From water waves to combinatorics: The mathematics of peaked solitons

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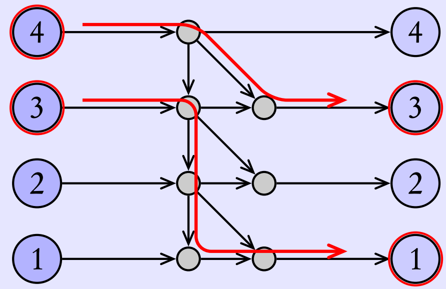
What does counting paths through a graph…



…have to do with water waves?

Not *very* much, to be honest!

(If you’re a physicist.)

(But maybe a little, if you’re a mathematician.)

Anyway, let us begin with a famous episode that took place near Edinburgh in 1834, by the Union Canal…

John Scott Russell

Scottish naval engineer

"I believe I shall best describe this phænomenon by describing the circumstances of my own first acquaintance with it. I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped—not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phænomenon which I have called the Wave of Translation [. . .]."

Report on waves (1845), p. 13

Reconstruction of a solitary wave near the site of discovery:

Naming ceremony for the Scott Russell Aqueduct over the Edinburgh City Bypass
(July 12, 1995)

(Photo from the web page of Chris Eilbeck, Heriot-Watt University, Edinburgh)

Some results of Scott Russell's water tank experiments:

- Solitary waves are very stable.
- The **speed** of a solitary wave depends on its **height**:

$$v = \sqrt{g(d+h)}$$

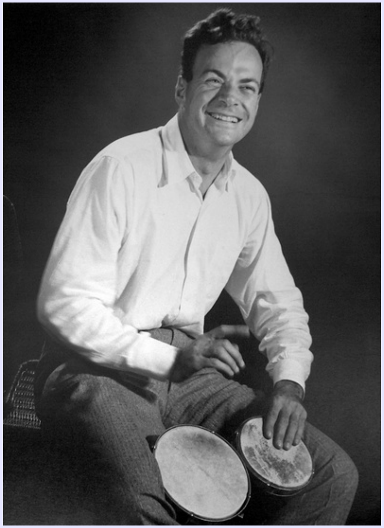
(*d* = depth of undisturbed water, *h* = wave height)

- If one tries to generate a wave which is "too big", it will decompose into several solitary waves, each with its own speed (the largest first).
- During the passage of the wave, the water particles are displaced. (In contrast to ordinary oscillatory waves, where they return to their original position.)

Scott Russell also found a way to make canal boats go faster, with less effort for the horses: pull the boat at just the right speed, and it will “surf” on the solitary wave that it generates!

But some influential scientists (Airy, Stokes) were still sceptical, since the solitary wave didn’t fit with their theories about water waves.

Water wave theory is hard!



"Now, the next waves of interest, that are easily seen by everyone and which are usually used as an example of waves in elementary courses, are water waves. As we shall soon see, they are the worst possible example, because they are in no respects like sound and light; they have all the complications that waves can have."

The Feynman Lectures on Physics
Vol. I, Sect. 51–4

Fluid flow basics

Velocity vector field: $\mathbf{v}(x, y, z, t)$

Basic equation: the **Navier–Stokes** equation.

(Newton's law applied to each "fluid particle".)

$$\text{Acceleration} = \frac{\text{Forces (pressure, viscosity, gravity)}}{\text{Mass}}$$

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{v} - g \hat{\mathbf{z}}$$

Water is (nearly) incompressible $\implies \begin{cases} \text{Constant density } \rho \\ \nabla \cdot \mathbf{v} = 0 \end{cases}$

In water wave problems, certain **boundary conditions** must hold at the free surface and at the bottom.

Major difficulty:

It is not known in advance where the surface is!

Some possible simplifying assumptions:

- Neglect viscosity.
  (**Euler equation**. John von Neumann: "Dry water.")
- Neglect surface tension.
- Assume that the vorticity $\omega = \nabla \times \mathbf{v}$ is zero.
  (Then $\mathbf{v} = \nabla\phi$, and $\nabla \cdot \mathbf{v} = 0$ becomes the Laplace equation $\nabla^2\phi = 0$.)
- One-dimensional wave propagation.
- Shallow water (compared to the wavelength).
- Small amplitude.
  (Neglect higher order terms. Evaluate surface boundary condition at $z = d$ instead of at some unknown height $z = \eta(x, y)$.)

