Peakons in pictures

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Dedicated to my mentor and friend, Jacek Szmigielski.

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Multipeakons and the Classical Moment Problem

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The Camassa–Holm equation

$$\boxed{u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}}$$

Equivalently (easier to remember):

$$\boxed{m_t + m_x u + 2mu_x = 0} \quad \text{where} \quad \boxed{m = u - u_{xx}}$$

Or rewrite the equation like this:

$$
\begin{aligned}
0 &= (u - u_{xx})_t + 3uu_x - 2u_x u_{xx} - uu_{xxx} \\
&= (1 - \partial_x^2)\left[u_t + \left(\tfrac{1}{2}u^2\right)_x\right] + \left(u^2 + \tfrac{1}{2}u_x^2\right)_x
\end{aligned}
$$

Taking $(1-\partial_x^2)^{-1}$ to be convolution with $\frac{1}{2}e^{-|x|}$ gives

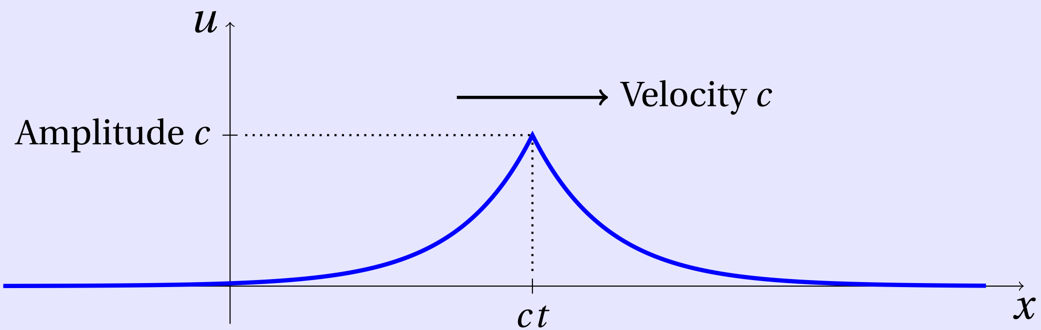
$$
\boxed{u_t + \partial_x\left[\tfrac{1}{2}u^2 + \tfrac{1}{2}e^{-|x|} * \left(u^2 + \tfrac{1}{2}u_x^2\right)\right] = 0}
$$

(Better formulation for rigorously defining weak solutions.)

The travelling wave

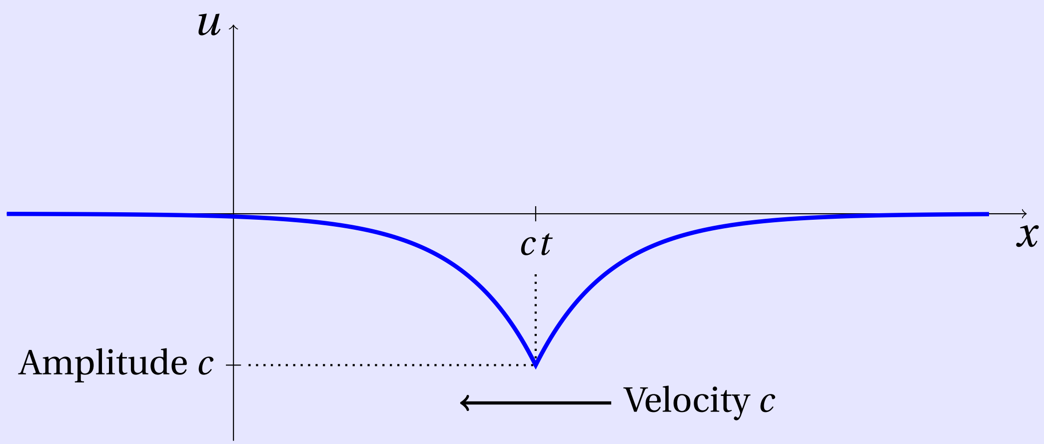
$$\boxed{u(x,t) = c\,e^{-|x-ct|}}$$

is a weak solution of the CH equation.

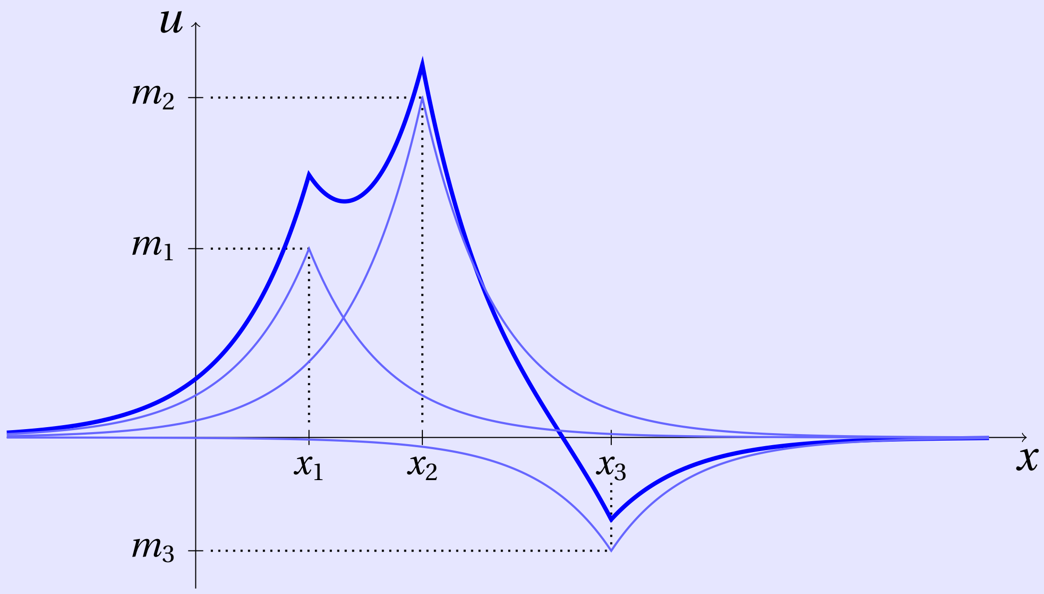


$\boxed{\textbf{Peakon} = \text{peaked soliton}}$

A peakon with $c < 0$ is sometimes called an **antipeakon**:



Multipeakon solutions



$$u(x,t) = \sum_{i=1}^{N} m_i(t)\, e^{-|x-x_i(t)|}$$

The multipeakon *Ansatz*

$$\boxed{u(x,t) = \sum_{i=1}^{N} m_i(t)\, e^{-|x-x_i(t)|}}$$

is a weak solution of the CH equation iff

$$\boxed{\dot{x}_k = u(x_k)} \qquad \boxed{\dot{m}_k = -m_k \big\langle u_x(x_k) \big\rangle}$$

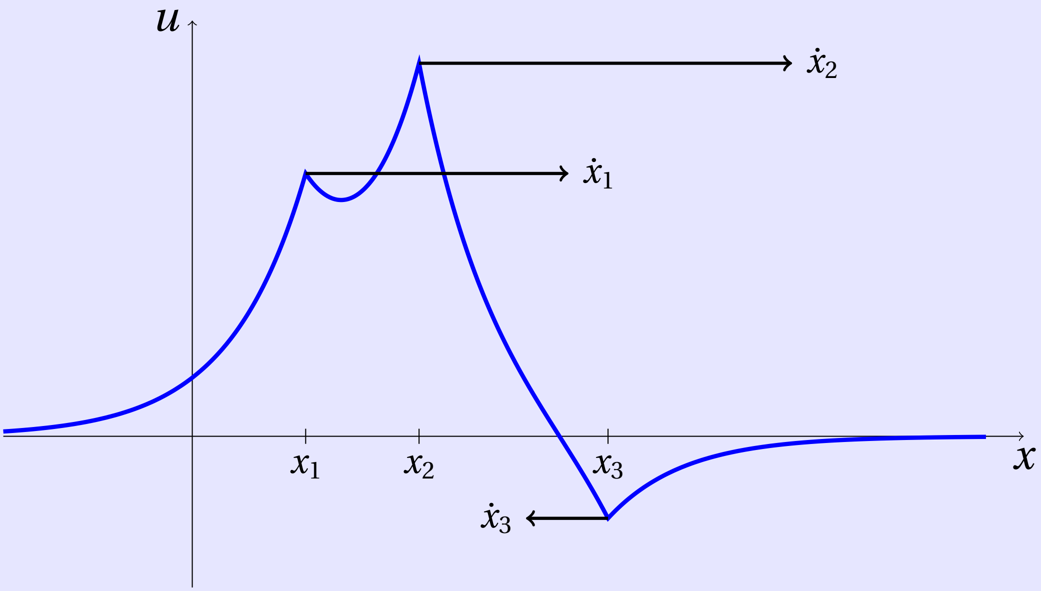
for $k = 1, \dots, N$.

So the PDE is reduced to a **finite-dimensional system of ODEs** for the positions x_k and the amplitudes m_k.

(Hamilton's equations with $H = \frac{1}{2} \sum_{i,j} m_i m_j\, e^{-|x_i - x_j|}$.)

ODE for positions

$$\boxed{\dot{x}_k = u(x_k)}$$

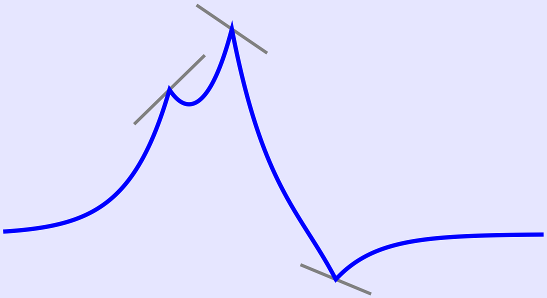


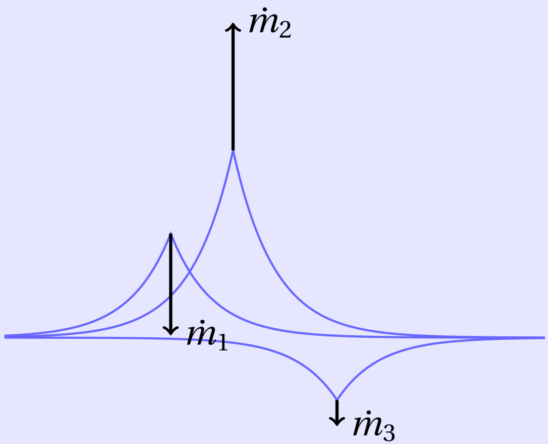
Velocity of kth peakon = elevation of the wave at at that point.

ODE for amplitudes

$$\boxed{\dot m_k = -m_k \left\langle u_x(x_k) \right\rangle}$$

$$\left\langle u_x(x_k) \right\rangle = \frac{u_x(x_k^-) + u_x(x_k^+)}{2} = \text{average slope of the wave at } x_k.$$





$$\boxed{\frac{d}{dt}\ln|m_k| = \frac{\dot m_k}{m_k} = -\left\langle u_x(x_k) \right\rangle} \qquad (\text{or } m_k(t) \equiv 0)$$

Positive/negative slope $\Longrightarrow$ $|m_k|$ decreasing/increasing.

