PARTIAL BALAYAGE AND THE EXTERIOR INVERSE PROBLEM OF POTENTIAL THEORY

STEPHEN J GARDINER AND TOMAS SJÖDIN

1. Introduction

Let B be an open ball in Euclidean space $\mathbb{R}^N$ $(N \geq 2)$ and let σ^* denote normalised surface area measure on ∂B. For any function h that is harmonic on a neighbourhood of $\overline{B}$ we have the spherical mean value property

$$h(x_0) = \int_{\partial B} h d\sigma^*, \tag{1}$$

where x_0 denotes the centre of B. Let

$$U_x(y) = \begin{cases} -\log|x-y| & (N=2) \\ |x-y|^{2-N} & (N \geq 3) \end{cases}$$

and (for a suitable measure μ) $U\mu(x) = \int U_x d\mu$. Using (1) with $h = U_x$ $(x \in \mathbb{R}^N \backslash \overline{B})$, we obtain the familiar fact that $U\delta_{x_0} = U\sigma^*$ outside $\overline{B}$, where δ_{x_0} is the Dirac measure at x_0. In this context we regard σ^* as the result of sweeping δ_{x_0} out of B.

Now let λ denote Lebesgue measure on $\mathbb{R}^N$ and let $\lambda^* = (\lambda|_B)/\lambda(B)$. For any integrable harmonic function h on B we have the volume mean value property $h(x_0) = \int_B h d\lambda^*$ and, arguing as before, this yields $U\delta_{x_0} = U\lambda^*$ outside B. By analogy with the previous paragraph we can regard λ^* as the result of a "partial sweeping" (or partial balayage) of δ_{x_0}, in which a certain density of measure is allowed to remain. Such a notion has been developed by Gustafsson and Sakai (see, for example, [8]) in connection with the construction of certain types of quadrature domains for harmonic functions. To be more specific, let μ be a measure with compact support, and suppose we have a procedure for arriving at a bounded open set Ω containing the support of μ such that $U\mu = U(\lambda|_\Omega)$ outside Ω. (Not every measure μ will have such an associated set Ω.) Arranging, by the addition to Ω of a null set if necessary, that the Lebesgue points of $\mathbb{R}^N \backslash \Omega$ are dense in $\mathbb{R}^N \backslash \Omega$, it follows that $\nabla U\mu = \nabla U(\lambda|_\Omega)$ outside Ω. Thus

$$\int_\Omega h d\lambda = \int h d\mu \quad \text{whenever} \quad h \in \{U_x, \partial_i U_x : x \in \mathbb{R}^N \backslash \Omega,\ i = 1, ..., N\}.$$

An approximation result of Sakai [11] now shows that $\int_\Omega h d\lambda = \int h d\mu$ for every λ-integrable harmonic function h on Ω; that is, Ω is a quadrature domain with respect to the measure μ.

---

[0]2000 Mathematics Subject Classification: 31B20.

This research was supported by Science Foundation Ireland under Grant 06/RFP/MAT057, and is also part of the programme of the ESF Network "Harmonic and Complex Analysis and Applications" (HCAA).

Partial balayage is closely related to the obstacle problem, and to the abstract theory of potential cones (see, for example, [2]). However, in this article, we will not rely on these connections. Instead we will present a concise, self-contained exposition of partial balayage from the viewpoint of classical potential theory. Along the way we will obtain a new proof of the singularity of harmonic measure with respect to Lebesgue measure and a general version of the Hele-Shaw law for fluid flow. The paper concludes with an outline of how partial balayage has recently been employed to make progress on the exterior inverse problem of potential theory [6].

2. Partial reductions of superharmonic functions

Now let $\Omega \subseteq \mathbb{R}^N$ be an open set that possesses a Green function (that is, Ω must have nonpolar complement in the case where $N = 2$). Further, let u and v be superharmonic functions on Ω, where $u > 0$ and $v < +\infty$, and let $\mathcal{S}(\Omega)$ denote the collection of all subharmonic functions on Ω. We write

$$^{\Omega}\Phi_u^v = \{s \in \mathcal{S}(\Omega) : s \leq u - v\}$$

(the prefix Ω may be omitted where confusion is unlikely to occur), and will use this class to define the "partial reduction" of u with respect to v.

If B is an open ball such that $\overline{B} \subset \Omega$, we denote by $\tau_B u$ the Poisson integral of u in B, and extend it to all of Ω by putting $\tau_B u = u$ elsewhere.

Lemma 1. $-v \leq \sup \Phi_u^v \in \Phi_u^v$.

Proof. Clearly $-v \in \Phi_u^v$. The family Φ_u^v is locally uniformly bounded above since, if B is any open ball such that $\overline{B} \subset \Omega$, then $s \leq \tau_B s \leq \tau_B u - \inf_B v$ on B for every $s \in \Phi_u^v$. Let w denote the upper semicontinuous regularization of $\sup \Phi_u^v$. Then $w \in \mathcal{S}(\Omega)$ and $w \geq \sup \Phi_u^v$, with equality λ-almost everywhere (see Theorem 3.7.5 or 5.7.1 of [1]). It follows that $w \leq u - v$ almost everywhere, and hence, by consideration of means over arbitrarily small balls, everywhere on Ω. Thus $w \in \Phi_u^v$, and the result follows. □

Next we define the *partial reduction of* u *relative to* v by

$$^{\Omega}P_u^v = \sup \Phi_u^v + v,$$

where, again, the prefix Ω may be omitted. It is clear that $0 \leq P_u^v \leq u$.