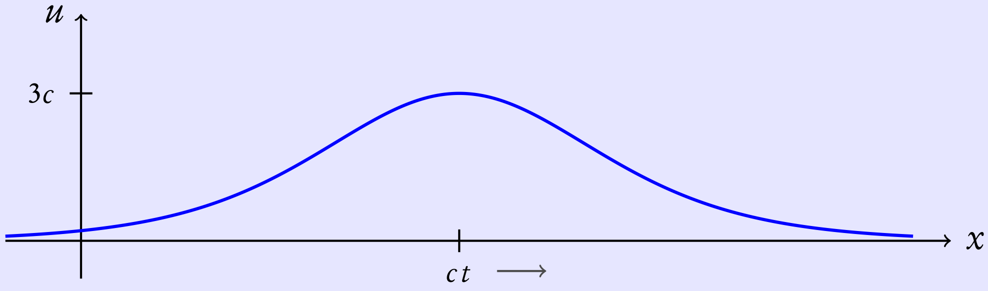
Different assumptions lead to different "water wave models", which (hopefully) describe some aspect or another of real water.

One such model is the **KdV equation** for shallow water waves:

$$u_t + u u_x + u_{xxx} = 0$$

(Boussinesq 1871, Lord Rayleigh 1876, Korteweg & de Vries 1895)

Travelling wave solution: $\quad u(x,t) = 3c \Big/ \cosh^2\left[\tfrac{1}{2}(x - ct)\sqrt{c}\right]$



(First theoretical explanation of the solitary wave.)

Much later: surprising discoveries

Kruskal & Zabusky 1965

> Numerical studies of the KdV equation.
> Found that several solitary waves could coexist, almost unaffected by each other: “**solitons**”.

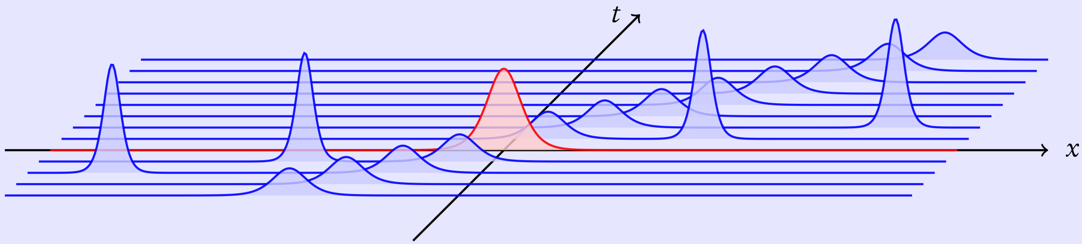
Gardner, Green, Kruskal & Miura 1967

> The **Inverse Scattering Transform (IST)**, a method which gives exact formulas for the multi-soliton solutions of the KdV equation.

(Nonlinear PDEs were considered “impossible” to solve analytically until then.)

Example: a two-soliton solution of the KdV equation

$$u(x,t) = \frac{72\left[3+4\cosh(2x-8t)+\cosh(4x-64t)\right]}{\left[3\cosh(x-28t)+\cosh(3x-36t)\right]^2}$$



Shown in red: $u(x,0) = 36/\cosh^2 x$

(Cf. Scott Russell's big wave splitting into several solitary waves.)

