A Time-Indexed Generalized Vehicle Routing Model and Stabilized Column Generation for Military Aircraft Mission Planning

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Abstract We introduce a time-indexed mixed-integer linear programming model for a military aircraft mission planning problem, where a fleet of cooperating aircraft should attack a number of ground targets so that the total expected effect is maximized. The model is a rich vehicle routing problem and the direct application of a general solver is practical only for scenarios of very moderate sizes. We propose a Dantzig–Wolfe reformulation and column generation approach. A column here represents a specific sequence of tasks at certain times for an aircraft, and to generate columns a longest path problem with side constraints is solved. We compare the column generation approach with the time-indexed model with respect to upper bounding quality of their linear programming relaxations and conclude that the former provides a much stronger formulation of the problem.

1 Introduction

We study a military aircraft mission planning problem (MAMPP), which was introduced by Quttineh *et al.* [26]. In general, a military aircraft mission might involve various tasks, such as surveillance, backup support, rescue assistance or an attack. We only consider the situation where a set of ground targets needs to be attacked with a fleet of aircraft. The planning of such aircraft missions is still to a large extent carried out manually, and it takes an experienced planner several hours to create a feasible plan.

The research presented here has been performed in collaboration with an industrial partner, and is a continuation of the work by Quttineh *et al.* [26, 25].

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The MAMPP is recognized as a generalized vehicle routing problem (GVRP) with precedence relationships and synchronization in time and position between multiple vehicles. Examples of mathematical optimization approaches to military routing problems can be found in [33, 29, 28, 7]. To the best of our knowledge, the MAMPP has not been analyzed by optimization methods by others.

Synchronization in a vehicle routing problem (VRP) might be exhibited with regard to spatial, temporal, and load aspects. A recent survey of VRPs with synchronization constraints (VRPS) is given in Drexl [10] and shows that this topic is challenging and emerging. Following the definitions from this paper, the synchronization in our problem can be classified as operation synchronization, in which one has to decide about time and location of some interaction between vehicles. In [11], Drexl presents modeling techniques for a VRP with trailers and transshipments (VRPTT), which is an application of the VRP with all the previously mentioned synchronization constraints. Different transformations of classic VRPs and of several types of VRPSs are described. Recently, Drexl [12] presented two mixed-integer programming formulations and five branch-and-cut algorithms for the VRPTT.

Bredström and Rönnqvist [6] give a daily homecare planning problem, which is modeled as a vehicle routing and scheduling problem with precedence constraints on visits as well as time windows and pairwise synchronization (because two staff members are required to visit an elderly person simultaneously). Redjem *et al.* [27] also consider routing with time windows and synchronized visits for a homecare planning problem. Synchronized routing and scheduling problems need to be solved also in the forestry industry. El Hachemi *et al.* [14], for instance, include multiple aspects such as pick-up and delivery, and inventory stock, and solve the decomposed problem using constraint-based local search. Other examples of work on routing with synchronization are [21, 3, 1].

Already in the 1970s, Golden [17] touched the GVRP as a variation of the classic VRP. One of the first dedicated papers on GVRP is by Ghiani and Improta [16], who give a transformation to the capacitated arc routing problem. Baldacci *et al.* [4] discuss some applications for the GVRP, whereas formulations and branch-and-cut algorithms are given in the recent paper of Bektaş *et al.* [5]. Hà *et al.* [18] solve the GVRP with the number of vehicles as a decision variable, both heuristically and exact using a branch-and-cut approach. For the same problem, Afsar *et al.* [2] present an exact method based on column generation, and two metaheuristics.

In Sigurd *et al.* [30], vehicle routing with precedence constraints and time windows is considered in order to schedule transportation of live animals to avoid the spread of diseases. A general framework for VRP with time windows and temporal dependencies, including exact synchronization, is given in Dohn *et al.* [9]. In the context of GVRP, a time windows extension is considered by Moccia *et al.* [24], who suggest a metaheuristic solution method. Their work concerns an application to the design of home-to-work transportation plans.

By taking into account multiple non-standard characteristics of the GVRP, such as precedence relationships and operation synchronization, we believe to contribute to the existing literature. Our paper reads as follows. In Section 2, the problem setting is described, followed by a time-indexed mathematical formulation in Section 3.

Section 4 develops a column generation method for a Dantzig–Wolfe reformulation of the time-indexed model, followed in Section 5 by a description of a stabilized column generation method. In Section 6, we give theoretical bounding results. Further, in Section 7, numerical results of our approach are discussed, followed by a conclusion in Section 8.

2 Problem Setting

This section provides a concise description of the problem setting. A detailed report on the complex problem characteristics and how to transform them into a mathematical formulation can be found in Quttineh *et al.* [26]. As mentioned above, we only consider military aircraft missions involving attacks. The geographical area of interest, referred to as the target scene, includes the targets that need to be attacked and other objects such as enemy defense positions, like surface-to-air missiles (SAMs), and protected objects, like hospitals and schools. We consider all objects to be stationary with known positions. The target scene is defined by a line of entrance and a line of exit for the aircraft. These are typically deployed from a base situated far away from the target scene and enter the scene by the entry line, carry out the mission and return to a base after leaving the scene at the exit line. The diameter of a target scene is usually of the order of 100 km, the distances between targets are of the order of a few kilometers, and the timespan of the attacks is around a quarter of an hour. Typically, a mission involves 6–8 targets and 4–6 aircraft. At the end of this section, an example of a target scene is depicted, together with a solution.