Remark 2. If (v_n) is an increasing sequence of superharmonic functions on Ω with limit v, then $\sup \Phi_u^{v_n} \downarrow \sup \Phi_u^v$ and $P_u^{v_n} \to P_u^v$. To see this, we observe that the sequence $(\sup \Phi_u^{v_n})$ of subharmonic functions decreases to a subharmonic limit w satisfying $w \geq \sup \Phi_u^v$ and, since $w \leq u - v_n$ for each n, we also have $w \leq \sup \Phi_u^v$.

Let $B(x, r)$ denote the open ball of centre x and radius r in $\mathbb{R}^N$.

Lemma 3. *If the superharmonic function* v *is continuous, then*
(a) P_u^v *is continuous, and*
(b) $\sup \Phi_u^v$ *is harmonic on the open set* $\omega = \{x \in \Omega : u(x) > P_u^v(x)\}$.

Proof. (a) Let $y \in \Omega$ and $\varepsilon > 0$. Since $\tau_{B(y,\delta)} v \uparrow v$ as $\delta \downarrow 0$, it follows from Dini's theorem that we can choose $\eta > 0$ such that $\overline{B(y, \eta)} \subset \Omega$ and

$$v - \tau_{B(y,\eta)} v < \varepsilon \quad \text{on} \quad \Omega. \tag{2}$$

Let $w = \sup \Phi_u^v$. Then

$$\tau_{B(y,\eta)} w \leq \tau_{B(y,\eta)}(u - v) \leq u - \tau_{B(y,\eta)} v < u - v + \varepsilon \quad \text{on} \quad \Omega,$$

so $\tau_{B(y,\eta)} w - \varepsilon \in \Phi_u^v$ and hence

$$\liminf_{x \to y} w(x) \geq \liminf_{x \to y} \tau_{B(y,\eta)} w(x) - \varepsilon = \tau_{B(y,\eta)} w(y) - \varepsilon \geq w(y) - \varepsilon.$$

Since ε and y are arbitrary, w is lower (as well as upper) semicontinuous and so P_u^v is continuous. (The same argument shows more generally that, if v is continuous on an open subset U of Ω, then P_u^v is also continuous on U.)

(b) Let B be any open ball such that $\overline{B} \subset \omega$ and let $\varepsilon = \inf_{\overline{B}}(u - P_u^v)$. It follows from (a) that $\varepsilon > 0$. Let $w = \sup \Phi_u^v$ and $y \in B$, and choose $\eta > 0$ such that $B(y, \eta) \subset B$ and (2) holds. Since $u - v - w = u - P_u^v \geq \varepsilon$ on $\overline{B}$, we have

$$w \leq \tau_{B(y,\eta)} w \leq \tau_{B(y,\eta)}(u - v) - \varepsilon \leq u - \tau_{B(y,\eta)} v - \varepsilon < u - v \quad \text{on } \Omega.$$

Since $\tau_{B(y,\eta)} w \in \mathcal{S}(\Omega)$, it follows that $\tau_{B(y,\eta)} w = w$. Hence w is harmonic on $B(y, \eta)$, and (b) follows in view of the arbitrary choice of B and y. □

Theorem 4. *The function P_u^v is superharmonic on Ω.*

Proof. In the case where v is continuous, we note from Lemma 3(b) that P_u^v is superharmonic on ω. Since $P_u^v \leq u$ on Ω with equality on $\Omega \backslash \omega$, we now see that P_u^v is supermeanvalued at points of $\Omega \backslash \omega$. This, together with Lemma 3(a), shows that P_u^v is superharmonic on all of Ω.

In the general case we can find an increasing sequence (v_n) of continuous superharmonic functions on Ω with limit v. The nonnegative functions $P_u^{v_n}$ are superharmonic, by the preceding paragraph, and $P_u^{v_n} \to P_u^v$ by Remark 2. Hence the lower semicontinuous regularization $\widehat{P}_u^v$ is superharmonic on Ω and coincides with P_u^v almost everywhere (λ) (see Corollary 5.7.2 in [1]). Since P_u^v is the sum of a subharmonic and a superharmonic function, consideration of means over arbitrarily small balls now shows that $\widehat{P}_u^v = P_u^v$ everywhere, whence P_u^v is superharmonic on Ω. □

Lemma 5. *(a) If (u_n) is an increasing sequence of superharmonic functions on Ω with limit u and, for each n, there is a superharmonic function w_n such that $u = u_n + w_n$, then $P_{u_n}^v \uparrow P_u^v$.*
(b) If (u_n) is a decreasing sequence of superharmonic functions on Ω such that $u_n \downarrow u$ quasi-everywhere, then $P_{u_n}^v \downarrow P_u^v$.
(c) If (Ω_n) is an exhaustion of Ω by open sets, then ${}^{\Omega_n}P_u^v \downarrow {}^{\Omega}P_u^v$.