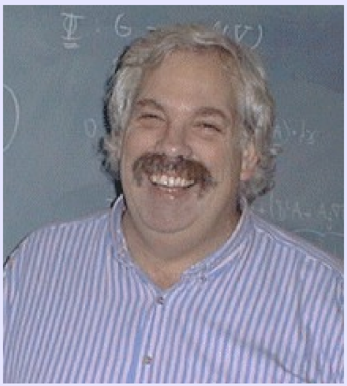
Martin Kruskal (right) playing chess against Francesco Calogero

(NEEDS ’97 conference, Kolymbari, Crete)

Here is a more recent shallow water wave model, derived in 1993 by Roberto Camassa and Darryl Holm:

$$u_t - u_{txx} + k u_x = 2 u_x u_{xx} - 3 u u_x + u u_{xxx}$$

(u = horizontal fluid velocity, $k > 0$ some physical parameter)



They are smiling, because their paper generates lots of citations:

AMERICAN MATHEMATICAL SOCIETY
MathSciNet *Mathematical Reviews on the Web*

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Citations
From References: 354
From Reviews: 21

MR1234453 (94f:35121) 35Q51 (58F07 76B15 76B25)
Camassa, Roberto (1-LANL-TD); **Holm, Darryl D.** (1-LANL-NL)
An integrable shallow water equation with peaked solitons. (English summary)
Phys. Rev. Lett. **71** (1993), *no. 11,* 1661–1664.

Summary: "We derive a new completely integrable dispersive shallow water equation that is bi-Hamiltonian and thus possesses an infinite number of conservation laws in involution. The equation is obtained by using an asymptotic expansion directly in the Hamiltonian for Euler's equations in the shallow water regime. The soliton solution for this equation has a limiting form that has a discontinuity in the first derivative at its peak."

(And this count doesn't even include the physics literature!)

Why all this interest?

The Camassa–Holm equation is an **integrable system**.

- Lots of mathematical structure.
- IST and other methods are available.

It admits **non-smooth solutions** (in a suitable weak sense).

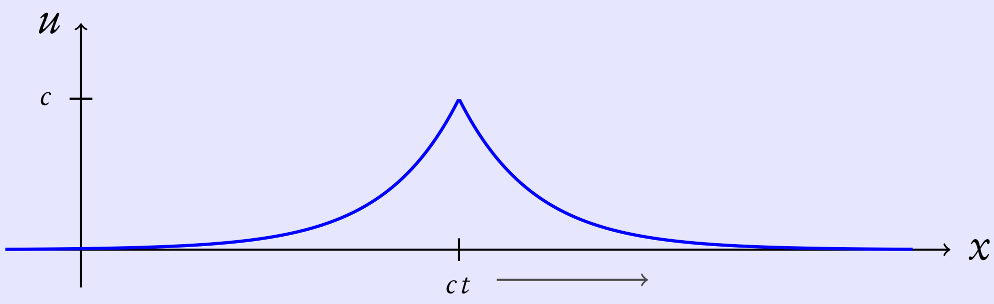
- Initially smooth solutions may lose smoothness after finite time.
- Possible model for *wave breaking*?
- In the limiting case $k \to 0$, it has peaked-shaped solitons with a particularly straightforward structure: "**peakons**".

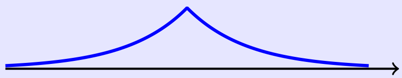
The one-peakon solution

The CH equation with $k = 0$ can be written as

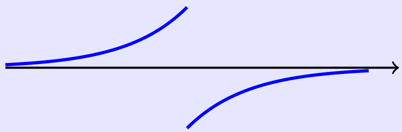
$$m_t + m_x u + 2 m u_x = 0 \qquad m = u - u_{xx}$$

Travelling wave solution: $\boxed{u(x,t) = c\, e^{-|x-ct|}}$

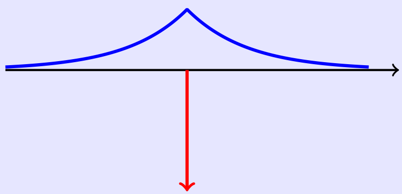




$$u = e^{-|x|} = \begin{cases} e^{x}, & x \leq 0 \\ e^{-x}, & x \geq 0 \end{cases}$$

