a block matrix system:

$$\begin{pmatrix} \langle \Psi_2 \rangle \\ \Psi_3 \end{pmatrix} = z \begin{pmatrix} 0 & TP \\ -EP & 0 \end{pmatrix} \begin{pmatrix} \langle \Psi_2 \rangle \\ \Psi_3 \end{pmatrix}$$

where

$$\langle \Psi_2 \rangle = \big(\langle \psi_2(x_1) \rangle, \dots, \langle \psi_2(x_n) \rangle\big)^t \qquad \Psi_3 = \big(\psi_3(x_1), \dots, \psi_3(x_n)\big)^t$$

$$P = \operatorname{diag}(m_1, \dots, m_n)$$

$$E = (E_{jk}) = \big(e^{-\left|x_j - x_k\right|}\big)$$

$$T = (T_{jk}) = \big(1 + \operatorname{sgn}(j-k)\big)$$

$$= \text{triangular matrix with 1 on diagonal, 0 above, 2 below}$$

In terms of $\langle \Psi_2 \rangle$ alone, we have $\langle \Psi_2 \rangle = -z^2 TPEP \langle \Psi_2 \rangle$ so the eigenvalues (which are constants of motion) are given by

$$0 = \det(I + z^2 TPEP) = \det(I - \lambda TPEP)$$

The coefficient of $(-\lambda)^k$ in this time-invariant polynomial equals the sum of the **principal** $k \times k$ minors of the $n \times n$ matrix $TPEP$.

Before we found this, we had computed constants of motion in other ways, and conjectured that they should be the sum of **all** $k \times k$ minors of PEP. Even more turned out to be true:

> **"The Canada Day Theorem"**
> For any symmetric $n \times n$ matrix X, the sum of the **principal** $k \times k$ minors of TX equals the sum of **all** $k \times k$ minors of X.

(Recall: $T = n \times n$ triangular matrix with 1 on diagonal, 0 above, 2 below)

Illustration: $n = 2$

$$X = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \qquad TX = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} a & b \\ 2a+b & 2b+c \end{pmatrix}$$

$$\det(I + sTX) = 1 + (a + 2b + c)s + (ac - b^2)s^2$$

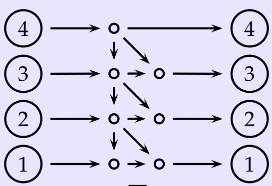
Coeff. of s: $\sum$ (**diagonal** entries of TX) $= \sum$ (**all** entries of X)

Coeff. of s^2: $\det TX = \det X$.

(But this example is really too simple! For $n \geq 4$ it starts to get much more complicated.)

Explaining the proof of the Canada Day Theorem would take another lecture. Main ingredients:

- The Cauchy–Binet formula for the minors of a product.
- Some rather intricate dependencies among the minors of a symmetric matrix.
- Lindström's Lemma (a.k.a. the Gessel–Viennot Theorem), for evaluating minors by counting non-intersecting families of paths through a planar network.



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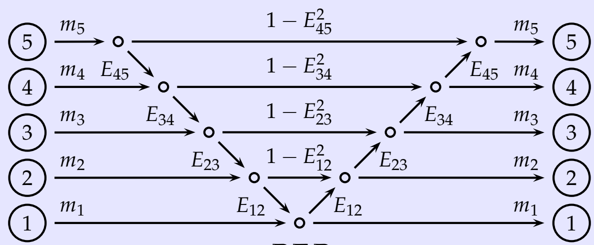

PEP