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Satellite Thematic Session: Integrable Systems

Peaked solitons

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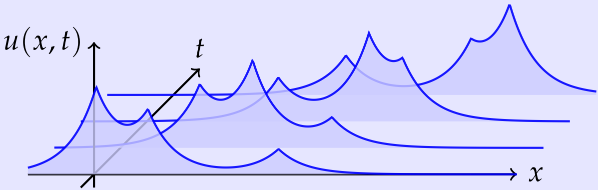
Some PDEs have peaked soliton solutions:



$$\boxed{u(x,t) = \sum_{i=1}^{N} m_i(t)\, e^{-|x-x_i(t)|}}$$

Main example:

$$\boxed{m_t + m_x u + 2 m u_x = 0 \qquad (m = u - u_{xx})}$$

Camassa–Holm shallow water equation (1993)

Positions and amplitudes governed by ODEs:

$$\boxed{\dot{x}_k = u(x_k) \qquad \dot{m}_k = -m_k \langle u_x(x_k) \rangle}$$

In detail:

$$\dot{x}_k = \sum_{i=1}^{N} m_i \, e^{-|x_k - x_i|}$$

$$\dot{m}_k = \sum_{i=1}^{N} m_k \, m_i \; \operatorname{sgn}(x_k - x_i) \, e^{-|x_k - x_i|}$$

Integrable Hamiltonian system with

$$H = \frac{1}{2} \sum_{i,j=1}^{N} m_i m_j \, e^{-|x_i - x_j|}$$

(Geodesics for metric $g^{ij} = e^{-|x_i - x_j|}$.)

Beals, Sattinger & Szmigielski (2000) solved these ODEs explicitly for arbitrary N.

(Rational functions of exponentials $e^{c_k t}$, where $c_1, \ldots, c_N$ are the asymptotic velocities of the peakons as $t \to \pm\infty$.)

- Inverse spectral methods.
- Spatial Lax equation $(\partial_x^2 - \frac{1}{4})\psi = -\frac{1}{2}\, z\, m\, \psi$ can be transformed to **string equation** $\phi'' = -z\, g\, \phi$ on the finite interval $(-1, 1)$.
- For peakons, m and g are discrete measures (lin. comb. of Dirac deltas). Inverse spectral problem for **discrete string** (Stieltjes, Krein).

Other integrable PDEs with peakon solutions:

- Degasperis–Procesi (1998):

$$m_t + m_x u + 3 m u_x = 0 \qquad (m = u - u_{xx})$$

- V. Novikov (2008):

$$m_t + (m_x u + 3 m u_x) u = 0 \qquad (m = u - u_{xx})$$

- Geng–Xue (2009):

$$\begin{aligned} m_t + m_x u v + 3 m v u_x &= 0 \qquad (m = u - u_{xx}) \\ n_t + n_x v u + 3 n u v_x &= 0 \qquad (n = v - v_{xx}) \end{aligned}$$

Remarkably, these three equations are all related to the so-called **cubic string**.

Spatial Lax equation for DP, $(\partial_x^3 - \partial_x)\psi = -z\, m\, \psi$, can be transformed to $\phi''' = -z\, g\, \phi$ on the finite interval $(-1, 1)$.

Inverse spectral problem for discrete cubic string with Dirichlet-like boundary conditions $\phi(-1) = \phi'(-1) = 0 = \phi(+1)$ gives explicit solution to DP N-peakon ODEs.