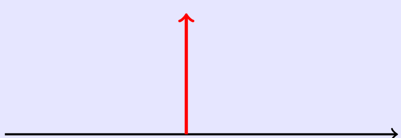


$$u_x = \begin{cases} +e^{x}, & x < 0 \\ -e^{-x}, & x > 0 \end{cases}$$



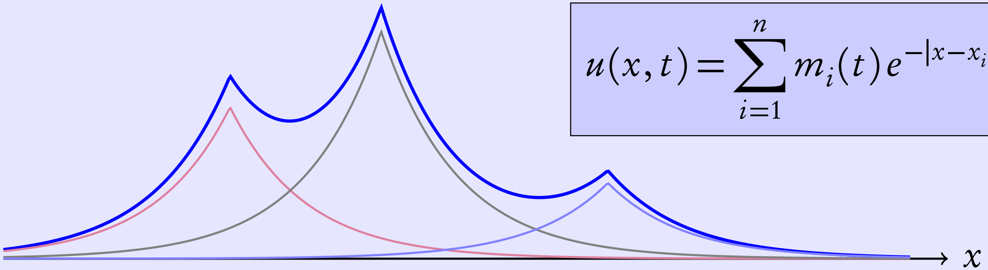
$$u_{xx} = e^{-|x|} - 2\,\delta(x)$$

(δ is the Dirac delta at $x = 0$, the distributional derivative of the Heaviside step function)

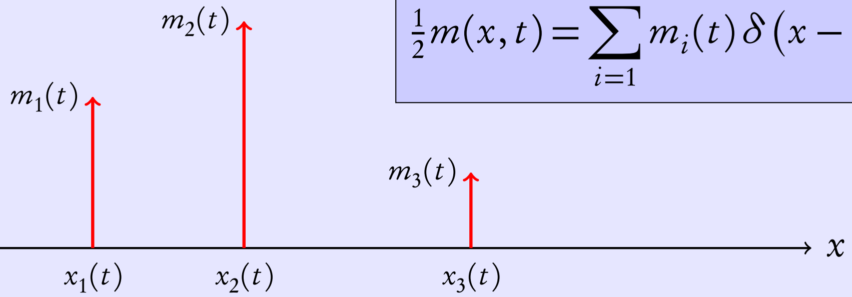


$$\Longrightarrow \quad m = u - u_{xx} = 2\,\delta$$

The multi-peakon solution



$$u(x,t) = \sum_{i=1}^{n} m_i(t)\, e^{-|x - x_i(t)|}$$



$$\tfrac{1}{2} m(x,t) = \sum_{i=1}^{n} m_i(t)\, \delta\big(x - x_i(t)\big)$$

This *Ansatz* satisfies the CH equation $m_t + m_x u + 2m u_x = 0$ iff

$$\dot{x}_k = \sum_{i=1}^{n} m_i \, e^{-|x_k - x_i|}$$

$$\dot{m}_k = \sum_{i=1}^{n} m_k \, m_i \, \operatorname{sgn}(x_k - x_i) \, e^{-|x_k - x_i|}$$

($2n$-dimensional dynamical system for positions $x_k(t)$ and amplitudes $m_k(t)$.)

Shorthand notation:

$$\dot{x}_k = u(x_k) \qquad\qquad \dot{m}_k = -\, m_k \, \langle u_x(x_k) \rangle$$

(Note: **Speed** $\dot{x}_k$ of the kth peakon = **wave height** at that point.)

The travelling wave is the case $n = 1$:

$$\dot{x}_1 = m_1 \qquad \dot{m}_1 = 0$$

The case $n = 2$, with $E_{12} = e^{-|x_1 - x_2|}$, reads

$$\begin{aligned} \dot{x}_1 &= m_1 + m_2 E_{12} \\ \dot{x}_2 &= m_1 E_{12} + m_2 \\ \dot{m}_1 &= -m_1 m_2 E_{12} \\ \dot{m}_2 &= m_1 m_2 E_{12} \end{aligned}$$

(Camassa & Holm solved this by elementary methods, using the constants of motion $M = m_1 + m_2$ and $H = \frac{1}{2} m_1 m_2 E_{12}$.)

The Camassa–Holm peakon equations for arbitrary n were solved using IST by Richard Beals, David Sattinger and Jacek Szmigielski.

Advances in Mathematics **154**, 229–257 (2000)
doi:10.1006/aima.1999.1883, available online at http://www.idealibrary.com on IDEAL®

Multipeakons and the Classical Moment Problem

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Suppose μ is a measure on the real line such that all its **moments** $\beta_k = \int x^k \, d\mu(x)$ $(k = 0, 1, 2, \ldots)$ are finite.