Peakon ODEs, special cases

For $N = 1$, we get the travelling wave (single peakon or antipeakon):

$$\dot{x}_1 = m_1 \qquad \dot{m}_1 = 0$$

For $N = 2$, with $E_{ij} = e^{-|x_i - x_j|} = e^{x_i - x_j}$ for $i < j$:

$$\begin{aligned} \dot{x}_1 &= m_1 + m_2 E_{12} & \qquad \dot{m}_1 &= -m_1 m_2 E_{12} \\ \dot{x}_2 &= m_1 E_{12} + m_2 & \qquad \dot{m}_2 &= m_1 m_2 E_{12} \end{aligned}$$

Can be integrated directly in new variables $x_1 \pm x_2$ and $m_1 \pm m_2$, using constants of motion:

$$M = m_1 + m_2 \qquad H = \tfrac{1}{2}(m_1^2 + m_2^2 + 2 m_1 m_2 E_{12})$$

(We always assume $x_1 < x_2 < \cdots < x_N$. This is preserved at least locally in time, and in fact globally for **pure** peakon solutions where all $m_k > 0$.)

For $N = 3$, direct integration seems virtually impossible:

$$\dot{x}_1 = m_1 + m_2 E_{12} + m_3 E_{13}$$
$$\dot{x}_2 = m_1 E_{12} + m_2 + m_3 E_{23}$$
$$\dot{x}_3 = m_1 E_{13} + m_2 E_{23} + m_3$$

$$\dot{m}_1 = -m_1(m_2 E_{12} + m_3 E_{13})$$
$$\dot{m}_2 = -m_2(-m_1 E_{12} + m_3 E_{23})$$
$$\dot{m}_3 = -m_3(-m_1 E_{13} - m_2 E_{23})$$

But with the help of inverse spectral methods, the general solution (for arbitrary N) can be found explicitly.

(Beals, Sattinger & Szmigielski 2000)

Remark. $E_{ij} \approx 0$ if the peakons are far apart, so in that situation each peakon is approximately a travelling wave:

$$\dot{x}_k \approx m_k \qquad \dot{m}_k \approx 0$$

For $N = 3$, the **exact solution** looks like this:

$$x_1(t) = \ln \frac{(\lambda_1-\lambda_2)^2(\lambda_1-\lambda_3)^2(\lambda_2-\lambda_3)^2 a_1 a_2 a_3}{\sum_{j<k} \lambda_j^2 \lambda_k^2 (\lambda_j-\lambda_k)^2 a_j a_k}$$

$$x_2(t) = \ln \frac{(\lambda_1-\lambda_2)^2 a_1 a_2 + (\lambda_1-\lambda_3)^2 a_1 a_3 + (\lambda_2-\lambda_3)^2 a_2 a_3}{\lambda_1^2 a_1 + \lambda_2^2 a_2 + \lambda_3^2 a_3}$$

$$x_3(t) = \ln(a_1 + a_2 + a_3)$$

$$m_1(t) = \frac{\sum_{j<k} \lambda_j^2 \lambda_k^2 (\lambda_j-\lambda_k)^2 a_j a_k}{\lambda_1\lambda_2\lambda_3 \sum_{j<k} \lambda_j \lambda_k (\lambda_j-\lambda_k)^2 a_j a_k}$$

$$m_2(t) = \frac{\left(\lambda_1^2 a_1 + \lambda_2^2 a_2 + \lambda_3^2 a_3\right) \sum_{j<k} (\lambda_j-\lambda_k)^2 a_j a_k}{(\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3) \sum_{j<k} \lambda_j \lambda_k (\lambda_j-\lambda_k)^2 a_j a_k}$$

$$m_3(t) = \frac{a_1 + a_2 + a_3}{\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3}$$

where $\lambda_k = \text{constant}$ and $a_k(t) = a_k(0)\, e^{t/\lambda_k}$ (for $k = 1, 2, 3$).

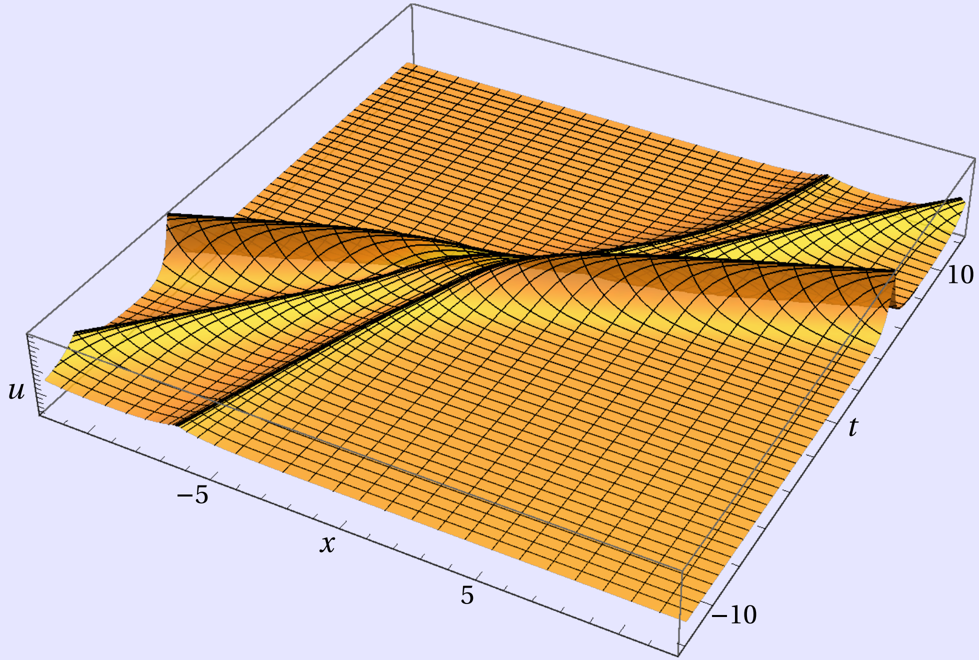
The asymptotic velocities (and amplitudes) are $\frac{1}{\lambda_k}$.

For example, suppose $\boxed{\frac{1}{\lambda_1}} > \frac{1}{\lambda_2} > \frac{1}{\lambda_3}$. Then, as $\boxed{t \to +\infty}$:

$$
\begin{aligned}
\boxed{x_3(t)} &= \ln\Big(a_1(0)\,\boxed{e^{t/\lambda_1}} + a_2(0)\,e^{t/\lambda_2} + a_3(0)\,e^{t/\lambda_3}\Big)\\
&= \ln\Big(a_1(0)\,e^{t/\lambda_1}\,\big(1+o(1)\big)\Big)\\
&= \boxed{\frac{t}{\lambda_1}} + \ln a_1(0) + o(1)
\end{aligned}
$$

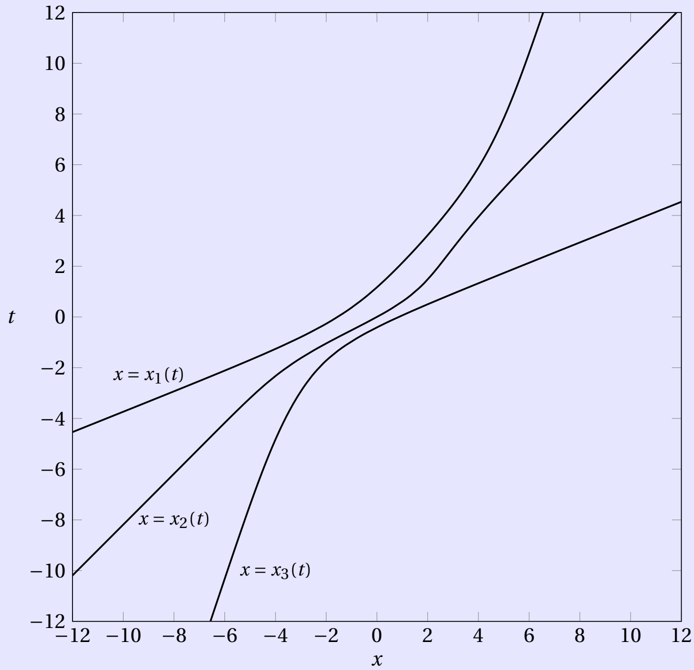
$$
\begin{aligned}
\boxed{m_3(t)} &= \frac{a_1(0)\,\boxed{e^{t/\lambda_1}} + a_2(0)\,e^{t/\lambda_2} + a_3(0)\,e^{t/\lambda_3}}{\lambda_1 a_1(0)\,\boxed{e^{t/\lambda_1}} + \lambda_2 a_2(0)\,e^{t/\lambda_2} + \lambda_3 a_3(0)\,e^{t/\lambda_3}}\\
&= \frac{a_1(0)\,e^{t/\lambda_1}\,\big(1+o(1)\big)}{\lambda_1 a_1(0)\,e^{t/\lambda_1}\,\big(1+o(1)\big)}\\
&= \boxed{\frac{1}{\lambda_1}} + o(1)
\end{aligned}
$$

A pure 3-peakon solution of the CH equation:



Asymptotic velocities: $\frac{1}{\lambda_1} = 3 \qquad \frac{1}{\lambda_2} = 1 \qquad \frac{1}{\lambda_3} = \frac{2}{5}$

Viewing this from above, we see the positions $x = x_k(t)$:



Incoming velocity for x_1 = outgoing velocity for x_3. (Etc.)

Always $x_1(t) < x_2(t) < x_3(t)$, so no overtaking.
More like runners in a relay race.

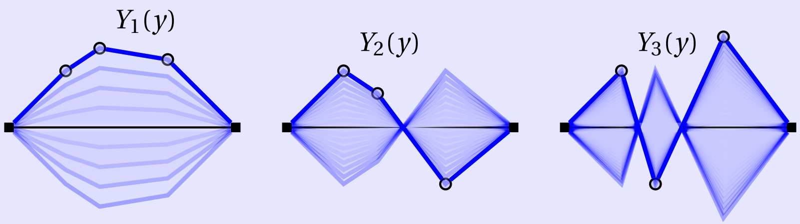
Sketch of derivation of N-peakon solution formulas

Consider the **usual linear wave equation** $U_{yy} = g(y)\,U_{\tau\tau}$ for vibrations of a string with **mass distribution** $g(y)$. Assume ends are attached at $y = \pm 1$.

Separation of variables $U(y,\tau) = Y(y)\,T(\tau)$ gives a selfadjoint problem for the vibrational eigenmodes of the string:

$$\boxed{-Y''(y) = z\,g(y)\,Y(y)} \qquad \boxed{Y(-1) = Y(1) = 0} \qquad \Big(\text{and } \ddot{T}(\tau) = -z\,T(\tau)\Big)$$

In particular: **discrete** mass distributions $g(y) = \sum_{k=1}^{N} g_k \delta(y - y_k)$ give eigenfunctions $Y(y)$ that are **piecewise linear**.



Eigenvalues (squared eigenfrequencies): $z = \lambda_1, \ldots, \lambda_N$. Real, nonzero, distict.

Number of pos./neg. eigenvalues λ_k = number of pos./neg. weights g_k.

Residues of Weyl function: $a_1, \ldots, a_N$. Positive. (Constant times $Y_k'(1)/Y_k'(0)$.)

Integrability magic (thanks to the **Lax pair** for the CH equation):

- Let $\{x_k(t), m_k(t)\}_{k=1}^N$ evolve in time according to the Camassa–Holm peakon ODEs.
- Consider an associated discrete string where the positions $y_k \in (-1, 1)$ and weights g_k of the point masses are given by $y_k = \tanh(x_k/2)$ and $g_k = 2m_k/(1 - y_k^2)$.
- Then this string deforms **isospectrally**, meaning that the eigenvalues λ_k stay constant as t changes.