The goal of a mission is to find an attack plan where maximal total expected effect is gained within short timespan. The mission time is defined by the time the first aircraft passes the entry line and the time the last aircraft passes the exit line. Since the entire target scene is located in hostile area, the mission time needs to be minimized. To take into account the threat from defense positions, aircraft are restricted not to fly through defended airspace. Weapons, on the other hand, are allowed to pass through defended airspace, but at the risk of being shot down, that is, with a lower expected effect on the target.

In order to plan a mission, the aircraft characteristics need to be taken into account. Each aircraft has an armament capacity, limiting the number of attacks it can perform. It can also be equipped with an illumination laser pod to guide weapons. Each target needs to be attacked exactly once, and requires one aircraft that illuminates the target with a laser beam and one aircraft that launches the weapon. Since an attack requires continuous illumination from the launch of the weapon until its impact, the two aircraft need to team up. This rendez-vous not only depends on the time but also on the location of both aircraft, so that the illumination is continuously visible for the weapon.

Figure 1 illustrates how a target is modeled. The feasible attack space can be derived from the type of aircraft and the type of weapon being used, and is represented by the inner and outer radii. This attack space is then divided into six sectors, which each holds at most three discretized attack positions and two compatible illumination positions. If a protected object is inside the estimated area of risk for collateral damage of a given attack position, this position is considered unfeasible. For any attack position, the expected effect on the target can be calculated. It depends on the kind of weapon being used, which is decided in advance, and on the direction of the impact and the weapon's kinetic energy. The two illumination alternatives per sector differ in flight direction, roughly clockwise or counter-clockwise, but are both compatible with all attack positions of the sector. In our problem setting we consider only one altitude layer, but one could of course extend the target modeling by allowing attack options on different discrete altitude layers.

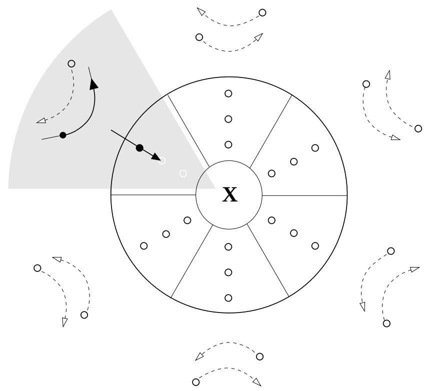

Fig. 1 The feasible attack space defined by inner and outer radii, and divided into six sectors, each with three attack and two illumination alternatives. A pair of compatible attack and illumination positions is marked, where the arrows indicate the flight directions.

Not all attack sequences are allowed. Depending on the wind direction and the proximity between targets, dust and debris might reduce the visibility and hinder an attack. Hence, we assume that precedence constraints are given, specifying which targets are not allowed to be attacked before other targets.

In summary, the problem involves three types of decisions. First, the choice of attack direction against each target. Second, which two aircraft shall be assigned against the targets. Third, the order in which each aircraft fulfils its assigned tasks in the mission. Now it is clear that the problem belongs to the class of vehicle routing problems, describing the attack and illumination positions by nodes, each of which being associated with an expected effect on the target. By further introducing dummy nodes associated with the crossings of the entry and exit lines of the target scene, and modeling possible aircraft movements by arcs, the mission planning problem can partly be represented by a network. Because of the precedence relationships, some arcs are eliminated from the network. The restriction that every target should be attacked exactly once results in a network that only contains arcs between different targets, or from or to the dummy nodes.

Each of the arcs has two attributes: an expected effect and a travel time. The effect attribute is different from zero only for an arc that is leaving an attack node, and it then equals the resulting expected effect against the target. A flight path between two positions has to comply with restrictions on the aircraft dynamics and that the aircraft cannot pass through defended airspace. By using a flight path generator provided by our industrial partner, we are able to find the path with minimal time between any pair of positions. In general, travel times will be asymmetric because each position is also associated with a flight direction.

To illustrate the essential aspects of a solution to the MAMPP, Figure 2 depicts a target scene and an optimal solution. For this problem instance, two aircraft are used, there are no precedence constraints on the targets, and each aircraft can attack at most two targets. All numerical data used in the scenario were provided by our industrial partner.