The **Classical Moment Problem** is the problem of recovering the measure μ from its moments. (Existence? Uniqueness?)

This problem was important in the development of functional analysis.

Basic object: the Hilbert space $L^2(\mu)$ with inner product

$$\langle f, g \rangle = \int \overline{f(x)} g(x) \, d\mu(x)$$

The moments β_k determine the inner product of any two polynomials:

$$\left\langle \sum_{i=0}^{r} a_i x^i, \sum_{j=0}^{s} b_j x^j \right\rangle = \sum_{i=0}^{r}\sum_{j=0}^{s} \overline{a_i} b_j \underbrace{\int x^{i+j}\, d\mu(x)}_{\beta_{i+j}}$$

(Important question: Are the polynomials dense in $L^2(\mu)$?)

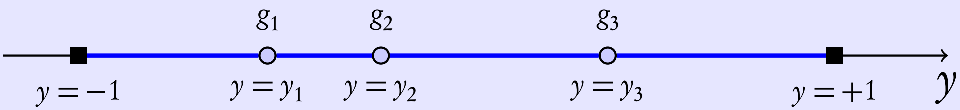
First step: define **orthogonal polynomials** by doing Gram–Schmidt on the standard basis $1, x, x^2, \ldots$

And so on. Big theory. (Very pretty, very classical, many connections to other parts of mathematics.)

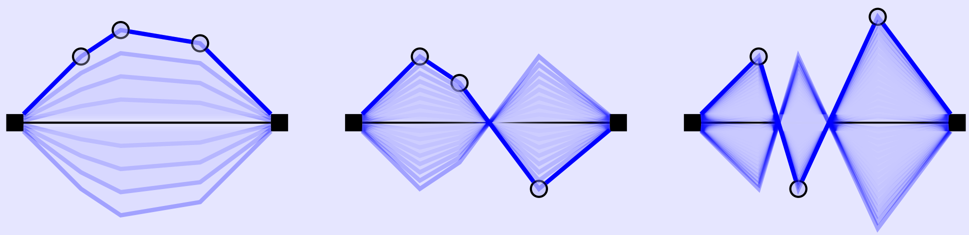
What does this have to do with peakons?

To any peakon configuration (values of the x_k's and m_k's) Beals, Sattinger & Szmigielski associated a **discrete string**:

> For $k = 1, 2, \ldots, n$, let $y_k = \tanh x_k$ (thus $-1 < y_k < 1$) and $g_k = 2m_k/(1-y_k^2)$. At the positions y_k, put point masses g_k and connect them by weightless thread. Attach the ends of the string to the points $y = \pm 1$.



Such a string has vibrational modes (just like a guitar string has a fundamental frequency and overtones), but only as many as there are point masses.



The squared eigenfrequencies are denoted $\lambda_1, \ldots, \lambda_n$.

One also defines numbers $b_1, \ldots, b_n$ which contain some information about the shape of the eigenfunctions.

(Basically the ratio between the slopes at the endpoints.)

We can view this as a **change of variables**: from $\{x_k, m_k\}$ to "spectral variables" $\{\lambda_k, b_k\}$.

Miracle: In terms of the spectral variables, the complicated nonlinear CH peakon equations simply become

$$\dot{\lambda}_k = 0 \qquad \dot{b}_k = b_k/\lambda_k$$

("Isospectral deformation" – the string's spectrum doesn't change!)

This system is of course easily solved:

$$\lambda_k = \text{constant} \qquad b_k(t) = b_k(0)\, e^{t/\lambda_k}$$

The **inverse change of variables** is given by formulas found by Thomas Joannes Stieltjes in his study of the moment problem.

(*Recherches sur les fractions continues*, 1894)

This connection between the vibrating string and the work of Stieltjes was noticed by Mark Grigorievich Krein around 1950.