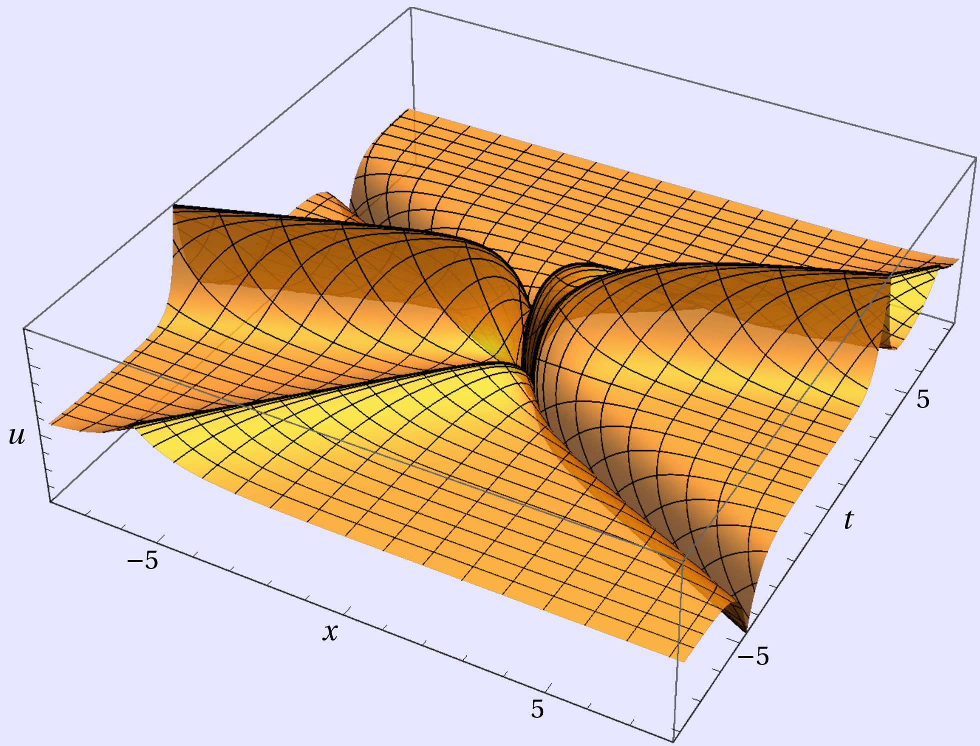
  Moreover, the residues a_k satisfy $\frac{d}{dt} a_k = a_k/\lambda_k \quad \big(\Longleftrightarrow\ a_k(t) = a_k(0)\, e^{t/\lambda_k}\big)$.

The **inverse spectral problem** is to **reconstruct the string** from the spectral data λ_k and a_k. The solution involves Stieltjes continued fractions, orthogonal polynomials, Padé approximation, etc.

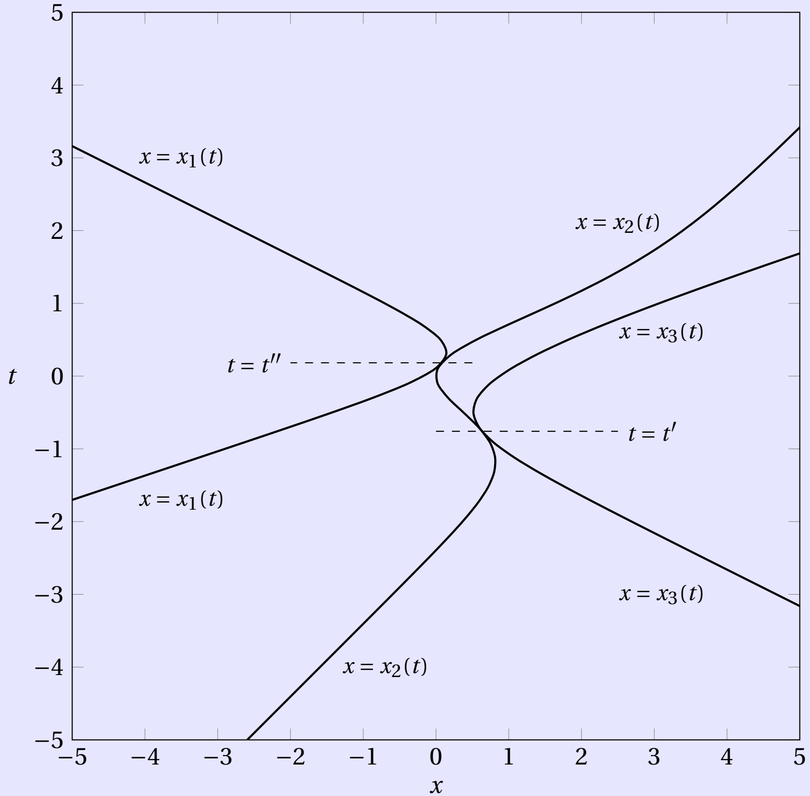
(Stieltjes 1894, Krein 1951, Moser 1975)

The time evolution of the spectral data is known, so reconstructing the string gives the time evolution of the string variables $y_k(t)$ and $g_k(t)$, and hence of the peakon variables $x_k(t)$ and $m_k(t)$. Done!

A mixed CH solution (two peakons, one antipeakon):

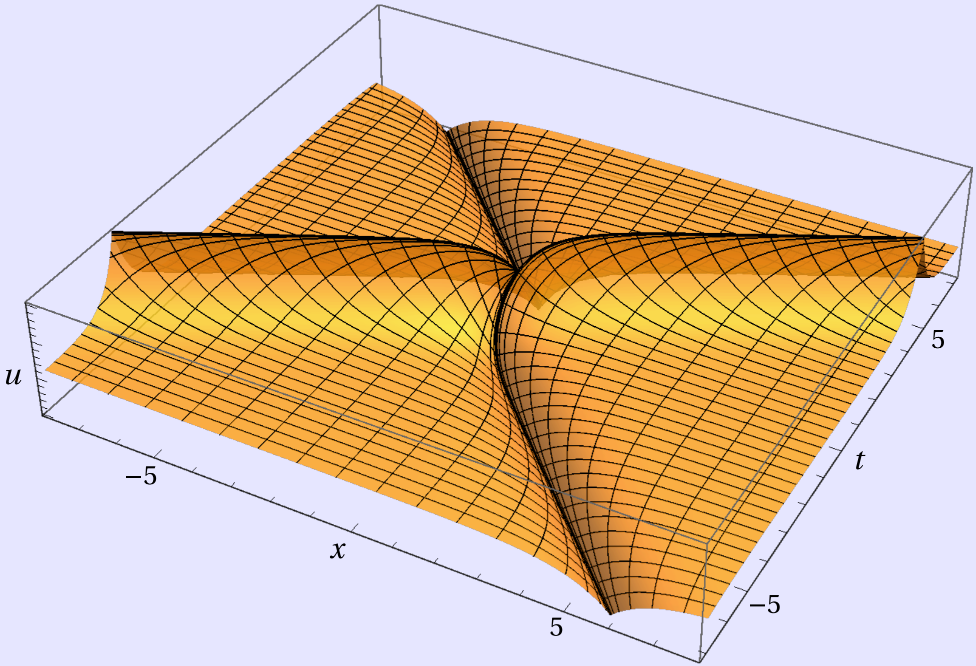


$\frac{1}{\lambda_1} > \frac{1}{\lambda_2} > 0 > \frac{1}{\lambda_3}$ $\qquad$ $\frac{1}{\lambda_1} = 3$ $\qquad$ $\frac{1}{\lambda_2} = 1$ $\qquad$ $\frac{1}{\lambda_3} = -2$

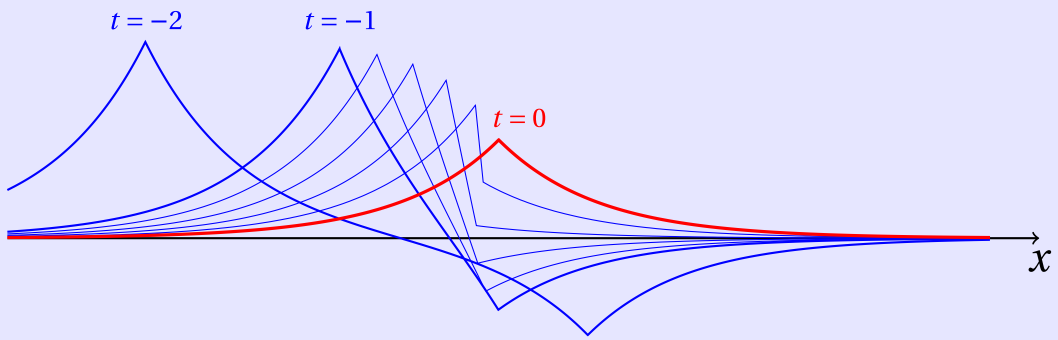


Collisions at $t = t'$ and $t = t''$. Derivative u_x blows up, u doesn't.

Easier to see with just one peakon and one antipeakon:



$\frac{1}{\lambda_1} > 0 > \frac{1}{\lambda_2}$ $\frac{1}{\lambda_1} = 2$ $\frac{1}{\lambda_2} = -1$



At the collision, there is only *one* peakon

$$u(x,0) = \lim_{t \nearrow 0} u(x,t) = m_0\, e^{-|x|}$$

of amplitude

$$m_0 = \lim_{t \nearrow 0} \big(m_1(t) + m_2(t) \big)$$

where $m_1(t) \to +\infty$ and $m_2(t) \to -\infty$.

(Actually $M = m_1 + m_2$ is a constant of motion for $N = 2$, so $m_0 = M$.)

Similarly for collisions in general ($N > 2$).

Weak solutions of the CH eqn are **not unique**.

In particular, continuation past the collision is **not unique**.

- Above: **Conservative** solution (Constantin & Escher 1998).

  Peakon & antipeakon reappear.

  The energy $E(t) = \frac{1}{2}\int_{\mathbf{R}}(u^2 + u_x^2)\,dx$ drops at $t = t_0$, and then immediately returns to its previous value.

Also possible:

- **Dissipative** solution (Bressan & Constantin 2007).

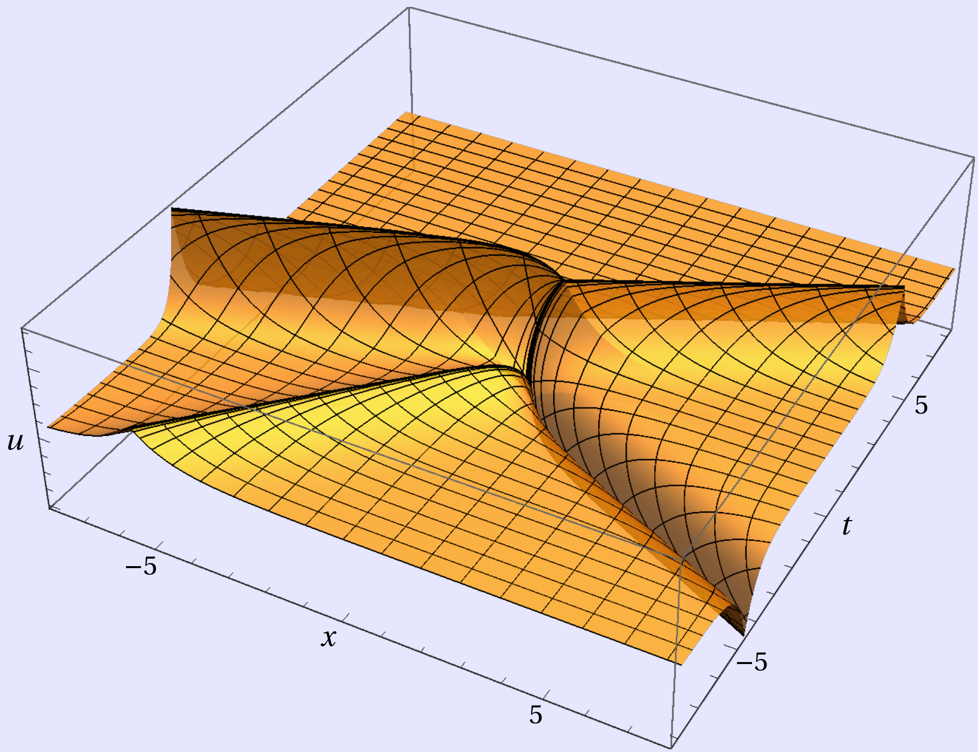
  Peakons stay merged. $E(t)$ only drops, never increases.