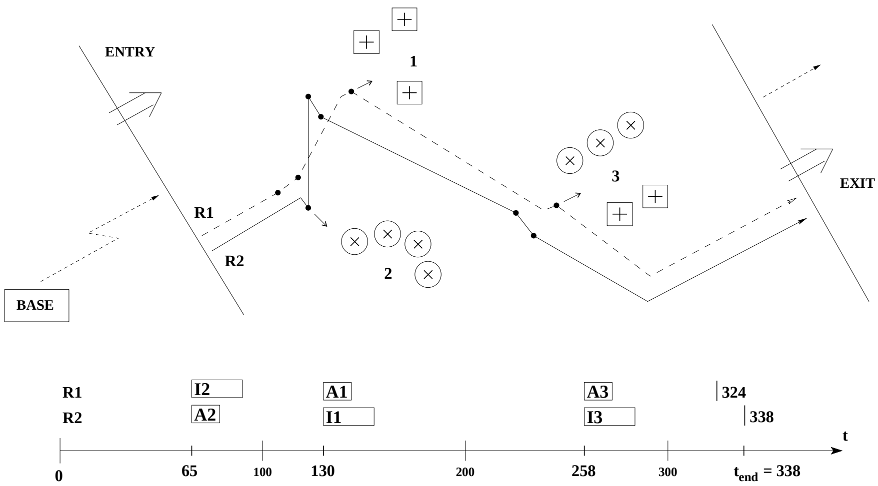

Fig. 2 Optimal solution to a problem instance that includes three targets and nearby SAMs (×) and hospitals (+). Shown are aircraft routes, chosen attack and illumination positions against each target, the times of the attacks, and the times when the two aircraft pass the exit line.

The aircraft routes are shown as solid and dashed lines. The attack sequence is 2–1–3, with a total mission time of $t_{end} = 338$ seconds. The expected effects of the attacks on targets 2 and 3 are maximal, among the available attack positions for these targets, while the attack position against target 1 renders an effect that is slightly below the maximal possible. Achieving maximal effect against this target would require a longer tour for both aircraft, which makes this alternative non-optimal.

3 A Time-Indexed Mathematical Model

We here present a time-indexed mixed-integer linear programming (MILP) mathematical model of the MAMPP. This MILP model can be derived from the one introduced by Quttineh *et al.* [26], through a discretization of time. In particular, this discretization allows an alternative modeling of the time propagation constraints.

We divide the nomenclature into indices and sets, parameters and coefficients, and decision variables, given in Tables 1, 2 and 3.

Table 1 Indices and sets

$\mathbf{R}$	fleet of aircraft, r
$\mathbf{M}$	set of targets, m, to be attacked
$\mathbf{N}$	set of nodes in the network, excluding the origin (o) and destination (d) nodes
$\mathbf{G}, \mathbf{G}_m$	set of all sectors for all targets and for target m, respectively
$\mathbf{N}_m^A, \mathbf{N}_m^I$	set of feasible attack (A) and illumination (I) nodes, respectively, for target m
$\mathbf{A}, \mathbf{A}_g, \mathbf{I}_g$	set of arcs in the network (including from o and to d) and sets of arcs (i, j) such that node j is an attack (A) node or illumination (I) node in sector g, respectively
$\mathbf{P}$	set of ordered pairs (m, n) of targets such that the attack on target m cannot preceed the attack on target n
$\mathbf{S}$	set of time periods within a discretized planning horizon, each of step length Δt

Table 2 Parameters

c_{ij}^r	for arcs (i, j) with $i \in N_m^A$, that is, for arcs leaving attack nodes, the value of c_{ij}^r is the expected effect of the attack, and otherwise the value is zero
S_{ij}^r	the time needed for aircraft r to traverse arc (i, j), expressed in number of time periods; equals actual time to traverse the arc divided by Δt, rounded upwards
T_s	the ending time of period s, which equals $s \cdot \Delta t$, $s = 0, 1, \ldots, \lvert\mathbf{S}\rvert$
Γ^r	armament capacity of aircraft r
q_m	weapon capacity needed towards target m
μ	positive parameter that weights mission timespan against expected effect on targets

Table 3 Decision variables

x_{ij}^r	routing variable, equals 1 if aircraft r traverses arc (i, j), and 0 otherwise
y_{is}^r	time indicator variable, equals 1 if node i is visited by aircraft r in time period s, and 0 otherwise
t_{end}	the time that the last aircraft passes the exit line

The primary objective is to maximize the total expected effect against all the targets. However, in order to acheive this effect, the use of long flight paths within the target scene might be neccessary, which exposes the aircraft to a higher risk of being detected and engaged by enemy defense. A secondary objective is therefore to limit the mission timespan. We thus have a multi-objective optimization problem, with two objectives that are typically in conflict.

Since the maximal allowed mission timespan is given by $\lvert\mathbf{S}\rvert \cdot \Delta t$, an explicit way of limiting the mission timespan is to reduce the cardinality of $\mathbf{S}$, which might however cause the MAMPP to become infeasible. A further drawback of this approach is that it can allow mission timespans that are unneccessarily long with respect to the obtained target effect.

Instead, we have chosen to optimize a weighted combination of the two objectives, using the positive parameter μ which reflects the trade-off between effect on target and mission timespan. This yields a solution that is Pareto optimal. As part of a decision support tool, the value of μ can either be chosen by a mission planner or varied systematically in order to generate a population of mission plans with different properties with respect to effect and time, to be further evaluated by a mission planner. Since target effect is the primary goal, the value of μ is typically small.