The change of variables in a nutshell (with $l_k = y_{k+1} - y_k$):

$$\frac{1/2}{z} + \frac{b_1}{z-\lambda_1} + \frac{b_2}{z-\lambda_2} + \cdots + \frac{b_n}{z-\lambda_n} =$$

$$= \cfrac{1}{z l_n + \cfrac{1}{-g_n + \cfrac{1}{z l_{n-1} + \cfrac{1}{\ddots + \cfrac{1}{-g_2 + \cfrac{1}{z l_1 + \cfrac{1}{-g_1 + \cfrac{1}{z l_0}}}}}}}}$$

It corresponds to
converting between two ways
of representing a rational function:

LHS = partial fractions
RHS = Stieltjes continued fraction

Example: The Camassa–Holm three-peakon solution

$$x_1(t) = \ln \frac{(\lambda_1 - \lambda_2)^2 (\lambda_1 - \lambda_3)^2 (\lambda_2 - \lambda_3)^2 b_1 b_2 b_3}{\sum_{j<k} \lambda_j^2 \lambda_k^2 (\lambda_j - \lambda_k)^2 b_j b_k}$$

$$x_2(t) = \ln \frac{\sum_{j<k} (\lambda_j - \lambda_k)^2 b_j b_k}{\lambda_1^2 b_1 + \lambda_2^2 b_2 + \lambda_3^2 b_3}$$

$$x_3(t) = \ln(b_1 + b_2 + b_3)$$

$$\begin{pmatrix} \lambda_k = \text{constant} \\ b_k(t) = b_k(0)\, e^{t/\lambda_k} \end{pmatrix}$$

$$m_1(t) = \frac{\sum_{j<k} \lambda_j^2 \lambda_k^2 (\lambda_j - \lambda_k)^2 b_j b_k}{\lambda_1 \lambda_2 \lambda_3 \sum_{j<k} \lambda_j \lambda_k (\lambda_j - \lambda_k)^2 b_j b_k}$$

$$m_2(t) = \frac{\left(\lambda_1^2 b_1 + \lambda_2^2 b_2 + \lambda_3^2 b_3\right) \sum_{j<k} (\lambda_j - \lambda_k)^2 b_j b_k}{(\lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3) \sum_{j<k} \lambda_j \lambda_k (\lambda_j - \lambda_k)^2 b_j b_k}$$

$$m_3(t) = \frac{b_1 + b_2 + b_3}{\lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3}$$

I have worked (mostly together with Jacek Szmigielski) on peakon solutions of some other integrable PDEs:

$$m_t + m_x u + 3 m u_x = 0 \qquad m = u - u_{xx}$$

(Antonio Degasperis & Michela Procesi, 1998)

$$m_t + (m_x u + 3 m u_x) u = 0 \qquad m = u - u_{xx}$$

(Vladimir Novikov, 2008)

Camassa–Holm, for comparison:

$$m_t + m_x u + 2 m u_x = 0 \qquad m = u - u_{xx}$$

The vibrational modes of a string are given by the eigenvalue problem

$$-\phi''(y) = z\, g(y)\, \phi(y)$$
$$\phi(-1) = \phi(+1) = 0$$

where $g(y)$ describes the distribution of mass.

Peakon solution of the new equations (DP & Novikov) are instead related to the third order eigenvalue problem

$$-\phi'''(y) = z\, g(y)\, \phi(y)$$
$$\phi(-1) = \phi'(-1) = \phi(+1) = 0$$

which we call the **cubic string.**

The cubic string

Not selfadjoint. The eigenvalues λ_k are real anyway (if g is positive).

(Total positivity, Gantmacher–Krein theory of oscillatory kernels.)

Inverse spectral problem for discrete case gives explicit formulas for the peakon solutions.

(Quite a lot harder than for the ordinary string.)