Proof. (a) Since $P_u^v - v - w_n \in \Phi_{u_n}^v$, we see that

$$P_u^v \geq P_{u_n}^v \geq P_u^v - w_n \uparrow P_u^v \quad \text{quasi-everywhere on} \quad \Omega.$$

The superharmonic functions $\lim_{n\to\infty} P_{u_n}^v$ and P_u^v thus agree quasi-everywhere, and hence everywhere, on Ω.

(b) The sequence $(\sup \Phi_{u_n}^v)$, of subharmonic functions on Ω, decreases to a subharmonic function w on Ω. Since $\sup \Phi_{u_n}^v \leq u_n - v$, we see that $w \leq u - v$ quasi-everywhere, and hence everywhere, on Ω. Thus

$$w \leq \sup \Phi_u^v \leq \sup \Phi_{u_n}^v \to w \quad \text{as} \quad n \to \infty,$$

so

$$P_{u_n}^v \downarrow P_u^v \quad \text{on} \quad \Omega.$$

(c) If $n \geq m$ and $s \in {}^{\Omega_n}\Phi_u^v$, then it is clear that $s|_{\Omega_m} \in {}^{\Omega_m}\Phi_u^v$. Hence ${}^{\Omega_n}P_u^v$ is decreasing in n. Since $\left({}^{\Omega_n}P_u^v - v\right)_{n \geq m}$ is a decreasing sequence of subharmonic functions on Ω_m, it is clear that the limit s is subharmonic and $s \leq u - v$ on Ω, which yields the desired result. □

Remark 6. (a) If $u' = u + h$, where h is positive and harmonic on Ω, then $\Phi_{u'}^v = \{w + h : w \in \Phi_u^v\}$ and so $P_{u'}^v = P_u^v + h$.
(b) If $v' = v + h$, where h is a harmonic function on Ω, then $\Phi_u^{v'} = \{w - h : w \in \Phi_u^v\}$ and so $P_u^{v'} = P_u^v$. Thus P_u^v depends only on the Riesz measure associated with v.

We will build on this last observation in the next section.

3. Partial balayage of measures

Let $G_\Omega\mu$ be the potential on Ω of a measure μ, and let ν denote the Riesz measure associated with v. (Thus, for any ball B satisfying $\overline{B} \subset \Omega$, the functions v and $U(\nu|_B)$ differ on B by a harmonic function.) We denote the Riesz measure associated with the superharmonic function $P_{G_\Omega\mu}^v$ by B_μ^ν and call it the *partial balayage* of μ onto ν (in Ω). It is well-defined, in view of Remark 6(b). Since, by definition, P_u^v is the sum of v and a subharmonic function, we see that $B_\mu^\nu \leq \nu$ on Ω.

We will denote the (classical) sweeping of a measure η onto a set A by η^A; that is, η^A is defined by the equation $G_\Omega\eta^A = \widehat{R}_{G_\Omega\eta}^A$. In particular, if U is an open set compactly contained in Ω, then $\delta_x^{\Omega\setminus U}$ is harmonic measure for U and x.

The next result concerns the structure of the measure B_μ^ν.

Theorem 7. *Suppose v is continuous and let ω be the open set defined by*

$$\omega = \{x \in \Omega : G_\Omega\mu(x) > P_{G_\Omega\mu}^v(x)\}. \tag{3}$$

Then $(\mu|_\omega)^{\Omega\setminus\omega} \geq (\nu|_\omega)^{\Omega\setminus\omega}$ and

$$B_\mu^\nu = \nu|_\omega + \mu|_{\Omega\setminus\omega} + (\mu|_\omega)^{\Omega\setminus\omega} - (\nu|_\omega)^{\Omega\setminus\omega}. \tag{4}$$

Proof. Since $P_{G_\Omega\mu}^v < \infty$ on Ω, by definition, it follows that μ does not charge the polar set of irregular boundary points of ω. Hence (see Theorem 7.5.2 in [1]) the greatest harmonic minorant of $P_{G_\Omega\mu}^v$ on ω is given by $R_{G_\Omega\mu}^{\Omega\setminus\omega}$. We know from Lemma 3(b) that the Riesz measure associated with $P_{G_\Omega\mu}^v$ on ω is $\nu|_\omega$. Hence, by the Riesz decomposition,

$$P_{G_\Omega\mu}^v = G_\omega(\nu|_\omega) + R_{G_\Omega\mu}^{\Omega\setminus\omega} \text{ on } \omega. \tag{5}$$

Thus $P_{G_\Omega\mu}^v = G_\Omega(\nu|_\omega) - \widehat{R}_{G_\Omega(\nu|_\omega)}^{\Omega\setminus\omega} + \widehat{R}_{G_\Omega\mu}^{\Omega\setminus\omega}$ quasi-everywhere on Ω, and (4) follows.