The time-indexed mathematical model for the MAMPP is given below.

$$z^*_{IP} = \max \sum_{r\in\mathbf{R}} \sum_{(i,j)\in\mathbf{A}} c^r_{ij} x^r_{ij} - \mu t_{end} \qquad \text{[TI–MAMPP]}$$

subject to

$$\sum_{(o,j)\in\mathbf{A}} x^r_{oj} = 1, \qquad r\in\mathbf{R} \qquad (1)$$

$$\sum_{(i,d)\in\mathbf{A}} x^r_{id} = 1, \qquad r\in\mathbf{R} \qquad (2)$$

$$\sum_{(i,k)\in\mathbf{A}} x^r_{ik} = \sum_{(k,j)\in\mathbf{A}} x^r_{kj}, \qquad k\in\mathbf{N},\ r\in\mathbf{R} \qquad (3)$$

$$\sum_{r\in\mathbf{R}} \sum_{g\in\mathbf{G}_m} \sum_{(i,j)\in\mathbf{A}_g} x^r_{ij} = 1, \qquad m\in\mathbf{M} \qquad (4)$$

$$\sum_{r\in\mathbf{R}} \sum_{g\in\mathbf{G}_m} \sum_{(i,j)\in\mathbf{I}_g} x^r_{ij} = 1, \qquad m\in\mathbf{M} \qquad (5)$$

$$\sum_{r\in\mathbf{R}} \sum_{(i,j)\in\mathbf{A}_g} x^r_{ij} = \sum_{r\in\mathbf{R}} \sum_{(i,j)\in\mathbf{I}_g} x^r_{ij}, \qquad g\in\mathbf{G} \qquad (6)$$

$$\sum_{g\in\mathbf{G}_m} \sum_{(i,j)\in\mathbf{A}_g\cup\mathbf{I}_g} x^r_{ij} \le 1, \qquad m\in\mathbf{M},\ r\in\mathbf{R} \qquad (7)$$

$$\sum_{m\in\mathbf{M}} \sum_{g\in\mathbf{G}_m} \sum_{(i,j)\in\mathbf{A}_g} q_m x^r_{ij} \le \Gamma^r, \qquad r\in\mathbf{R} \qquad (8)$$

$$y^r_{o0} = 1, \qquad r\in\mathbf{R} \qquad (9)$$

$$\sum_{t=s+S^r_{ij}}^{|\mathbf{S}|} y^r_{jt} \ge x^r_{ij} + y^r_{is} - 1, \qquad (i,j)\in\mathbf{A},\ s\in\{0\}\cup\mathbf{S},\ r\in\mathbf{R} \qquad (10)$$

$$\sum_{s\in\mathbf{S}} y^r_{ks} = \sum_{(k,j)\in\mathbf{A}} x^r_{kj}, \qquad k\in\mathbf{N},\ r\in\mathbf{R} \qquad (11)$$

$$\sum_{r\in\mathbf{R}} \sum_{i\in\mathbf{N}^A_m} y^r_{is} = \sum_{r\in\mathbf{R}} \sum_{i\in\mathbf{N}^I_m} y^r_{is}, \qquad m\in\mathbf{M},\ s\in\mathbf{S} \qquad (12)$$

$$\sum_{r\in\mathbf{R}} \sum_{t=s}^{|\mathbf{S}|} \sum_{i\in\mathbf{N}^A_m} y^r_{it} \ge \sum_{r\in\mathbf{R}} \sum_{i\in\mathbf{N}^A_n} y^r_{is}, \qquad (m,n)\in\mathbf{P},\ s\in\mathbf{S} \qquad (13)$$

$$\sum_{i\in\mathbf{N}^A_m} \sum_{t=1}^{s-1} y^r_{it} + \sum_{i\in\mathbf{N}^A_n} y^r_{is} \le 1, \qquad (m,n)\in\mathbf{P},\ s\in\mathbf{S},\ r\in\mathbf{R} \qquad (14)$$

$$\sum_{s\in\mathbf{S}} y^r_{is} \le 1, \qquad i\in\mathbf{N}\cup\{o,d\},\ r\in\mathbf{R} \qquad (15)$$

$$\sum_{s\in\{0\}\cup\mathbf{S}} T_s y^r_{ds} \le t_{end}, \qquad r\in\mathbf{R} \qquad (16)$$

$$x^r_{ij} \in \{0,1\}, \qquad (i,j)\in\mathbf{A},\ r\in\mathbf{R} \qquad (17)$$

$$y^r_{is} \in \{0,1\}, \qquad i\in\mathbf{N}\cup\{o,d\},\ s\in\{0\}\cup\mathbf{S},\ r\in\mathbf{R} \qquad (18)$$

Constraints (1) and (2) describe that each aircraft leaves and enters the target scene via the origin and destination nodes, respectively, while constraint (3) is the node balance equation for each aircraft. The requirement that each target shall be attacked and illuminated exactly once is modeled by constraints (4) and (5), respectively, while constraint (6) synchronizes these tasks to the same sector. Constraint (7) states that each aircraft can visit each target at most once. This constraint is actually redundant, but it strengthens the column generation problems to be presented. The armament limitation is modeled by constraint (8).

