Exercises IV

1. Derive boundary conditions such that the initial-boundary value problem for

$$u_t = -u_{xxxx}\,, \quad 0 \le x \le 1\,,\ t \ge 0$$

is well posed.

2. Consider the approximation

$$\begin{aligned} \frac{du_j}{dt} &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} D_0 u_j\,, \quad j = 1, 2, \ldots, N-1\,, \\ \frac{du_0^{II}}{dt} &= D_+ u_0^I\,, \\ u_0^I &= 0\,, \\ \frac{du_N^{II}}{dt} &= D_- u_N^I\,, \\ u_N^I &= 0\,. \end{aligned}$$

Prove that the norm

$$||u||_h = \Big(\frac{h}{2}(|u_0|^2 + |u_N|^2) + ||u||_{1,N-1}^2\Big)^{1/2}$$

is independent of t.

3. (Stronger version of Assignment 2, part 1.)

Let Q be a semibounded difference operator satisfying

$$(v, Qv)_h \le 0$$

for all v satisfying certain boundary conditions. The θ-scheme is defined by

$$(I - \theta k Q)u^{n+1} = (I + (1-\theta)kQ)u^n\,.$$

Prove that $||u^n||_h \le ||u^0||_h$ for $\frac{1}{2} \le \theta \le 1$.

Hint: Take the scalar product of $u^{n+1} - u^n$ with $kQ(\theta u^{n+1} + (1-\theta)u^n)$.