Instead of orthogonal polynomials: **Cauchy biorthogonal polynomals**

$$\langle f, g \rangle = \iint \frac{f(x)\,g(y)}{x+y}\,d\alpha(x)\,d\beta(y) \qquad \left\langle p_i, q_j \right\rangle = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

(Bertola–Gekhtman–Szmigielski, 2009, 2010)

Example: The Degasperis–Procesi three-peakon solution

$$x_1(t) = \ln \frac{U_3}{V_2} \qquad x_2(t) = \ln \frac{U_2}{V_1} \qquad x_3(t) = \ln U_1$$

$$m_1(t) = \frac{U_3 (V_2)^2}{V_3 W_2} \quad m_2(t) = \frac{(U_2)^2 (V_1)^2}{W_2 W_1} \quad m_3(t) = \frac{(U_1)^2}{W_1}$$

where

$$U_1 = b_1 + b_2 + b_3 \qquad V_1 = \lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3$$

$$U_2 = \frac{(\lambda_1 - \lambda_2)^2}{\lambda_1 + \lambda_2} b_1 b_2 + \frac{(\lambda_1 - \lambda_3)^2}{\lambda_1 + \lambda_3} b_1 b_3 + \frac{(\lambda_2 - \lambda_3)^2}{\lambda_2 + \lambda_3} b_2 b_3$$

$$V_2 = \frac{(\lambda_1 - \lambda_2)^2}{\lambda_1 + \lambda_2} \lambda_1 \lambda_2 b_1 b_2 + \frac{(\lambda_1 - \lambda_3)^2}{\lambda_1 + \lambda_3} \lambda_1 \lambda_3 b_1 b_3 + \frac{(\lambda_2 - \lambda_3)^2}{\lambda_2 + \lambda_3} \lambda_2 \lambda_3 b_2 b_3$$

$$U_3 = \frac{(\lambda_1 - \lambda_2)^2 (\lambda_1 - \lambda_3)^2 (\lambda_2 - \lambda_3)^2}{(\lambda_1 + \lambda_2)(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3)} b_1 b_2 b_3 \qquad V_3 = \lambda_1 \lambda_2 \lambda_3 U_3$$

$$W_1 = U_1 V_1 - U_2 = \lambda_1 b_1^2 + \lambda_2 b_2^2 + \lambda_3 b_3^2 + \frac{4\lambda_1 \lambda_2}{\lambda_1 + \lambda_2} b_1 b_2 + \frac{4\lambda_1 \lambda_3}{\lambda_1 + \lambda_3} b_1 b_3 + \frac{4\lambda_2 \lambda_3}{\lambda_2 + \lambda_3} b_2 b_3$$

$$W_2 = U_2 V_2 - U_3 V_1 = \frac{(\lambda_1 - \lambda_2)^4}{(\lambda_1 + \lambda_2)^2} \lambda_1 \lambda_2 (b_1 b_2)^2 + \cdots + \frac{4\lambda_1 \lambda_2 \lambda_3 (\lambda_1 - \lambda_2)^2 (\lambda_1 - \lambda_3)^2 b_1^2 b_2 b_3}{(\lambda_1 + \lambda_2)(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3)} + \dots$$

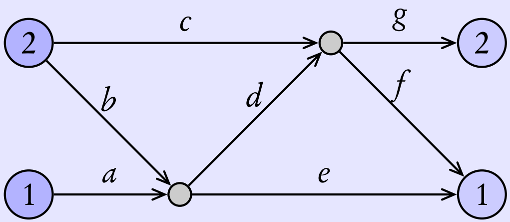
Now, finally, some combinatorics!

Lindström's Lemma

If X is the **path matrix** of a planar network, then the minor (=subdeterminant) taken from rows $i_1, \ldots, i_k$ and columns $j_1, \ldots, j_k$ equals the sum of the weights of the **non-intersecting** path families from the source nodes $i_1, \ldots, i_k$ to the sink nodes $j_1, \ldots, j_k$.

(Karlin & McGregor 1959, Lindström 1973, Gessel & Viennot 1985)

Example:

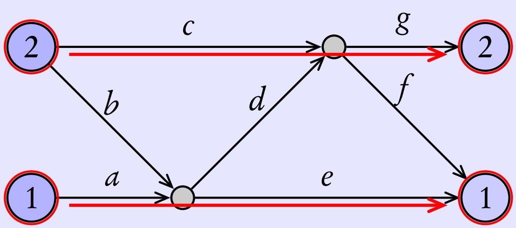


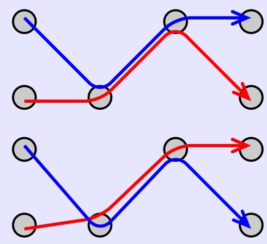
Path matrix:

$$X = \begin{pmatrix} ae + adf & adg \\ be + bdf + cf & bdg + cg \end{pmatrix}$$

$$\det X = ae \cdot cg$$

All other terms cancel in pairs.
("Switch paths at first intersection"
is a sign-reversing involution.)