Further, constraint (9) states that each aircraft is leaving the origin at time zero. Constraint (10) ensures that if aircraft r is visiting node j directly after node i, then the time of visiting node j cannot be earlier than the time of visiting node i plus the time needed to traverse arc (i,j). Constraint (11) enforces that if node i is not visited by an aircraft, no outgoing arc (i,j) from that node can be traversed by the aircraft.

Constraint (12) states that the attack and the illumination of a target need to be synchronized in time. Constraint (13) imposes the precedence restrictions on the attacking times of pairs of targets. Similarly, constraint (14) imposes the precedence restrictions for an individual aircraft. This constraint is also redundant, but it strengthens the column generation problems. Constraint (15) states that each aircraft can visit each node in at most one time period, and constraint (16) defines the total mission time, since all aircraft end up at the destination node. Finally, (17) and (18) are definitional constraints.

The optimal value of the linear programming (LP) relaxation of TI–MAMPP is denoted z^*_{LP}.

4 Column Generation

The planning of a military aircraft mission is typically made close to when the mission actually takes place (say, within 24 hours); one reason for this is that the planning can then be based on the most recent information. The time needed for the chain of planning is of the order of several hours. Solving the continuous time version of MAMPP presented in Quttineh *et al.* [26] to optimality takes a general MIP solver several hours for already moderate-sized problem instances. This is also the case for the model TI–MAMPP presented above. Hence, efficient algorithms are needed to meet the needs and expectations in a real-life setting. We propose a column generation method based on a Dantzig–Wolfe reformulation [8] of the model TI–MAMPP. For overviews of column generation, see for example [22] and [32].

The Dantzig–Wolfe reformulation is defined in the following steps. Suppose that the constraints (1)–(3), (7)–(11), (14)–(15), and (17)–(18) have N_r feasible solutions for aircraft $r \in \mathbf{R}$. Each of these describes a possible route for the aircraft, involving specific tasks at specific targets at certain times. Assume that $n_r < N_r$ of the routes for aircraft $r \in \mathbf{R}$ are explicitly available. Typically, $n_r \ll N_r$ holds. Let the values of the variables for each feasible solution to the above-mentioned constraints be denoted by x^{rk}_{ij} and $y^{rk}_{is},\ k = 1,\ldots,n_r$.

Next, we relax the binary variable restrictions from the TI–MAMPP and introduce variables z_k^r as convexity weights on the solutions x_{ij}^{rk} and y_{is}^{rk}, $k = 1, \dots, n_r$. Further, we impose the relationships

$$x_{ij}^r \;=\; \sum_{k=1}^{n_r} x_{ij}^{rk} z_k^r \quad \text{and} \quad y_{is}^r \;=\; \sum_{k=1}^{n_r} y_{is}^{rk} z_k^r.$$

Substitution of these relationships into the objective function and into the constraints (4)–(6), (12), (13) and (16) yields the following restricted Dantzig–Wolfe master problem.

$$z_{RMP}^* \;=\; \max \quad \sum_{r\in\mathbf{R}} \sum_{k=1}^{n_r} \left(\sum_{(i,j)\in\mathbf{A}} c_{ij}^r x_{ij}^{rk} \right) z_k^r - \mu t_{end} \qquad \text{[DW–RMP]}$$

subject to

$$[\alpha_m] \quad \sum_{r\in\mathbf{R}} \sum_{k=1}^{n_r} \left(\sum_{g\in\mathbf{G}_m} \sum_{(i,j)\in\mathbf{A}_g} x_{ij}^{rk} \right) z_k^r \;=\; 1, \qquad m \in \mathbf{M} \qquad (19)$$

$$[\beta_m] \quad \sum_{r\in\mathbf{R}} \sum_{k=1}^{n_r} \left(\sum_{g\in\mathbf{G}_m} \sum_{(i,j)\in\mathbf{I}_g} x_{ij}^{rk} \right) z_k^r \;=\; 1, \qquad m \in \mathbf{M} \qquad (20)$$

$$[\gamma_g] \quad \sum_{r\in\mathbf{R}} \sum_{k=1}^{n_r} \left(\sum_{(i,j)\in\mathbf{A}_g} x_{ij}^{rk} \right) z_k^r \;=\; \sum_{r\in\mathbf{R}} \sum_{k=1}^{n_r} \left(\sum_{(i,j)\in\mathbf{I}_g} x_{ij}^{rk} \right) z_k^r, \qquad g \in \mathbf{G} \qquad (21)$$