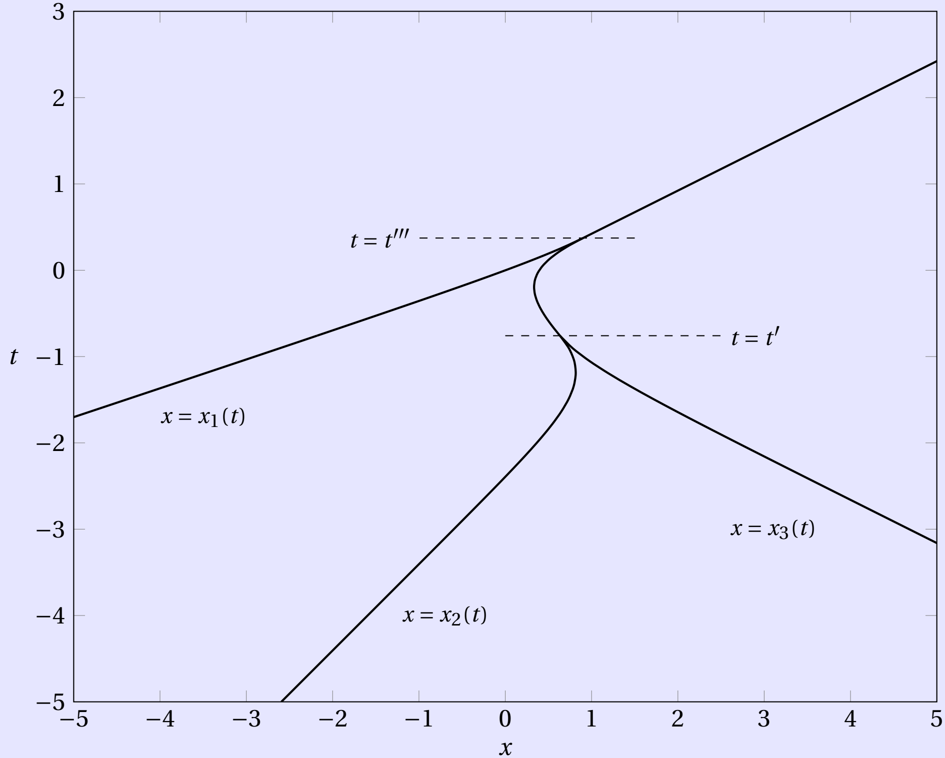
- α**-dissipative** solution (Grunert, Holden & Raynaud 2015).

  Peakon & antipeakon reappear, but a fraction $0 < \alpha < 1$ of the energy concentrated at the collision is lost).

  (Conservative if $\alpha = 0$, dissipative if $\alpha = 1$.)

Dissipative continuation of the solution starting out as above:





Peakons merge at collisions.

$$\boxed{m_t + m_x u + 2 m u_x = 0}$$ Camassa–Holm (1993)

Some other integrable PDEs with peakon solutions:

$$\boxed{m_t + m_x u + 3 m u_x = 0}$$ Degasperis–Procesi (1998)

$$\boxed{m_t + (m_x u + 3 m u_x) u = 0}$$ V. Novikov (2008)

$$\boxed{\begin{aligned} m_t + (m_x u + 3 m u_x) v = 0 \\ n_t + (n_x v + 3 n v_x) u = 0 \end{aligned}}$$ Geng–Xue (2009)

where $\boxed{m = u - u_{xx}}$ and $\boxed{n = v - v_{xx}}$

Multipeakon solutions have the form $u = \sum_{i=1}^{N} m_i \, e^{-|x-x_i|}$ for all these equations, but the ODEs differ:

CH	$\boxed{\dot{x}_k = u(x_k)}$	$\boxed{\dot{m}_k = -m_k \langle u_x(x_k) \rangle}$
DP	$\boxed{\dot{x}_k = u(x_k)}$	$\boxed{\dot{m}_k = -2m_k \langle u_x(x_k) \rangle}$
Novikov	$\boxed{\dot{x}_k = u(x_k)^2}$	$\boxed{\dot{m}_k = -m_k \, u(x_k) \langle u_x(x_k) \rangle}$

GX

$$\boxed{\dot{x}_k = u(x_k)\, v(x_k)} \quad \boxed{\dot{m}_k = m_k \big(u(x_k) \langle v_x(x_k) \rangle - 2 \langle u_x(x_k) \rangle \, v(x_k) \big)}$$

$$\boxed{\dot{y}_k = u(y_k)\, v(y_k)} \quad \boxed{\dot{n}_k = n_k \big(\langle u_x(y_k) \rangle \, v(y_k) - 2u(y_k) \langle v_x(y_k) \rangle \big)}$$

where $u = \sum_{i=1}^{N_1} m_i \, e^{-|x-x_i|}$, $v = \sum_{j=1}^{N_2} n_j \, e^{-|x-y_j|}$, $x_i \neq y_j$.

Degasperis–Procesi peakons

Explicit N-peakon solution formulas via the inverse spectral problem for the discrete **cubic string**:

$$-\boxed{Y'''(y)} = z\,g(y)\,Y(y)$$

$$Y(-1) = Y'(-1) = 0 \qquad Y(1) = 0$$

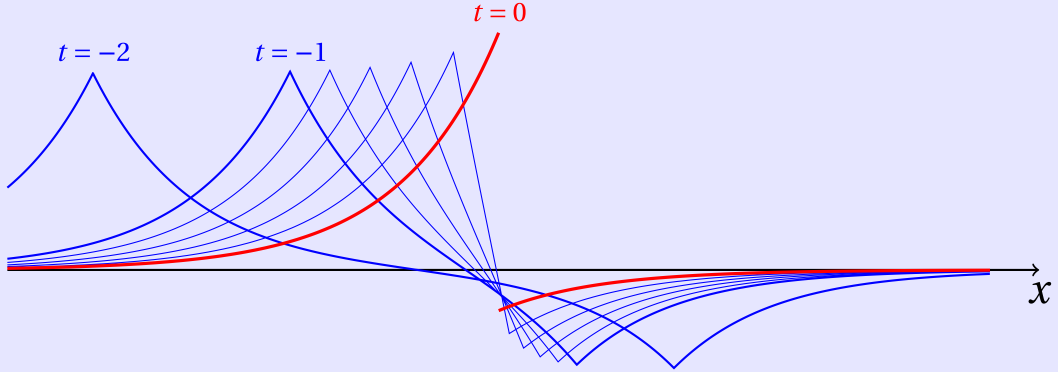
(Lundmark & Szmigielski 2005)

Positive & simple eigenvalues for **pure** peakon solutions (Gantmacher–Krein theory of oscillatory kernels). Eigenvalues can be complex in general.

Cauchy biorthogonal polynomials (Bertola, Gekhtman & Szmigielski 2009).

Pure peakon solutions for DP look similar to CH.

But **peakon–antipeakon collisions** for DP behave very differently:



In the limit, $u(x, t)$ develops a **shock**.

(Lundmark 2007, Szmigielski & Zhou 2013)

The DP equation rewritten as

$$\boxed{u_t + \partial_x \left[\tfrac{1}{2} u^2 + \tfrac{1}{2} e^{-|x|} * \tfrac{3}{2} u^2 \right] = 0}$$

admits weak solution that need not be continuous. Entropy condition gives existence and uniqueness.

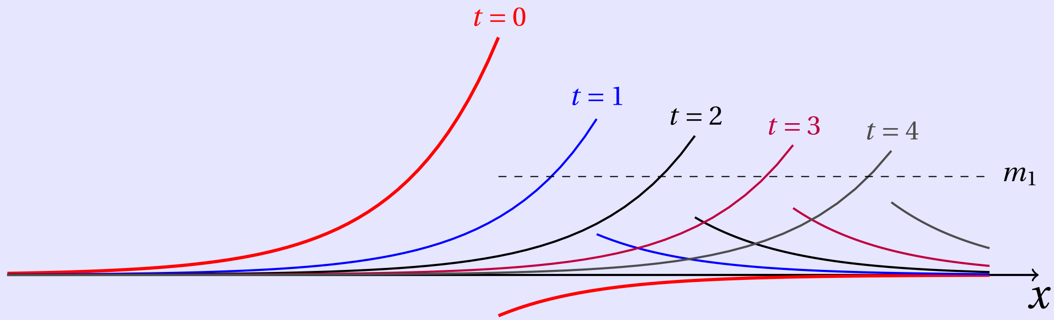
(Coclite & Carlsen 2006)

The entropy solution after the collision (for $t \geq 0$) is a **shockpeakon**:

$$\boxed{u(x,t) = \big(m_1(t) - s_1(t) \operatorname{sgn}(x - x_1(t)) \big)\, e^{-|x - x_1(t)|}}$$

$$\boxed{\dot{x}_1 = m_1} \qquad \boxed{\dot{m}_1 = 0} \qquad \boxed{\dot{s}_1 = -s_1^2}$$

(Lundmark 2007)



Constant velocity $\dot{x}_1$
= average amplitude m_1 at the jump.

Shock strength $s_1(t) = \dfrac{s_1(0)}{t + s_1(0)}$ decays like $\dfrac{1}{t}$.

More generally: **multishockpeakon solutions**.

$$u(x,t) = \sum_{k=1}^{N} \big(m_k(t) - s_k(t)\,\mathrm{sgn}(x - x_k(t)) \big)\, e^{-|x - x_k(t)|}$$

Governed by a system of $3N$ ODEs for positions x_k, amplitudes m_k and shock strengths s_k.

(Lundmark 2007)

Open problem:

Are these ODEs **integrable**?

(Trivial for $N = 1$, but very little is known for $N \geq 2$.)

Novikov peakons

Explicit N-peakon solution using **dual** discrete cubic string (swap roles of distances and masses).

Positive & simple eigenvalues in pure peakon case.

(Hone, Lundmark & Szmigielski 2009)

For mixed peakon–antipeakon solutions, eigenvalues need not be real, nor simple.

The real part is always nonnegative.

Since $\dot{x}_k = u(x_k)^2$, both peakons and antipeakons move to the right, but they collide anyway.

(Kardell & Lundmark, in preparation)

At collisions, u stays continuous, and

$$E(t) = \frac{1}{2}\int_{\mathbf{R}} (u^2 + u_x^2)\,dx$$

is preserved.