Finally we note from (5) that

$$P_{G_\Omega\mu}^v = G_\omega(\nu|_\omega) + R_{G_\Omega(\mu|_\omega)}^{\Omega\setminus\omega} + G_\Omega(\mu|_{\Omega\setminus\omega}) \text{ on } \omega.$$

Since the function $w = P_{G_\Omega\mu}^v - G_\Omega(\mu|_{\Omega\setminus\omega})$ satisfies $w \leq G_\Omega(\mu|_\omega)$ on Ω with equality on $\Omega \setminus \omega$, we have

$$R_{G_\Omega(\mu|_\omega)}^{\Omega\setminus\omega} \leq w \leq G_\Omega(\mu|_\omega) \text{ on } \Omega.$$

Noting that w is superharmonic on ω, we see that w is locally super-meanvalued on Ω, and also lower semicontinuous except perhaps at a polar subset of $\Omega \cap \partial\omega$. Thus $\widehat{w}$ is superharmonic, and it follows from (4) that $(\mu|_\omega)^{\Omega\setminus\omega} \geq (\nu|_\omega)^{\Omega\setminus\omega}$. □

Our next aim is to show that (4) can be simplified when $\nu \ll \lambda$. To this end, we need to borrow some results from the theory of Sobolev spaces. We will do this by proving what we need directly using the same methods. The interested reader can consult [7] for further information about Sobolev spaces. We write U^f in place of $U(f\lambda)$ when f is a bounded Borel measurable function with compact support.

Lemma 8. *Let B_1, B_2, B_3 be open balls such that $\overline{B}_1 \subset B_2$ and $\overline{B}_2 \subset B_3$. There exist constants C_1, C_2, C_3 such that, for any $g \in C_c^\infty(B_1)$,*

$$\left\| D^2 U^g \right\|_{L^2(B_2)} \leq C_1 \left\| g \right\|_{L^2(B_1)} + C_2 \left\| \nabla U^g \right\|_{L^2(B_3)} + C_3 \left\| U^g \right\|_{L^2(B_3)} . \tag{6}$$

Proof. For any $\psi \in C_c^3(\mathbb{R}^N)$ we have

$$\begin{aligned} \left\| D^2 \psi \right\|^2_{L^2(\mathbb{R}^N)} &= \int \sum_{i=1}^N \sum_{j=1}^N \left(\frac{\partial^2 \psi}{\partial x_i \partial x_j} \right)^2 d\lambda = - \int \sum_{i=1}^N \sum_{j=1}^N \frac{\partial^3 \psi}{\partial x_i^2 \partial x_j} \frac{\partial \psi}{\partial x_j} d\lambda \\ &= \int \sum_{i=1}^N \frac{\partial^2 \psi}{\partial x_i^2} \sum_{j=1}^N \frac{\partial^2 \psi}{\partial x_j^2} d\lambda = \left\| \Delta \psi \right\|^2_{L^2(\mathbb{R}^N)} . \end{aligned}$$

Thus, if $\phi \in C_c^\infty(B_3)$ and $\phi = 1$ on B_2, we obtain

$$\begin{aligned} \left\| D^2 U^g \right\|_{L^2(B_2)} &\leq \left\| D^2 \left(U^g \phi \right) \right\|_{L^2(B_3)} = \left\| \Delta \left(U^g \phi \right) \right\|_{L^2(B_3)} \\ &= \left\| -a_N g \phi + 2 \left\langle \nabla U^g, \nabla \phi \right\rangle + U^g \Delta \phi \right\|_{L^2(B_3)} \quad (g \in C_c^\infty(B_1)), \end{aligned}$$

where a_N is a dimensional constant. It follows that (6) holds with $C_1 = a_N \sup |\phi|$, $C_2 = 2 \sup |\nabla \phi|$ and $C_3 = \sup |\Delta \phi|$. □

Theorem 9. *Let h be a harmonic function on an open ball B and let $A = \{x \in B : U^f(x) = h(x)\}$, where f is a bounded Borel function with compact support. Then $f = 0$ almost everywhere (λ) on A.*

Proof. We choose B_1, B_2, B_3 as in the preceding lemma, with the additional condition that $\overline{B} \subset B_1$. There is no loss of generality in assuming that h is harmonic on B_1 and that $f = 0$ outside B_1. Further, since there exists f_1 in $L^\infty(\mathbb{R}^N)$ such that $f_1 = 0$ outside $B_1 \backslash B$ and $h = U^{f_1}$ in B, we may assume that $h = 0$.

We will first show that, for each i in $\{1, ..., N\}$, there exists v_i in $L^2(B_2)$ such that $\partial^2 U^f / \partial x_i^2 = v_i$ in the sense of distributions. To see this, we use smoothing to find a sequence (g_n) of functions in $C_c^\infty(B_1)$ such that $g_n \to f$ almost everywhere (λ) and $\sup |g_n| \leq \|f\|_\infty$ for each n. Let x_0 denote the centre of B_1. Since

$$\begin{aligned} \int_{B_1} \left| \nabla_x U_x(y) \right| \left| g_n(y) \right| d\lambda(y) &\leq \|f\|_\infty \int_{B_1} \left| \nabla_x U_x(y) \right| d\lambda(y) \\ &\leq \|f\|_\infty \max\{N-2, 1\} \int_{B_1} \left| x_0 - y \right|^{1-N} d\lambda(y) \\ &< +\infty, \end{aligned}$$