$$[\eta_{ms}] \quad \sum_{r\in\mathbf{R}} \sum_{k=1}^{n_r} \left(\sum_{i\in\mathbf{N}_m^A} y_{is}^{rk} \right) z_k^r \;=\; \sum_{r\in\mathbf{R}} \sum_{k=1}^{n_r} \left(\sum_{i\in\mathbf{N}_m^I} y_{is}^{rk} \right) z_k^r, \qquad m \in \mathbf{M},\; s \in \mathbf{S} \qquad (22)$$

$$[\lambda_{mns}] \quad \sum_{r\in\mathbf{R}} \sum_{k=1}^{n_r} \left(\sum_{t=s}^{|\mathbf{S}|} \sum_{i\in\mathbf{N}_m^A} y_{it}^{rk} \right) z_k^r \;\geq\; \sum_{r\in\mathbf{R}} \sum_{k=1}^{n_r} \left(\sum_{i\in\mathbf{N}_n^A} y_{is}^{rk} \right) z_k^r, \qquad s \in \mathbf{S},\; (m,n) \in \mathbf{P} \qquad (23)$$

$$[\tau_r] \quad \sum_{k=1}^{n_r} \left(\sum_{s\in\{0\}\cup\mathbf{S}} T_s \cdot y_{ds}^{rk} \right) z_k^r \;\leq\; t_{end}, \qquad r \in \mathbf{R} \qquad (24)$$

$$[\nu_r] \quad \sum_{k=1}^{n_r} z_k^r \;=\; 1, \qquad r \in \mathbf{R} \qquad (25)$$

$$z_k^r \;\geq\; 0, \qquad k = 1, \dots, n_r,\;\; r \in \mathbf{R} \qquad (26)$$

Each column of this problem represents a route for a specific aircraft, and the restricted master problem is to find the best way to combine all available routes into a solution that is feasible and optimal with respect to the restrictions that couple all aircraft, in a linear programming sense.

Comparing DW–RMP with TI–MAMPP, constraints (19)–(21) correspond to the attack, illumination and synchronization constraints (4)–(6), while constraints (22) and (23) match the time synchronization and precedence constraints (12) and (13). Further, constraint (24) defines the total mission time, similarly to (16). Finally, constraints (25) and (26) are definitional.

If all feasible routes for each aircraft are known, that is, if $n_r = N_r$ holds for all $r \in \mathbf{R}$, the restricted master problem becomes a full master problem, with an optimal objective value denoted z^*_{MP}. Further, any optimal solution to DW–RMP that is integral yields a feasible solution to TI-MAMPP and a lower bound to z^*_{IP}, denoted $\underline{z}_{IP}$.

Assume that DW–RMP has a feasible solution. Each of its constraints is associated with a dual variable, indicated in the square brackets to the left. The optimal values of these dual variables are used to define a Dantzig–Wolfe subproblem, or column generation problem, for each aircraft $r \in \mathbf{R}$. The objective function in each subproblem describes the reduced cost of any feasible column, that is, any possible route for the aircraft. As long as there is a route with a positive reduced cost, such routes should be generated and their corresponding columns added to DW–RMP. Generating columns with positive reduced costs boils down to solving the following subproblem for each aircraft $r \in \mathbf{R}$.

$$\begin{aligned}
\overline{c}^r_{n_r+1} \;=\; \max \sum_{(i,j)\in\mathbf{A}} c^r_{ij} x^r_{ij} - \tau_r \sum_{s\in\{0\}\cup\mathbf{S}} T_s y^r_{ds} - \qquad\qquad\qquad & [\text{DW–SUB}_r] \\
- \sum_{m\in\mathbf{M}} \left(\alpha_m \sum_{g\in\mathbf{G}_m} \sum_{(i,j)\in\mathbf{A}_g} x^r_{ij} - \beta_m \sum_{g\in\mathbf{G}_m} \sum_{(i,j)\in\mathbf{I}_g} x^r_{ij} \right) - & \\
- \sum_{g\in\mathbf{G}} \gamma_g \left(\sum_{(i,j)\in\mathbf{A}_g} x^r_{ij} - \sum_{(i,j)\in\mathbf{I}_g} x^r_{ij} \right) - \sum_{m\in\mathbf{M}} \sum_{s\in\mathbf{S}} \eta_{ms} \left(\sum_{i\in\mathbf{N}^A_m} y^r_{is} - \sum_{i\in\mathbf{N}^I_m} y^r_{is} \right) - & \\
- \sum_{(m,n)\in\mathbf{P}} \sum_{s\in\mathbf{S}} \lambda_{mns} \left(\sum_{t=s}^{|\mathbf{S}|} \sum_{i\in\mathbf{N}^A_m} y^r_{it} - \sum_{i\in\mathbf{N}^A_n} y^r_{is} \right) - \nu_r &
\end{aligned}$$

subject to $(1), (2), (3), (7), (8), (9), (10), (11), (14), (15), (17), (18)$

The problem DW–SUB$_r$ can be described as a side constrained longest path problem in a time-layered network where all nodes in $\mathbf{N}$ have $|\mathbf{S}|$ time copies. The constraints (9)–(11) are taken into account implicitly in the construction of the network, while constraints (7), (8), (14) and (15) are side constraints. This problem does not possess the integrality property.