$+adf \cdot bdg$

$-adg \cdot bdf$

The peakon ODEs for Novikov's equation have constants of motion, but the structure is not obvious.

Example: the case $n = 3$

$$
\begin{aligned}
H_1 &= m_1^2 + m_2^2 + m_3^2 + 2m_1 m_2 E_{12} + 2m_1 m_3 E_{13} + 2m_2 m_3 E_{23} \\
H_2 &= (1 - E_{12}^2)\, m_1^2\, m_2^2 + (1 - E_{13}^2)\, m_1^2\, m_3^2 + (1 - E_{23}^2)\, m_2^2\, m_3^2 \\
&\qquad + 2(E_{23} - E_{12}E_{13})\, m_1^2\, m_2\, m_3 + 2(E_{12} - E_{13}E_{23})\, m_1\, m_2\, m_3^2 \\
H_3 &= (1 - E_{12}^2)(1 - E_{23}^2)\, m_1^2\, m_2^2\, m_3^2
\end{aligned}
$$

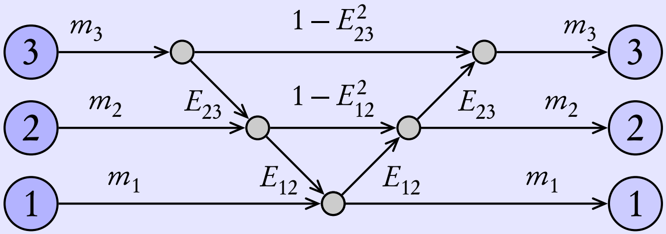
(Abbreviation: $E_{ij} = e^{-|x_i - x_j|}$. Assume $x_1 < x_2 < x_3$.)

Question: What's the pattern (for general n)?

Conjecture early on in our investigations:

$$H_k = \text{sum of all } k \times k \text{ minors (subdeterminants) of}$$
$$\text{the } n \times n \text{ matrix } X \text{ with entries } X_{ij} = m_i m_j E_{ij}$$

$$X = \begin{pmatrix} m_1^2 & m_1 m_2 E_{12} & m_1 m_3 E_{13} \\ m_1 m_2 E_{12} & m_2^2 & m_2 m_3 E_{23} \\ m_1 m_3 E_{13} & m_2 m_3 E_{23} & m_3^2 \end{pmatrix}$$

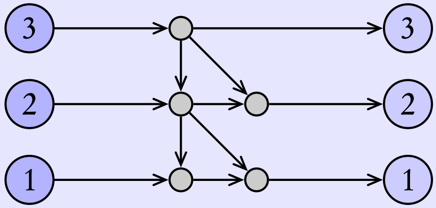


Later we found:

$$H_k = \text{sum of the } \textbf{principal } k \times k \text{ minors of } TX$$

$$\text{where } T_{ij} = \begin{cases} 0, & i < j \\ 1, & i = j \\ 2, & i > j \end{cases}$$

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 2 & 1 \end{pmatrix}$$



Does this result agree with the conjecture? Yes, and there is nothing special with our particular matrix X, except that it is symmetric.

The Canada Day Theorem

For any **symmetric** $n \times n$ matrix X, the sum of all $k \times k$ minors of X equals the sum of the principal $k \times k$ minors of TX.

Example: $n = 2$ (Too simple, really. For $n \geq 4$ it starts to get more interesting.)

$$X = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \qquad TX = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} a & b \\ 2a+b & 2b+c \end{pmatrix}$$

$k = 1$: $\quad a + b + b + c = a + (2b + c)$

$k = 2$: $\quad \det X = \det(TX) \quad$ (true since $\det T = 1$)

Like as the waves make towards the pebbled shore,
So do our minutes hasten to their end

William Shakespeare, Sonnet LX