the sequence (∇U^{g_n}) is uniformly bounded and, by dominated convergence, pointwise convergent to ∇U^f. In particular, $\nabla U^{g_n} \to \nabla U^f$ in $L^2(B_1)$, and similar reasoning shows that $U^{g_n} \to U^f$ in $L^2(B_2)$. By the preceding lemma, we know that $\left\| \partial^2 U^{g_n - g_m} / \partial x_i^2 \right\|_{L^2(B_2)} \to 0$ as $m, n \to \infty$ and hence, by completeness, $\left(\partial^2 U^{g_n} / \partial x_i^2 \right)$ converges in $L^2(B_2)$ to some v_i. Thus, as claimed,

$$\int_{B_2} U^f \frac{\partial^2 \phi}{\partial x_i^2} d\lambda = \lim_{n \to \infty} \int_{B_2} U^{g_n} \frac{\partial^2 \phi}{\partial x_i^2} d\lambda = \int_{B_2} \phi v_i d\lambda \quad (\phi \in C_c^\infty(B_2)).$$

Let $\varepsilon > 0$ and $\Phi_\varepsilon(t) = \operatorname{sgn}(t)\left(\sqrt{t^2+\varepsilon^2}-\varepsilon\right)$ for $t \in \mathbb{R}$. Then $\Phi_\varepsilon \in C^1(\mathbb{R})$ and

$$\begin{aligned}\int \Phi_\varepsilon\left(\frac{\partial U^f}{\partial x_i}\right)\frac{\partial \phi}{\partial x_i}d\lambda &= \lim_{n\to\infty}\int \Phi_\varepsilon\left(\frac{\partial U^{g_n}}{\partial x_i}\right)\frac{\partial \phi}{\partial x_i}d\lambda \\ &= -\lim_{n\to\infty}\int \Phi'_\varepsilon\left(\frac{\partial U^{g_n}}{\partial x_i}\right)\frac{\partial^2 U^{g_n}}{\partial x_i^2}\phi d\lambda \\ &= -\int \Phi'_\varepsilon\left(\frac{\partial U^f}{\partial x_i}\right)v_i\phi d\lambda \quad (\phi \in C_c^\infty(B_2)).\end{aligned}$$

Letting $\varepsilon \to 0+$ we see that

$$\int U^f \frac{\partial^2\phi}{\partial x_i^2}d\lambda = -\int \frac{\partial U^f}{\partial x_i}\frac{\partial \phi}{\partial x_i}d\lambda = \int_{\left\{\frac{\partial U^f}{\partial x_i}\neq 0\right\}} v_i\phi d\lambda,$$

and summing over i yields

$$\int U^f \Delta\phi d\lambda = \int \phi \sum v_i \chi_{\left\{\frac{\partial U^f}{\partial x_i}\neq 0\right\}}d\lambda \qquad (\phi \in C_c^\infty(B_2)).$$

Since $\nabla U^f = \nabla h = 0$ almost everywhere on A (a C^1 function has zero gradient at all Lebesgue points of its zero set), we have $f = 0$ almost everywhere on A, as required. □

We will now use partial balayage to give a new proof of the singularity of harmonic measure with respect to n-dimensional measure. For planar domains this was originally proved by Øksendal [10]. In the case of higher dimensions it is contained in work of Bourgain [3] (see also Hansen and Hueber [9]).

Theorem 10. *If U is an open set and $x_0 \in U$, then $\delta_{x_0}^{U^c} \perp \lambda$.*

Proof. We may assume that U is bounded, since the case of an unbounded set U will then follow by consideration of the intersection of U with arbitrarily large balls centred at x_0 We choose r_0 such that $U \subset B(x_0, r_0)$, and define $\Omega = \mathbb{R}^N$ if $N \geq 3$, or $\Omega = B(x_0, r_0+1)$ if $N = 2$. We may assume that U is connected.

Let (U_n) be a sequence of open sets with union U such that $x_0 \in U_1$ and $\overline{U}_n \subset U_{n+1}$ for all n. For $t > 0$ we define

$$\mu_t = \lambda|_U + t\delta_{x_0}, \quad \nu_{t,n} = \lambda|_U + t\delta_{x_0}^{U_n^c} \quad \text{and} \quad \nu_t = \lambda|_U + t\delta_{x_0}^{U^c}.$$

We now consider the partial balayage of these measures onto λ (so that the function v may be taken to be a suitable multiple of $-|x|^2$). Since $B_{\mu_t}^\lambda \leq \lambda$, we can write $B_{\mu_t}^\lambda = f_t\lambda$, where $0 \leq f_t \leq 1$. Let ω_t denote the open set $\{G_\Omega\mu_t > G_\Omega f_t\}$. Then $f_t = 1$ almost everywhere (λ) on ω_t, by Lemma 3(b). Since $G_\Omega\mu_t - G_\Omega f_t$ is nonnegative and superharmonic, but not harmonic, on U, we see that $U \subset \omega_t$. Since

$$G_\Omega(f_t - \chi_U) = G_\Omega(\mu_t - \lambda|_U) = tG_\Omega(x_0, \cdot) \;\text{ outside }\; \omega_t,$$

we see from Theorem 9 that $f_t = \chi_U = 0$ almost everywhere (λ) on $\mathbb{R}^N \backslash \omega_t$, and so $B_{\mu_t}^\lambda = \lambda|_{\omega_t}$.