An upper bound on z^*_{MP} is given by $z^*_{RMP} + \sum_{r\in\mathbf{R}} \overline{c}^r_{n_r+1}$. The lowest such upper bound ever found is denoted by $\overline{z}_{MP}$.

5 Stabilized Column Generation

As is well known, column generation methods are dually equivalent to cutting plane methods. The latter are known to be inherently instable [19], in the sense that successive iterates can be very distant, which may cause slow convergence. In column generation methods, the dual instability manifests itself as oscillations in the values of the dual variables, which slows down the convergence also in the primal space.

In order to improve the efficiency of the column generation scheme, it is therefore common to apply a stabilization of the values of the dual variables. This technique was introduced by Marsten *et al.* [23] back in 1975, and examples of applications from more recent years can be found in [13] and [31], to mention some.

The idea is to prevent the dual solution of the DW–RMP to fluctuate between successive iterations. This is accomplished by including a box-constraint for each dual variable, centered around its current value and preventing the value to change drastically from one iteration to the next. These additional constraints in the dual problem correspond to auxiliary variables in the primal problem, and the effect of these variables is a relaxation of the original primal constraints. Consequently, the parameters that specify the size of the box appear as penalty weights in the objective function for the auxiliary variables.

We stabilize constraints (19)–(23) in DW–RMP, and the optimal objective value of the stabilized DW–RMP is denoted z^*_{SRMP}. An upper bound on z^*_{MP} is calculated as $z^*_{SRMP} + \sum_{r\in\mathbf{R}} \bar{c}^r_{n_r+1}$. (The reason that a formula similar to the one used in non-stabilized column generation applies also in the stabilized case is that both formulas are in fact equivalent to a Lagrangian dual bound, and that it is of no significance how the dual point is obtained.)

The size of each box slowly shrinks every iteration, and it is re-centered every time it becomes binding (that is, every time an auxiliary variable becomes nonzero).

6 Bounding Properties

The relationships between the various optimal values and bounds in our column generation approach becomes rather intricate. These relationships are illustrated in Figure 3.

The optimal value z^*_{IP} for TI–MAMPP is trivially bounded from below by the objective value, $\underline{z}_{IP}$, of any feasible solution, and bounded above by the optimal LP value z^*_{LP}. This bound has proven to be very weak, see [26, 25]. Further, z^*_{IP} can be bounded from above by the optimal LP value of the full master problem, z^*_{MP}. It always holds that $z^*_{MP} \leq z^*_{LP}$, but since the column generation problem DW–SUB$_r$ does not have the integrality property, $z^*_{MP} < z^*_{LP}$ can be expected to hold.

Assume first that no stabilization is used. As routes are added to the restricted master problem, its optimal value z^*_{RMP} converges monotonically towards z^*_{MP}. Note that the relationship between z^*_{RMP} and z^*_{IP} is unknown. Further, $\overline{z}_{MP}$ is convergent towards z^*_{MP} from above.

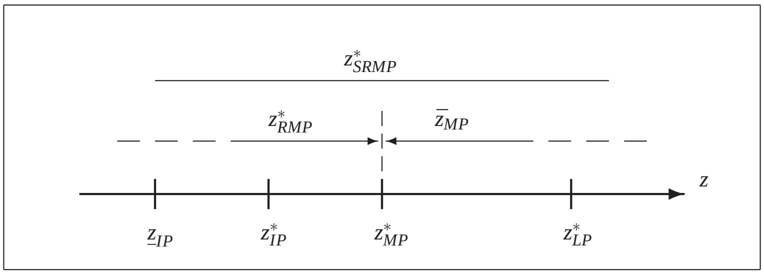

Fig. 3 Bounding relationships for the column generation approach.

Considering the case with stabilization, the relationship between z^*_{SRMP} and z^*_{MP} is unknown, since the stabilized restricted master problem includes both a restriction and a relaxation, as compared to the full master problem. However, the value z^*_{SRMP} becomes a lower bound for z^*_{MP} if the dual box is not binding (that is, all auxiliary variables in the primal problem are zero). Finally, $\bar{z}_{MP}$, as calculated in Section 5, is an upper bound for z^*_{MP}. Further, it converges towards z^*_{MP}.

7 Numerical Validation

We have made a preliminary assessment of TI–MAMPP and the column generation approach by using a few small problem instances that are identical to, or slight modifications of, instances used in [26]. All experiments have been carried out using the modeling language AMPL [15] and the solver CPLEX [20].

Table 4 shows problem characteristics and results obtained with the continuous-time model of MAMPP in [26] and TI–MAMPP. We observe that even for rather large time steps, the optimal solutions found by the continuous-time and time-indexed models are very similar, with respect to attack sequences and to attack and illumination nodes. Although not reported in the table, we also observe that the solution times of the continuous-time and time-indexed models are similar for large time steps, while the latter is much more demanding when the steps are small. Further, the upper bounds given by the linear programming relaxations of the continuous-time and time-indexed versions of the MAMPP are very similar, independent of the sizes of the time steps, and very weak.