Conservative solutions are instead defined using a conserved quantity of degree four,

$$F(t) = \int_{\mathbf{R}} (u^4 + 2u^2u_x^2 - \tfrac{1}{3}u_x^4)\,dx.$$

(Chen, Chen & Liu 2018)

Novikov two-peakon ODEs

$$\dot{x}_1 = (m_1 + m_2 E_{12})^2$$
$$\dot{x}_2 = (m_1 E_{12} + m_2)^2$$

$$\dot{m}_1 = -m_1 m_2 E_{12}(m_1 + m_2 E_{12})$$
$$\dot{m}_2 = m_1 m_2 E_{12}(m_1 E_{12} + m_2)$$

Constants of motion:

$$H_1 = c_1 + c_2 = m_1^2 + m_2^2 + 2 m_1 m_2 E_{12}$$
$$H_2 = c_1 c_2 = m_1^2 m_2^2 (1 - E_{12}^2)$$

where $c_{1,2}$ are asymptotic velocities (for pure peakons at least).

Already this two-peakon system is quite difficult to solve completely by direct integration!

- Hone & Wang (2008) give explicit expressions (in the pure peakon case) for $m_2(t)^2 - m_1(t)^2$, $m_1(t)\, m_2(t)$ and $x_2(t) - x_1(t)$, together with

$$x_1(t) + x_2(t) = (c_1 + c_2)(t - t_0) + \int f(T)\, dT$$

$$f(T) = \frac{2(c_1^2 - c_2^2)\left((c_1 - c_2)^2 + 8c_1 c_2 \cosh^2 T\right)}{(c_1 - c_2)^4 + 16 c_1 c_2 (c_1 + c_2)^2 \cosh^4 T} \qquad T = \tfrac{1}{2}(c_1 - c_2)(t - t_0)$$

"and the quadrature can be performed explicitly by partial fractions in $\tanh(T)$, but the answer is omitted here."

- Himonas, Holliman & Kenig (2018), looking specifically at the peakon–antipeakon case $x_1(0) = -a$, $x_2(0) = a$, $m_1(0) = -b-\delta$, $m_2(0) = b$, give formulas for $m_2(t) \pm m_1(t)$ and $m_1(t)\, m_2(t)$ in terms of $q(t) = x_2(t) - x_1(t)$, together with an ODE for $q(t)$,

$$\dot{q}(t) = -\left(w_0^2 + 2z_1\left(\frac{\sqrt{1-e^{-2q_0}}}{1+e^{-q_0}} - \frac{\sqrt{1-e^{-2q(t)}}}{1+e^{-q(t)}}\right)\right)^{1/2} \times \left(p_0^2 + 2z_1\left(\frac{1+e^{-q(t)}}{\sqrt{1-e^{-2q(t)}}} - \frac{1+e^{-q_0}}{\sqrt{1-e^{-2q_0}}}\right)\right)^{1/2} (1 - e^{-2q(t)})$$

They then estimate the solution in terms of solutions to a simpler ODE.

Novikov two-peakon solution, as obtained from the (almost) general N-peakon solution formulas:

$$x_1(t) = \tfrac{1}{2}\ln \frac{\frac{(\lambda_1-\lambda_2)^4}{(\lambda_1+\lambda_2)^2\lambda_1\lambda_2}\, b_1^2 b_2^2}{\lambda_1\, b_1^2 + \lambda_2\, b_2^2 + \frac{4\,\lambda_1\lambda_2}{\lambda_1+\lambda_2}\, b_1 b_2}$$

$$x_2(t) = \tfrac{1}{2}\ln\!\left(\frac{1}{\lambda_1}\, b_1^2 + \frac{1}{\lambda_2}\, b_2^2 + \frac{4}{\lambda_1+\lambda_2}\, b_1 b_2\right)$$

$$m_1(t) = \frac{\left(\frac{(\lambda_1-\lambda_2)^4}{(\lambda_1+\lambda_2)^2\lambda_1\lambda_2}\, b_1^2 b_2^2 \left(\lambda_1\, b_1^2 + \lambda_2\, b_2^2 + \frac{4\,\lambda_1\lambda_2}{\lambda_1+\lambda_2}\, b_1 b_2\right)\right)^{1/2}}{\frac{(\lambda_1-\lambda_2)^2}{\lambda_1+\lambda_2}\, b_1 b_2\,(b_1+b_2)}$$

$$m_2(t) = \frac{\left(\frac{1}{\lambda_1}\, b_1^2 + \frac{1}{\lambda_2}\, b_2^2 + \frac{4}{\lambda_1+\lambda_2}\, b_1 b_2\right)^{1/2}}{b_1+b_2}$$

where $\quad b_1(t) = b_1(0)\, e^{t/\lambda_1} \qquad b_2(t) = b_2(0)\, e^{t/\lambda_2}$

- m_1 & m_2 positive $\iff$ $0 < \lambda_1 < \lambda_2$, b_1 & b_2 positive.
- m_1 & m_2 negative $\iff$ $0 < \lambda_1 < \lambda_2$, b_1 & b_2 negative.
- m_1 & m_2 of opposite signs $\iff$

$$\boxed{0 < \lambda_1 < \lambda_2, \quad b_1 \text{ \& } b_2 \text{ of opposite signs.}}$$

or

$\lambda_2 = \overline{\lambda_1}$ with positive real part and nonzero imaginary part,
$b_2 = \overline{b_1}$ nonzero.

or

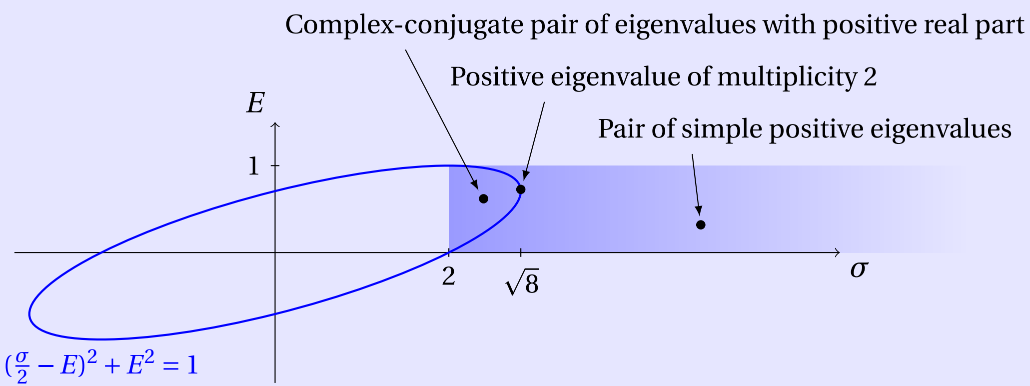
Double eigenvalue $0 < \lambda_1 = \lambda_2$ for which the above formulas are **not valid**, since b_1 & b_2 are not meaningful.

(**Separate solution formulas** for this case can be obtained by letting $\lambda_2 \to \lambda_1$ while handling $b_{1,2}$ properly.)

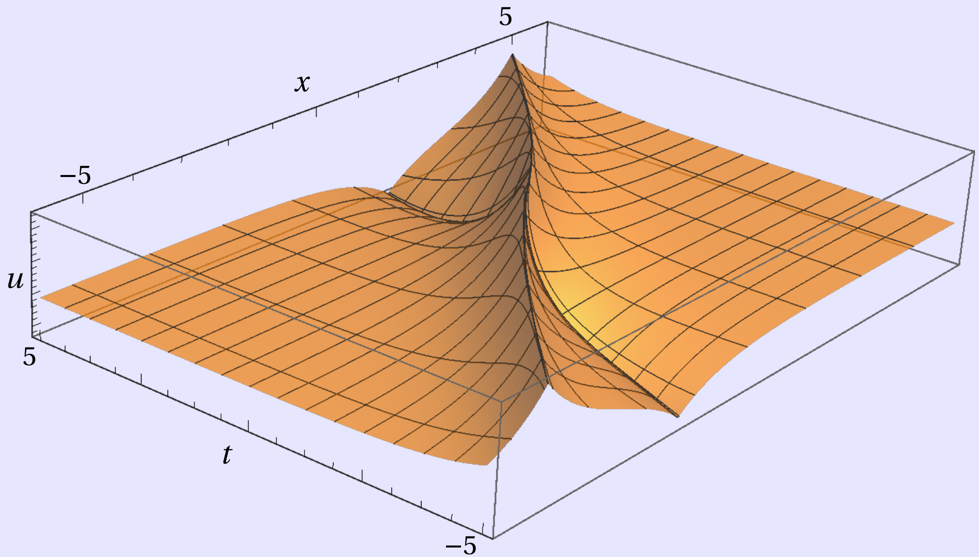
The character of the eigenvalues in the mixed peakon–antipeakon case are determined by

$$\boxed{E = e^{x_1 - x_2}} \qquad \text{and} \qquad \boxed{\sigma = \left|\frac{m_1}{m_2}\right| + \left|\frac{m_2}{m_1}\right|}$$

as follows:



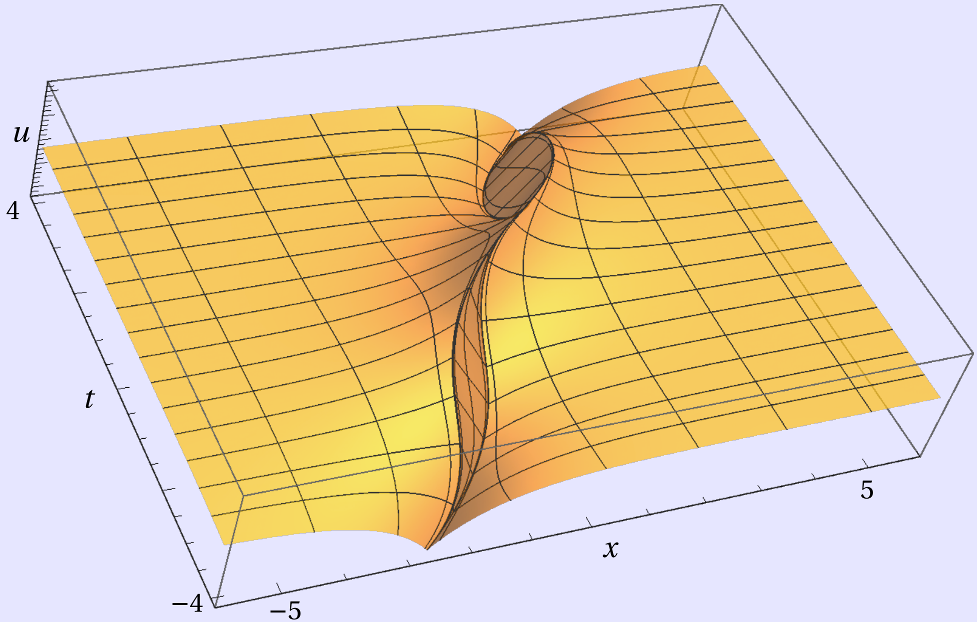
Real and simple eigenvalues (conservative solution)



$\boxed{\frac{1}{\lambda_1} = 1}$ $\boxed{\frac{1}{\lambda_2} = \frac{1}{2}}$ $b_{1,2}(0) = \pm\sqrt{6}$

Just one collision. Scattering. Asymptotic velocities 1 and $\frac{1}{2}$.