Similarly, writing $B_{\nu_{t,n}}^\lambda = f_{t,n}\lambda$ and $\omega_{t,n} = \{G_\Omega\nu_{t,n} > G_\Omega f_{t,n}\}$, we see that $U \subset \omega_{t,n}$ and $B_{\nu_{t,n}}^\lambda = \lambda|_{\omega_{t,n}}$. Since $G_\Omega\nu_{t,n} = G_\Omega\mu_t$ on $\mathbb{R}^N\backslash\overline{U}_n$ and $G_\Omega f_{t,n} \leq G_\Omega f_t$ we now have

$$\omega_{t,n} = U \cup \{x \in \mathbb{R}^N\backslash U : G_\Omega\mu_t(x) > G_\Omega f_{t,n}(x)\} \supseteq \omega_t$$

and so, in fact, $\lambda\left|_{\omega_{t,n}} = \lambda\right|_{\omega_t}$. Hence $B^{\lambda}_{\nu_{t,n}} = \lambda|_{\omega_t}$ for each n, and thus, by Lemma 5(b), $B^{\lambda}_{\nu_t} = \lambda|_{\omega_t}$. From Theorem 7 we now see that $\nu_t(\Omega\backslash\omega_t) = 0$, so $\delta^{U^c}_{x_0} = 0$ outside $(\cap_{t>0}\omega_t)\backslash U$.

Finally, let $A_t = B(x_0, r_t)\backslash B(x_0, r_0)$, where r_t is chosen such that $\lambda(A_t) = t$. Then $G_\Omega(\lambda|_{A_t\cup U}) \le G_\Omega\mu_t$, with equality on $\Omega\backslash\overline{B}(x_0, r_t)$. Since

$$G_\Omega(\lambda|_{A_t\cup U}) \le G_\Omega(\lambda|_{\omega_t}) \le G_\Omega\mu_t,$$

it follows that $\omega_t \subset B(x_0, r_t)$. It is now easy to see that $\lambda(\omega_t) = \mu_t(\Omega) = \lambda(U) + t$, and so $\lambda((\cap_{t>0}\omega_t)\backslash U) = 0$, as required. □

Corollary 11. *If $\nu \ll \lambda$, then $(\mu|_\omega)^{\Omega\backslash\omega} = (\nu|_\omega)^{\Omega\backslash\omega}$ on Ω, where ω is defined by (3), and so*

$$B^{\nu}_{\mu} = \nu|_\omega + \mu|_{\Omega\backslash\omega}.$$

Proof. Suppose that $\nu \ll \lambda$. Since $B^{\nu}_{\mu} \le \nu$ on Ω, we see from the above result and (4) that $(\mu|_\omega)^{\Omega\backslash\omega} = (\nu|_\omega)^{\Omega\backslash\omega}$ on Ω. □

4. The Hele-Shaw law

Let U be a smoothly bounded region in the narrow gap between two parallel plates, and suppose that U is filled with a viscous incompressible fluid, and that fluid is being steadily injected at $x_0 \in U$. This results in a growing family of domains $\{U_t : t \ge 0\}$ which represent the extent of the fluid at time t. The evolution of these domains is described by the Hele-Shaw equation, which says that at a given point of ∂U_t the normal velocity is proportional to the density of $\delta^{U_t^c}_{x_0}$ with respect to arc length measure on ∂U_t. From this it can be shown that the partial balayage of the measure $t\delta_{x_0} + \lambda|_U$ with respect to λ is $\lambda|_{U_t}$, and that the 'derivative' of $B^{\lambda}_{t\delta_{x_0}+\lambda|_U}$ with respect to t is $\delta^{U_t^c}_{x_0}$. In this section we will adapt the proof of Theorem 10 to show that, for a reasonably general family of domains U, the 'derivative' of $B^{\lambda}_{t\delta_{x_0}+\lambda|_U}$ at $t = 0$ is $\delta^{U^c}_{x_0}$.

Now let U be a bounded domain in $\mathbb{R}^N$ and let $x_0 \in U$. As before we choose r_0 such that $U \subset B(x_0, r_0)$, and define $\Omega = \mathbb{R}^N$ if $N \ge 3$, or $\Omega = B(x_0, r_0 + 1)$ if $N = 2$. We would like to be able to say that

$$\left(B^{\lambda}_{t\delta_{x_0}+\lambda|_U} - \lambda|_U\right)/t \xrightarrow{w^*} \delta^{U^c}_{x_0} \qquad (t \to 0+). \tag{7}$$

However, this is not true in general. For example, if L is the closed upper halfspace and $U = B(0, 2)\backslash(L \cap \partial B(0, 1))$, then

$$B^{\lambda}_{t\delta_0+\lambda|_U} = B^{\lambda}_{t\delta_0+\lambda|_{B(0,2)}} = \lambda|_{B(0,r_t)} \qquad (t > 0),$$

where r_t satisfies $\lambda(B(0, r_t)) = \lambda(B(0, 2)) + t$, and thus

$$\left(B^{\lambda}_{t\delta_0+\lambda|_U} - \lambda|_U\right)/t \xrightarrow{w^*} \delta_0^{B(0,2)^c} \ne \delta_0^{U^c} \qquad (t \to 0+).$$

The above example partially explains the formulation of the following generalisation of the Hele-Shaw law. We recall that the fine topology of classical potential theory is the coarsest topology which makes all superharmonic functions continuous. A set which is open with respect to this topology will be called finely open.