Table 5 shows a comparison between the time-indexed model and the column generation approach. Here, initial values for the dual variables for the stabilized constraints (19)–(23), used to initialize the dual boxes, are obtained by solving the LP relaxation of TI–MAMPP. (The radii of the boxes were initially set to 0.3 and shrinked by a factor of 0.97 in each iteration.) To create an initial set of routes and columns, the DW–SUB$_r$ problem is solved for an ad hoc fixed set of sectors to be visited, for each aircraft $r \in \mathbf{R}$.

Table 4 Problem characteristics and comparison of the continuous-time and time-indexed models. Here, $\mu = 0.005$ and all instances include two aircraft. The notation $\{1|23\}$ means that target 1 is attacked before targets 2 and 3. The maximal possible total effect on targets is 1.000.

	Problem			Cont.		$\Delta t = 60$		$\Delta t = 45$		$\Delta t = 30$	
No.	$\lvert\mathbf{M}\rvert$	Prec.	Γ	Eff.	t_{end}	Eff.	t_{end}	Eff.	t_{end}	Eff.	t_{end}
1	3	–	3	0.974	333	0.808	420	0.974	405	0.974	390
2	3	–	2	0.974	338	0.808	420	0.974	405	0.974	390
3	3	$\{1\|23\}$	3	0.863	352	0.808	420	0.863	405	0.808	390
4	4	$\{1\|2\|3\|4\}$	3	0.917	628	1.000	840	0.917	720	0.917	720
5	4	$\{1\|2\|3\|4\}$	2	0.917	638	1.000	840	0.917	720	0.917	720

Table 5 Comparison of the time-indexed model and column generation. The optimal LP value z^*_{LP} of the time-indexed model varies very little with the step size; we give the value for $\Delta t = 60$. The columns $z^*_{IP}[45]$ and $z^*_{IP}[30]$ are the optimal values of the time-indexed model with different time steps. Further, $\underline{z}_{IP}$ are the objective values obtained when solving the integer version of the final master problem (and a feasible solution exists), and Iter. is the number of column generation iterations needed to reach optimality.

	Time-indexed			CG: $\Delta t = 45$			CG: $\Delta t = 30$		
No.	z^*_{LP}	$z^*_{IP}[45]$	$z^*_{IP}[30]$	z^*_{MP}	$\underline{z}_{IP}$	Iter.	z^*_{MP}	$\underline{z}_{IP}$	Iter.
1	23.173	1.933	2.683	1.933	1.933	16	2.683	2.683	22
2	23.173	1.887	2.674	1.887	1.887	11	2.674	2.674	15
3	22.813	0.346	0.080	0.346	0.346	22	1.271	–	22
4	30.117	-7.677	-7.730	-6.532	–	37	-4.744	–	37
5	30.115	-7.730	-7.730	-7.083	–	29	-6.002	–	60

Comparing the columns z^*_{LP} and z^*_{MP} with the columns z^*_{IP}, we conclude that the upper bound on z^*_{IP} obtained from z^*_{MP} is much tighter than the bound z^*_{LP}. The bound z^*_{MP} is indeed very close to z^*_{IP} while the bound z^*_{LP} is very weak. Further, comparing the columns z^*_{IP} and $\underline{z}_{IP}$, we see that whenever the restricted master problem has an integral feasible solution, it is also of high quality.

8 Conclusion

Clearly, the Dantzig–Wolfe reformulation and column generation approach provide vastly superior upper bounds on the optimal value of TI–MAMPP. We conclude that the Dantzig–Wolfe reformulation gives rise to a very strong formulation of the TI–MAMPP. This model by itself is not very efficient in terms of solving the military aircraft mission planning problem, but it was helpful in the development of the column generation procedure.

The solution times of our implementation of the column generation approach are not competitive compared to direct methods. The solution of DW–RMP takes very little time. This holds even for the integer version of this problem. The column generation problem DW–SUB$_r$ is however very time-consuming to solve to optimality.

There are several opportunities for tailoring and streamlining the computations, and especially to reduce the computational burden of the column generation problem DW–SUB$_r$. For example, in early column generation iterations it might be more efficient to terminate the column generation solver as soon as the objective value gets positive, since this is enough to ensure progress. Further, a tailored solver for DW–SUB$_r$ can be developed by exploiting its underlying time-layered network structure. This is an interesting opportunity for further research.

The column generation approach can be applied to obtain an upper bound, to be used for assessing the quality of any feasible solution to TI–MAMPP, for example generated by a metaheuristic. Also, feasible solutions generated by metaheuristics can be used to provide high quality initial columns to the restricted master problem. This combination is another topic for further research.

A great advantage of the column generation approach to MAMPP in a real-life planning situation would be its creation of many possible routes for all aircraft. This is of practical interest since a real-life MAMPP can never be expected to include all possible aspects of the mission to be planned, and because of the multi-objective nature of the problem. The access to multiple aircraft routes can then be exploited in an interactive decision support system.

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