Complex eigenvalues (conservative solution)



$$\boxed{\frac{1}{\lambda_{1,2}} = \frac{2}{3} \pm i\frac{\pi}{4}} \qquad b_{1,2}(0) = \mp\frac{4i}{\pi\sqrt{3}}$$

Periodic + drift of velocity $\frac{2}{3}$. Frequency $\frac{\pi}{4}$ (i.e. period $\frac{2\pi}{\pi/4} = 8$).

The Novikov equation can be written as

$$\boxed{u_t + u^2 u_x + \tfrac{1}{2} e^{-|x|} * \Bigl(\bigl(u^3 + \tfrac{3}{2} u u_x^2 \bigr)_x + \tfrac{1}{2} u_x^3 \Bigr) = 0}$$

- Well-posed in H^s for $s > \frac{3}{2}$.

  Sobolev embedding: $H^s \subset C^1$. Smoother than peakons.

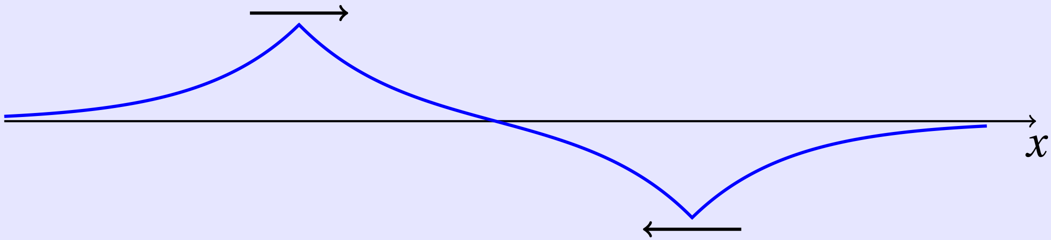
  (Tığlay 2011 for $s > \frac{5}{2}$, Himonas & Holliman 2012)

- But ill-posed for $s < \frac{3}{2}$ (peakons).

  Shown by studying the $N = 2$ peakon–antipeakon solution.

  (Himonas, Holliman & Kenig 2018)

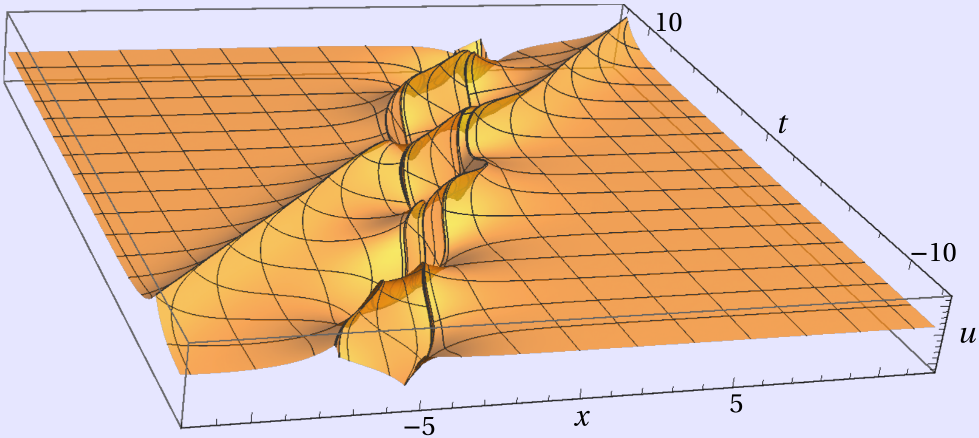
Similar results for CH and DP are much easier to obtain, since one can get them already from the particularly simple **antisymmetric** peakon–antipeakon solution:



$$x_1 + x_2 = m_1 + m_2 = 0$$

The Novikov equation has no such solution, since $\dot{x}_k = u(x_k)^2$ cannot be negative. Everything moves to the right.

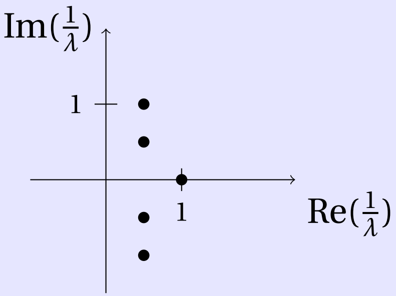
A conservative **5-peakon solution** of the Novikov equation:

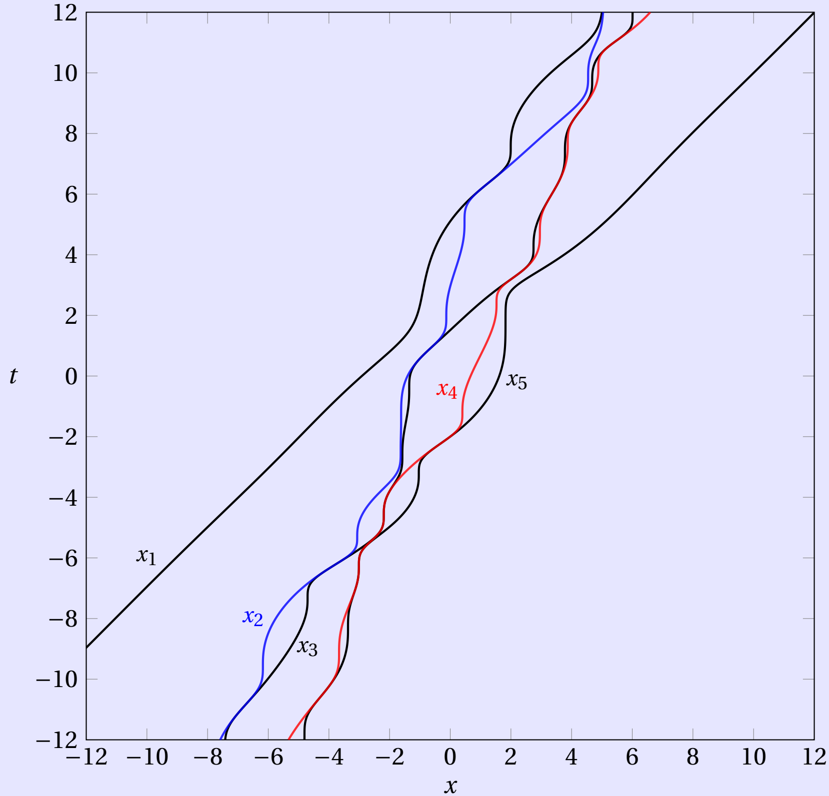


Three peakons, two antipeakons.

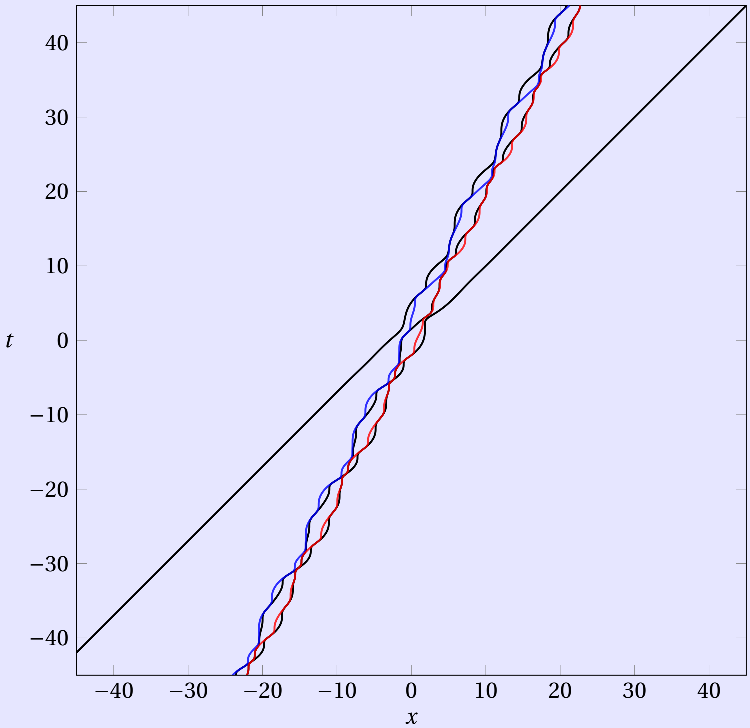
One real eigenvalue $\lambda = 1$.

Two complex-conjugate eigenvalue pairs with common value of $\mathrm{Re}(1/\lambda) = \frac{1}{2}$.





$x_{k+1}(t) - x_k(t) = O\big((t - t_0)^4\big)$ at a typical collision. Powers $4k$ can also occur.
(The power is always 2 for Camassa–Holm collisions.)



Asymptotically, a **solitary peakon** (velocity $\frac{1}{\lambda} = 1$) and a **four-peakon cluster** with overall drift velocity $\mathrm{Re}(\frac{1}{\lambda}) = \frac{1}{2}$ and two frequencies $\mathrm{Im}(\frac{1}{\lambda}) \in \{\frac{1}{2}, 1\}$.

To describe the **asymptotics** precisely, one needs the **exact** 4-peakon solution formulas!

Lots of possibilities:

- Arbitrarily many clusters, each with arbitrarily many peakons.
- Frequencies commensurable or not.

  (Periodic or quasi-periodic behaviour.)
- Eigenvalues of higher multiplicity.

  (Peakons separate at logarithmic rate as $t \to \pm\infty$.)
- Purely imaginary eigenvalues (if $N \geq 3$).

  (Peakons slow to a halt, amplitude tends to zero.)

(Kardell & Lundmark)

Geng–Xue peakons

Peakons in u and v must be **non-overlapping**.

First: Solution formulas for **interlacing** $K+K$ case

$$\boxed{u(x,t)} = \sum_{i=1}^{K} m_i(t)\, e^{-|x-x_i(t)|}$$

$$\boxed{v(x,t)} = \sum_{i=1}^{K} n_i(t)\, e^{-\left|x-y_i(t)\right|}$$

where

$$\boxed{x_1} < \boxed{y_1} < \boxed{x_2} < \boxed{y_2} < \cdots < \boxed{x_K} < \boxed{y_K}$$

(Lundmark & Szmigielski 2016, 2017)

The GX equation has **two** Lax pairs (swap u and v), leading to **two** spectral problems of cubic string type.

The solution formulas for the $K+K$ interlacing case contain two sets of constant eigenvalues

$$\{\lambda_i\}_{i=1}^{K}, \qquad \{\mu_j\}_{j=1}^{K-1}$$

with associated residues $\{a_i\}_{i=1}^{K}$ and $\{b_j\}_{j=1}^{K-1}$ such that

$$a_i(t) = a_i(0)\, e^{t/\lambda_i}, \qquad b_j(t) = b_j(0)\, e^{t/\mu_j},$$

plus two additional constants C and D also coming from the spectral problems.

($4K$ parameters in total, as it should be.)