Theorem 12. *Let U be a bounded domain in $\mathbb{R}^N$ and let $x_0 \in U$. Suppose that, for any finely open set V satisfying $\lambda(V\backslash U) = 0$, the set $V\backslash U$ is actually polar. Then (7) holds.*

Proof. Let μ_t, ν_t and ω_t be as in the proof of Theorem 10. We saw there that $B^\lambda_{\mu_t} = B^\lambda_{\nu_t} = \lambda|_{\omega_t}$ and $\lambda(\omega_t) = \lambda(U) + t$, that $U \subseteq \omega_t$ and that $\omega_t \subset B(x_0, r_0 + \frac{1}{2})$ for all sufficiently small t. Hence $B^\lambda_{\mu_t} - \lambda|_U = \lambda|_{\omega_t \backslash U}$ for all $t > 0$. Let (t_n) be a positive l^1 sequence. By weak* compactness we can replace (t_n) by an appropriate subsequence so that the associated sequence of probability measures $\left(t_n^{-1}\lambda|_{\omega_{t_n}\backslash U}\right)$ is weak* convergent to some measure ν. Since $G_\Omega\mu_t \geq G_\Omega\nu_t \geq G_\Omega\left(\lambda|_{\omega_t}\right)$, with equality outside ω_t, we have

$$G_\Omega\left(t^{-1}\lambda|_{\omega_t\backslash U}\right) \leq G_\Omega\left(t^{-1}\left(\nu_t - \lambda|_U\right)\right) = G_\Omega\delta^{U^c}_{x_0}, \quad \text{with equality outside } \omega_t.$$

It follows (by Exercise 5.13 in [1]) from weak* convergence that $G_\Omega\nu \leq G_\Omega\delta^{U^c}_{x_0}$ with equality quasi-everywhere on $\Omega\backslash\omega^*$, where $\omega^* = \cap_{m\geq 1}\cup_{n\geq m}\omega_{t_n}$. Clearly $\lambda(\omega^*) = \lambda(U)$. The finely open set $V = \{G_\Omega\nu < G_\Omega\delta^{U^c}_{x_0}\}$ satisfies $\lambda(V\backslash U) \leq \lambda(\omega^*\backslash U) = 0$ and so, by hypothesis, $V\backslash U$ is polar. It follows that $G_\Omega\nu = G_\Omega\delta^{U^c}_{x_0}$ quasi-everywhere on $\Omega\backslash U$. Since $\delta^{U^c}_{x_0}(U) = 0$, and $G_\Omega\delta^{U^c}_{x_0}$ is finite valued, the Maria-Frostman domination principle shows that $G_\Omega\nu \geq G_\Omega\delta^{U^c}_{x_0}$ and hence $\nu = \delta^U_{x_0}$. The result now follows, in view of the arbitrary nature of the sequence (t_n). □

The following example shows the necessity of the hypothesis in the above result.

Example 13. Let (x_k) be a sequence in $(B(0,1)\backslash B(0,\frac{1}{2})) \cap L$ with limit set $E = L \cap \partial B(0,1)$, and let

$$F = \{x \in B(0,1) : U\mu(x) \geq (1-|x|)^{2-N}\},$$

where $\mu = \sum m_k\delta_{x_k}$ and $0 < m_k \leq 2^{-k}$. (If $N = 2$, we replace $(1-|x|)^{2-N}$ by $-\log(1-|x|)$.) By choosing the m_k to be sufficiently small we can ensure that the open set $U = B(0,2)\backslash(E \cup F)$ is connected. The proof of Theorem 12 can be modified to show that

$$\begin{aligned} t^{-1}\left(B^\lambda_{t\delta_0+\lambda|_U} - \lambda\,|_U\right) &= t^{-1}\left(B^\lambda_{t\delta_0+\lambda|_{B(0,2)\backslash F}} - \lambda\,|_U\right) \\ &\xrightarrow{w^*} \delta_0^{V^c} \neq \delta_0^{U^c} \qquad (t \to 0+), \end{aligned}$$

where $V = B(0,2)\backslash F$, and so (7) fails to hold for this choice of U. The crucial point here is that the set V is finely open (compare Example 7.9.3 in [1]) and $\lambda(V\backslash U) = 0$, but $V\backslash U = L \cap \partial B(0,1)$, which is nonpolar.