(Lundmark & Szmigielski 2003, 2005)

Cubic string:

$$\phi''' = -z\, g\, \phi$$

Let $(\phi_1, \phi_2, \phi_3) = (\phi, \phi', \phi'')$ to get a system of first-order equations:

$$\frac{\partial}{\partial y}\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\end{pmatrix} = \begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ -\lambda g(y) & 0 & 0\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\end{pmatrix}$$

Novikov's equation comes with a 3×3 matrix Lax pair. After transformation to the finite interval, we obtain the **dual cubic string**:

$$\frac{\partial}{\partial y}\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\end{pmatrix} = \begin{pmatrix}0 & g(y) & 0\\ 0 & 0 & g(y)\\ -\lambda & 0 & 0\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\end{pmatrix}$$

(Duality: $\frac{d\tilde{y}}{dy} = g(y) = 1/\tilde{g}(\tilde{y})$.
In the discrete case $g = \sum_{k=1}^{N} g_k \, \delta(y - y_k)$, the roles played by masses g_k and distances $l_k = y_{k+1} - y_k$ are interchanged.)

Inverse spectral theory for this problem gives explicit Novikov N-peakon solution.

(Hone, Lundmark & Szmigielski 2009)

Spatial Lax equation for GX (transformed):

$$\frac{\partial}{\partial y}\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\end{pmatrix} = \begin{pmatrix} 0 & h(y) & 0 \\ 0 & 0 & g(y) \\ -\lambda & 0 & 0 \end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\end{pmatrix}$$

Or, equally well:

$$\frac{\partial}{\partial y}\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\end{pmatrix} = \begin{pmatrix} 0 & g(y) & 0 \\ 0 & 0 & h(y) \\ -\lambda & 0 & 0 \end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\end{pmatrix}$$

We need to use spectra from *both* Lax pairs together to compute the peakon solutions!

(Lundmark & Szmigielski, work in progress)

The Geng–Xue peakon solutions have the form

$$u(x,t) = \sum_{i=1}^{N} m_i(t)\, e^{-|x-x_i(t)|}$$

$$v(x,t) = \sum_{i=1}^{N} n_i(t)\, e^{-|x-x_i(t)|}$$

where $m_i n_i = 0$ for all i. (Disjoint support.)

We study an *interlacing* setup where odd-numbered m_i and even-numbered n_i are nonzero. This introduces an asymmetry which makes the two Lax pairs slightly different.

To handle non-interlacing cases:

- Introduce additional peakons so that the problem becomes interlacing.
- Let the amplitudes of these auxiliary peakons tend to zero and see what you get in the limit.

(Related to the "ghostpeakon" problem: how to solve CH/DP/VN peakon ODEs when some $m_k(0) = 0$ in the initial data? Even though $m_k(t) \equiv 0$, there is a nontrivial equation for $x_k(t)$. But point masses of weight zero are invisible to the inverse spectral methods!)

(Lundmark, Shuaib & Szmigielski, work in progress)

A nice feature of GX peakons:

The solution involves Cauchy biorthogonal polynomials with respect to two different measures α and β (one spectral measure from each Lax pair).

(For DP and Novikov peakons, $\alpha = \beta$.)

(CBOPs: Bertola, Gekhtman & Szmigielski 2009, 2010, 2012)

Finally, to complete the picture:

By substituting $(x,t) \mapsto (\varepsilon x, \varepsilon t)$ and letting $\varepsilon \to 0$, we get a couple of other related PDEs.

- Camassa–Holm gives

$$m_t + m_x u + 2mu_x = 0 \qquad (m = u_{xx})$$

  Hunter–Saxton equation: $(u_t + uu_x)_{xx} = u_x u_{xx}$.
  (Nematic liquid crystals.)

- Degasperis–Procesi gives

$$m_t + m_x u + 3mu_x = 0 \qquad (m = u_{xx})$$

  Derivative of inviscid Burgers: $(u_t + uu_x)_{xx} = 0$.

These admit piecewise linear solutions instead of peakons:

$$u(x,t) = \sum_{i=1}^{N} m_i(t)\ |x - x_i(t)|$$

Inverse spectral methods involve ordinary / cubic string with Neumann(-like) boundary conditions.

DP and derivative Burgers also admit discontinuous (shock) solutions. DP shockpeakons – integrable or not? (Lax pair relies on rules of calculus which are not applicable to discontinuous functions.)