Example. The solution formulas for the $3+3$ interlacing case:

$$X_1 = \tfrac{1}{2}e^{2x_1} = \frac{\mathrm{J}_{32}^{00}}{\mathrm{J}_{21}^{11} + C\,\mathrm{J}_{22}^{10}} \qquad Y_1 = \tfrac{1}{2}e^{2y_1} = \frac{\mathrm{J}_{32}^{00}}{\mathrm{J}_{21}^{11}}$$

$$X_2 = \tfrac{1}{2}e^{2x_2} = \frac{\mathrm{J}_{22}^{00}}{\mathrm{J}_{11}^{11}} \qquad Y_2 = \tfrac{1}{2}e^{2y_2} = \frac{\mathrm{J}_{21}^{00}}{\mathrm{J}_{10}^{11}}$$

$$X_3 = \tfrac{1}{2}e^{2x_3} = \mathrm{J}_{11}^{00} \qquad Y_3 = \tfrac{1}{2}e^{2y_3} = \mathrm{J}_{11}^{00} + D\,\mathrm{J}_{10}^{00}$$

$$Q_1 = 2m_1 e^{-x_1} = \frac{\mu_1\mu_2}{\lambda_1\lambda_2\lambda_3}\left(\frac{\mathrm{J}_{21}^{11}}{\mathrm{J}_{22}^{10}} + C\right) \qquad P_1 = 2n_1 e^{-y_1} = \frac{\mathrm{J}_{21}^{11}\,\mathrm{J}_{22}^{10}}{\mathrm{J}_{21}^{01}\,\mathrm{J}_{32}^{01}}$$

$$Q_2 = 2m_2 e^{-x_2} = \frac{\mathrm{J}_{11}^{11}\,\mathrm{J}_{21}^{01}}{\mathrm{J}_{11}^{10}\,\mathrm{J}_{22}^{10}} \qquad P_2 = 2n_2 e^{-y_2} = \frac{\mathrm{J}_{10}^{11}\,\mathrm{J}_{11}^{10}}{\mathrm{J}_{10}^{01}\,\mathrm{J}_{21}^{01}}$$

$$Q_3 = 2m_3 e^{-x_3} = \frac{\mathrm{J}_{10}^{01}}{\mathrm{J}_{11}^{10}} \qquad P_3 = 2n_3 e^{-y_3} = \frac{1}{\mathrm{J}_{10}^{00}}$$

where, for instance,

$$\begin{aligned}\mathrm{J}_{21}^{01} &= \frac{(\lambda_1-\lambda_2)^2\,\mu_1}{(\lambda_1+\mu_1)(\lambda_2+\mu_1)}\,a_1a_2b_1 + \frac{(\lambda_1-\lambda_3)^2\,\mu_1}{(\lambda_1+\mu_1)(\lambda_3+\mu_1)}\,a_1a_3b_1 + \frac{(\lambda_2-\lambda_3)^2\,\mu_1}{(\lambda_2+\mu_1)(\lambda_3+\mu_1)}\,a_2a_3b_1 \\ &+ \frac{(\lambda_1-\lambda_2)^2\,\mu_2}{(\lambda_1+\mu_2)(\lambda_2+\mu_2)}\,a_1a_2b_2 + \frac{(\lambda_1-\lambda_3)^2\,\mu_2}{(\lambda_1+\mu_2)(\lambda_3+\mu_2)}\,a_1a_3b_2 + \frac{(\lambda_2-\lambda_3)^2\,\mu_2}{(\lambda_2+\mu_2)(\lambda_3+\mu_2)}\,a_2a_3b_2\end{aligned}$$

Collisions lead to shockpeakon formation, so here we assume **pure** peakon solutions (no antipeakons).

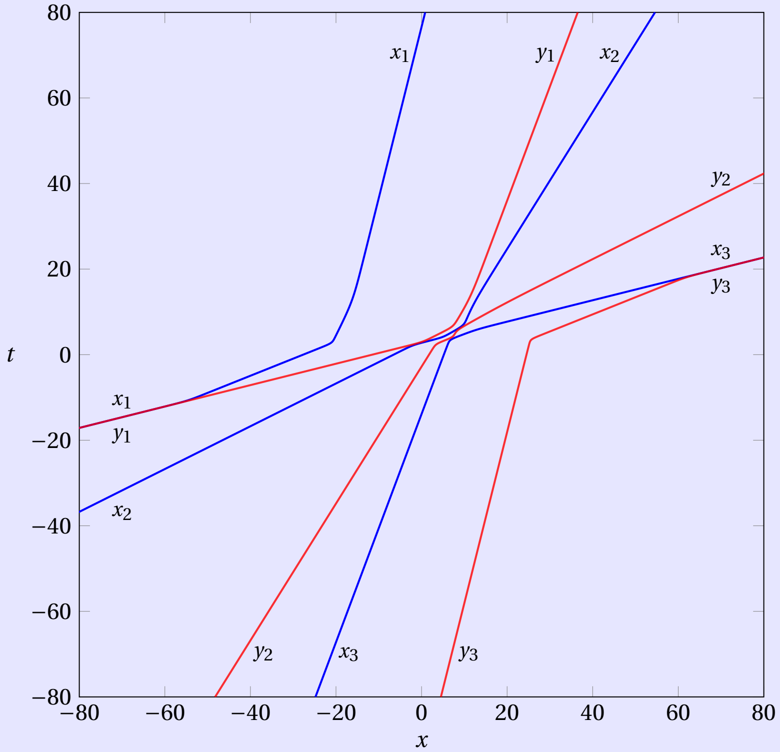
Then the eigenvalues are **positive** and **simple**:

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_K, \qquad 0 < \mu_1 < \mu_2 < \cdots < \mu_{K-1}$$

Asymptotic velocities as $t \to \pm\infty$ for $3+3$ interlacing solution (from fastest to slowest):

$$\underbrace{\frac{1}{2}\left(\frac{1}{\lambda_1}+\frac{1}{\mu_1}\right)}_{\text{twice}}, \quad \frac{1}{2}\left(\frac{1}{\lambda_2}+\frac{1}{\mu_1}\right), \quad \frac{1}{2}\left(\frac{1}{\lambda_2}+\frac{1}{\mu_2}\right), \quad \frac{1}{2}\left(\frac{1}{\lambda_3}+\frac{1}{\mu_2}\right), \quad \frac{1}{2}\left(\frac{1}{\lambda_3}\right)$$

Positions: $x = x_k(t)$ and $x = y_k(t)$



Incoming $\dot{x}_1$ & $\dot{y}_1$ = outgoing $\dot{x}_3$ & $\dot{y}_3$. Incoming $\dot{x}_2$ = outgoing $\dot{y}_2$. (Etc.)

The **amplitudes** m_k and n_k (typically) do **not** tend to constants as $t \to \pm\infty$.

Instead they **grow or decay exponentially**.

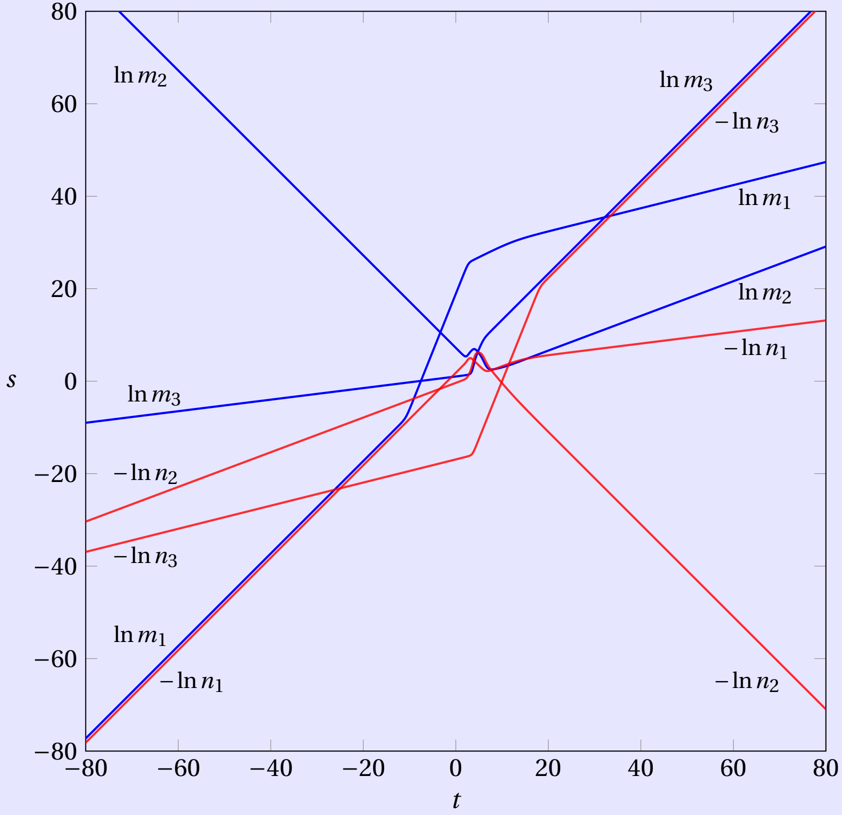
Thus, the curves

$$s = \ln m_k(t) \qquad s = -\ln n_k(t)$$

asymptotically approach straight lines as $t \to \pm\infty$, with slopes as follows (in order):

$$\underbrace{\frac{1}{2}\left(\frac{1}{\lambda_1} - \frac{1}{\mu_1}\right)}_{\text{twice}}, \quad \frac{1}{2}\left(\frac{1}{\lambda_2} - \frac{1}{\mu_1}\right), \quad \frac{1}{2}\left(\frac{1}{\lambda_2} - \frac{1}{\mu_2}\right), \quad \frac{1}{2}\left(\frac{1}{\lambda_3} - \frac{1}{\mu_2}\right), \quad \frac{1}{2}\left(\frac{1}{\lambda_3}\right)$$

Logarithms of amplitudes: $s = \ln m_k(t)$ and $s = -\ln n_k(t)$



Next, solution formulas for **arbitrary configurations**.

(Shuaib & Lundmark, preprint 2018)

Notation for positions:

$$\underbrace{x_{1,1} < x_{1,2} < \cdots < x_{1,N_1^X}}_{\text{First } X\text{-group}} < \underbrace{y_{1,1} < y_{1,2} < \cdots < y_{1,N_1^Y}}_{\text{First } Y\text{-group}} < \cdots$$

$$< \underbrace{x_{j,1} < x_{j,2} < \cdots < x_{j,N_j^X}}_{j\text{th } X\text{-group}} < \underbrace{y_{j,1} < y_{j,2} < \cdots < y_{j,N_j^Y}}_{j\text{th } Y\text{-group}} < \cdots$$

$$< \underbrace{x_{K,1} < x_{K,2} < \cdots < x_{K,N_K^X}}_{\text{Last } X\text{-group}} < \underbrace{y_{K,1} < y_{K,2} < \cdots < y_{K,N_K^Y}}_{\text{Last } Y\text{-group}}$$

Similarly for the amplitudes $m_{j,i}$ and $n_{j,i}$.

Inverse spectral technique does **not** work directly.

For non-interlacing configurations, the Lax pairs yield too few constants of motion.

Instead: use **ghostpeakon** technique.

Ghostpeakons are also useful for deriving exact formulas for the **characteristic curves** $x = \xi(t)$ of a peakon solution $u(x,t)$:

$$\begin{aligned} \dot{\xi}(t) &= u\big(\xi(t), t\big) && \text{for CH \& DP} \\ \dot{\xi}(t) &= u\big(\xi(t), t\big)^2 && \text{for Novikov} \end{aligned}$$

These curves were used for making the 3D plots of $u(x,t)$ above.

(Lundmark & Shuaib 2019)

- An arbitrary configuration is given.
- Pad it with auxiliary peakons to obtain a $K+K$ interlacing configuration. In the known solution formulas for that configuration, make a substitution of the form

$$\lambda_K = \frac{\text{constant}}{\varepsilon} \qquad a_K(0) = \text{constant} \times \varepsilon^{k_1}$$

$$\mu_{K-1} = \frac{\text{constant}}{\varepsilon} \qquad b_{K-1}(0) = \text{constant} \times \varepsilon^{k_2}$$

and let $\varepsilon \to 0^+$.

- With the powers k_1 & k_2 **suitably chosen**, this will turn one of the inserted auxiliary peakons into a "ghostpeakon" with amplitude zero.
- Repeat this, to "kill" all the inserted peakons, one by one.

Example. Solution sought for this config with $3+3$ groups:

$$x_1 < \underbrace{y_{1,1} < y_{1,2} < y_{1,3}}_{Y\text{-group}} < x_2 < y_2 < \underbrace{x_{3,1} < x_{3,2} < x_{3,3}}_{X\text{-group}} < \underbrace{y_{3,1} < y_{3,2} < y_{3,3} < y_{3,4}}_{Y\text{-group}}$$

Schematically: ●○○○●○●●●○○○○

Steps to obtain solution formulas:

Start (10 + 10 interlacing)	●○●○●○●○●○●○●○●○●○●○,
Step 1a	●○●○●○●○●○●○●○●○●○ ○,
Step 1b	●○●○●○●○●○●○●○●○ ○ ○,
Step 1c	●○●○●○●○●○●○●○ ○ ○ ○,
Step 2a	●○●○●○●○●○● ●○ ○ ○ ○,
Step 2b	●○●○●○●○● ● ●○ ○ ○ ○,
Step 3a	●○●○ ○●○● ● ●○ ○ ○ ○,
Step 3b (finish)	●○ ○ ○●○● ● ●○ ○ ○ ○.

Need to keep track of what happens to all the solution formulas at each step. At the end, the desired formulas remain.

Results

- Singletons obey the formulas from the interlacing case.
- Each group with $N \geq 2$ peakons has internal parameters

$$\tau_1, \ldots, \tau_{N-1} > 0 \qquad 0 < \sigma_1 < \cdots < \sigma_{N-1}$$

  appearing only in the solution formulas for that group.
- The two spectral problems only sense the **effective position and amplitude** of each group:

$$\widetilde{m}_j \, e^{\widetilde{x}_j} = \sum_{i=1}^{N} m_{j,i} e^{x_{j,i}} \qquad \widetilde{m}_j \, e^{-\widetilde{x}_j} = \sum_{i=1}^{N} m_{j,i} e^{-x_{j,i}}$$

  The additional constants of motion τ_i and σ_i (not coming from the Lax pairs) are needed in order to determine what happens inside each group.

Example. 3+3 groups, all singletons except one 5-peakon group.

$$X_1 = \frac{J^{00}_{32}}{J^{11}_{21} + C\, J^{10}_{22}} \qquad Y_1 = \frac{J^{00}_{32}}{J^{11}_{21}} \qquad X_2 = \frac{J^{00}_{22}}{J^{11}_{11}}$$

$$\boxed{Y_{2,1}} = \frac{J^{00}_{22} + \tau_1\, J^{00}_{21}}{J^{11}_{11} + \tau_1\, J^{11}_{10}} \qquad \left(\tau_i = \tau^Y_{2,i} \quad \& \quad \sigma_i = \sigma^Y_{2,i}\right)$$

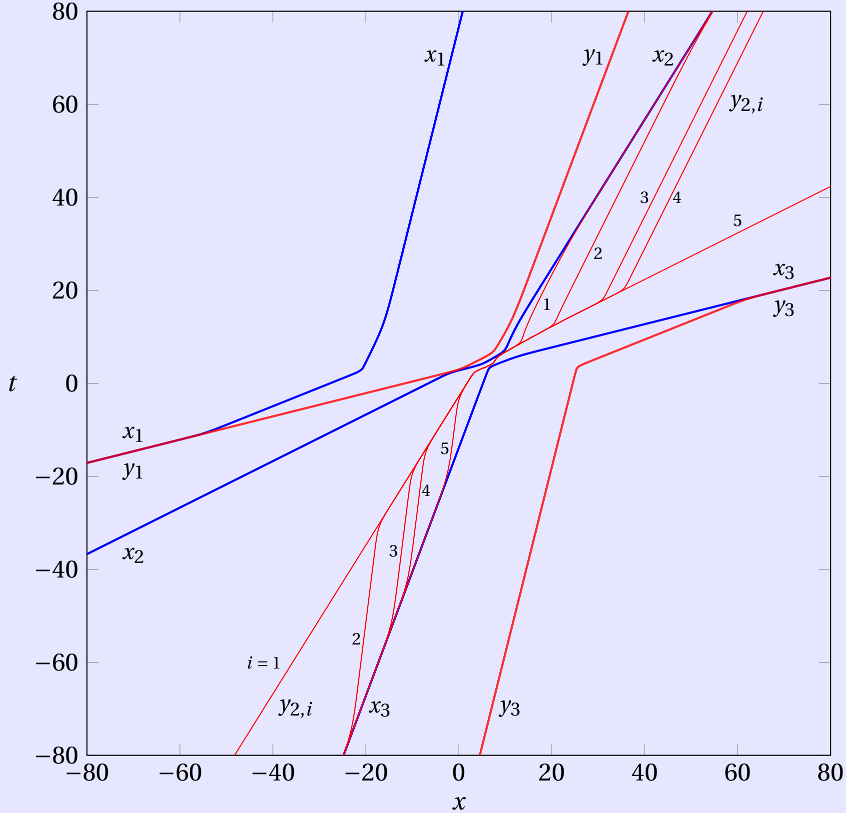
$$\boxed{Y_{2,2}} = \frac{J^{00}_{22} + (\tau_1 + \tau_2)\, J^{00}_{21} + (\tau_2\sigma_1)\, J^{00}_{11}}{J^{11}_{11} + (\tau_1 + \tau_2)\, J^{11}_{10} + (\tau_2\sigma_1)\, J^{11}_{00}}$$

$$\boxed{Y_{2,3}} = \frac{J^{00}_{22} + (\tau_1 + \tau_2 + \tau_3)\, J^{00}_{21} + (\tau_2\sigma_1 + \tau_3\sigma_2)\, J^{00}_{11}}{J^{11}_{11} + (\tau_1 + \tau_2 + \tau_3)\, J^{11}_{10} + (\tau_2\sigma_1 + \tau_3\sigma_2)\, J^{11}_{00}}$$

$$\boxed{Y_{2,4}} = \frac{J^{00}_{22} + (\tau_1 + \tau_2 + \tau_3 + \tau_4)\, J^{00}_{21} + (\tau_2\sigma_1 + \tau_3\sigma_2 + \tau_4\sigma_3)\, J^{00}_{11}}{J^{11}_{11} + (\tau_1 + \tau_2 + \tau_3 + \tau_4)\, J^{11}_{10} + (\tau_2\sigma_1 + \tau_3\sigma_2 + \tau_4\sigma_3)\, J^{11}_{00}}$$

$$\boxed{Y_{2,5}} = \frac{J^{00}_{21} + \sigma_4\, J^{00}_{11}}{J^{11}_{10} + \sigma_4\, J^{11}_{00}} \qquad X_3 = \frac{J^{00}_{11}}{J^{11}_{00}} = J^{00}_{11} \qquad Y_3 = J^{00}_{11} + D\, J^{00}_{10}$$

Positions for the config on the previous page: $\bullet\circ\bullet\underbrace{\circ\circ\circ\circ\circ}\bullet\circ$



Two main cases

- Even case: $K+K$ groups. $\{\lambda_i\}_{i=1}^{K}$ and $\{\mu_j\}_{j=1}^{K-1}$
- Odd case: $(K+1)+K$ groups. $\{\lambda_i\}_{i=1}^{K}$ and $\{\mu_j\}_{j=1}^{K}$

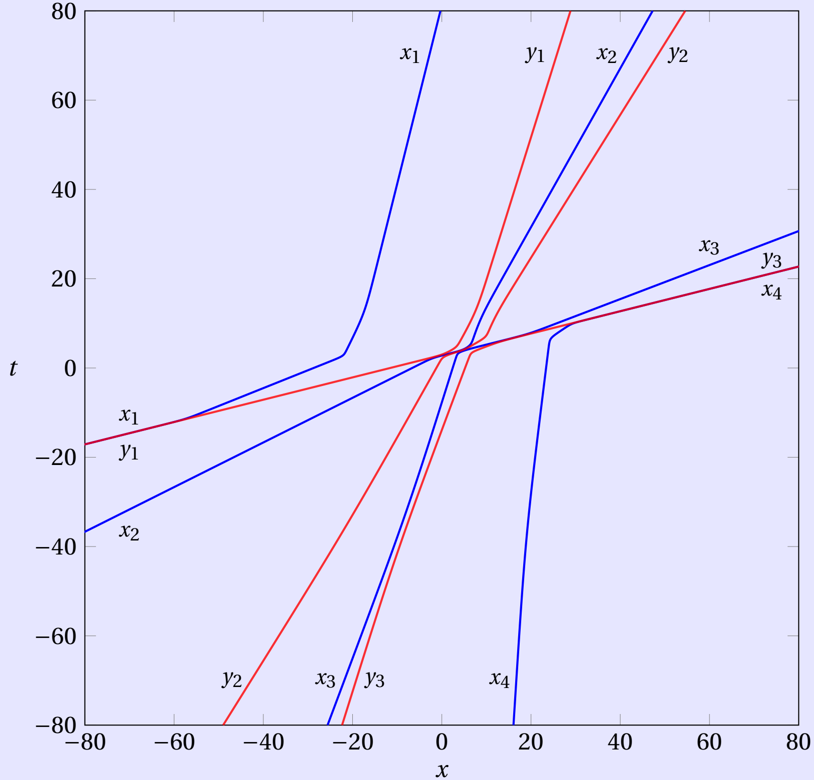
Already the **interlacing** odd case is a bit surprising:

Asymptotic velocities

(4 + 3 interlacing case)

$t \to -\infty$		$t \to +\infty$	
$\frac{1}{2}\left(\frac{1}{\lambda_1}+\frac{1}{\mu_1}\right)$	$=$	$\frac{1}{2}\left(\frac{1}{\lambda_1}+\frac{1}{\mu_1}\right)$	(twice)
$\frac{1}{2}\left(\frac{1}{\lambda_2}+\frac{1}{\mu_1}\right)$	$\neq$	$\frac{1}{2}\left(\frac{1}{\lambda_1}+\frac{1}{\mu_2}\right)$	
$\frac{1}{2}\left(\frac{1}{\lambda_2}+\frac{1}{\mu_2}\right)$	$=$	$\frac{1}{2}\left(\frac{1}{\lambda_2}+\frac{1}{\mu_2}\right)$	
$\frac{1}{2}\left(\frac{1}{\lambda_3}+\frac{1}{\mu_2}\right)$	$\neq$	$\frac{1}{2}\left(\frac{1}{\lambda_2}+\frac{1}{\mu_3}\right)$	
$\frac{1}{2}\left(\frac{1}{\lambda_3}+\frac{1}{\mu_3}\right)$	$=$	$\frac{1}{2}\left(\frac{1}{\lambda_3}+\frac{1}{\mu_3}\right)$	
$\frac{1}{2}\left(0+\frac{1}{\mu_3}\right)$	$\neq$	$\frac{1}{2}\left(\frac{1}{\lambda_3}+0\right)$	

Example. Positions for a $4+3$ interlacing configuration:



Incoming $\dot{x}_1$ & $\dot{y}_1$ = outgoing $\dot{y}_3$ & $\dot{x}_4$. But incoming $\dot{x}_2 \neq$ outgoing $\dot{x}_3$. (Etc.)

THE END