5. Application to the exterior inverse problem

A bounded domain Ω in $\mathbb{R}^N$ is called *solid* if $\mathbb{R}^N\backslash\overline{\Omega}$ is connected and $\Omega = \overline{\Omega}^\circ$. The *exterior inverse problem of potential theory*, which dates back to work of Novikov in the 1930's, is as follows:

If Ω_1 and Ω_2 are solid domains in $\mathbb{R}^N$ such that $\lambda|_{\Omega_1}$ and $\lambda|_{\Omega_2}$ produce the same potential in the complement of $\Omega_1 \cup \Omega_2$, must Ω_1 and Ω_2 coincide?

Novikov himself proved that the answer is yes if both domains are assumed to be convex or, more generally, starshaped with respect to a common point. Although it is nowadays suspected that the answer to the general question may be negative, it has long been conjectured that convexity of *one* of the domains should be enough for a positive answer. That this is, in fact, the case was recently proved in [6]. The proof, which relies on partial balayage and the 'moving plane' method, will be outlined in this section. We begin by formulating the main result.

Theorem 14. *If Ω_1, Ω_2 are bounded solid domains in $\mathbb{R}^N$, where Ω_2 is convex, and if*

$$U^{\chi_{\Omega_1}} = U^{\chi_{\Omega_2}} \text{ in } \mathbb{R}^N \setminus (\Omega_1 \cup \Omega_2),$$

then $\Omega_1 = \Omega_2$.

We will use partial balayage in the special case where $\nu = \lambda$, so $v(x)$ can be taken to be a suitable multiple of $-|x|^2$, and $\Omega = \mathbb{R}^N$. This creates a problem when $N = 2$, since $\mathbb{R}^2$ is not Greenian, but our measure μ will always have compact support and the construction and basic properties of partial balayage can be extended to cover this case (see [8], for instance). For ease of exposition we will assume below that $N \geq 3$.

In order to describe the moving plane method, we need some notation. Points in $\mathbb{R}^N$ will be denoted by $(x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$, and we will write

$$W_+ = \{(x', x_N) : x_N > 0\},\ W_- = \{(x', x_N) : x_N < 0\},\ H = \{(x', x_N) : x_N = 0\}.$$

Given a positive measure μ we denote by $\Omega(\mu)$ the largest open set U for which $(\lambda - B_\mu^\lambda)(U) = 0$. (Thus $\Omega(\mu)$ will contain the set ω in Theorem 7.) By Corollary 11 we know that

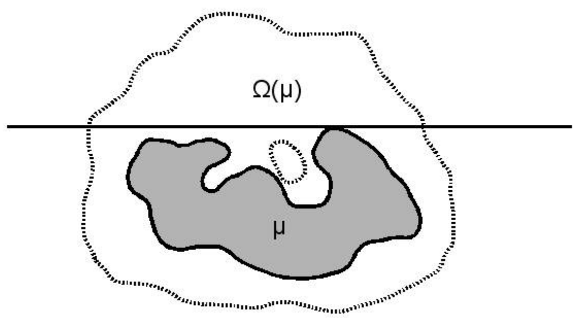
$$B_\mu^\lambda = \lambda|_{\Omega(\mu)} + \mu|_{\Omega(\mu)^c}, \tag{8}$$

where $\mu|_{\Omega(\mu)^c} \leq \lambda$.

Lemma 15. *Let μ be a measure with compact support contained in $W_- \cup H$ and let $A = \{x' : (x', 0) \in \Omega(\mu) \cap H\}$. Then there is a continuous function $g : A \to (0, \infty)$, continuously vanishing on ∂A, such that*

$$\Omega(\mu) \cap W_+ = \{(x', x_N) : x' \in A \text{ and } 0 < x_N < g(x')\}. \tag{9}$$

Since it is not immediately obvious why this should be true we will supply a proof below, based on [8]. However, the plausibility of the lemma is suggested by the intuition, using the connection with Hele-Shaw flow, that the measure B_μ^λ will be as 'close' to μ as possible subject to the constraint that it is majorized by λ. The name 'moving plane' comes from the fact that we can apply the above lemma to all hyperplanes disjoint from the convex hull K of the support of μ to see that $\Omega(\mu)$ cannot have any 'holes' outside K (see the following figure).



Proof. Let $u = U\mu - UB_\mu^\lambda$. Thus $u \geq 0$. We may assume, by means of a limiting argument, that supp$\mu \subset W_-$, and so u is continuously differentiable on an open set containing $W_+ \cup H$. Let $\overline{u}(x) = u(x', -x_N)$. We note that $U\mu - \overline{u} - v$ is subharmonic on W_+, and $UB_\mu^\lambda - v$ is subharmonic on all of $\mathbb{R}^N$. Since

$$U\mu - \overline{u} - v = U\mu - u - v = UB_\mu^\lambda - v \quad \text{on } H,$$

the function

$$w = \begin{cases} \max\{U\mu - \overline{u} - v, UB_\mu^\lambda - v\} & \text{on } W_+ \\ UB_\mu^\lambda - v & \text{on } W_- \cup H \end{cases}$$

is a subharmonic minorant of $U\mu - v$. Thus $w \leq UB_\mu^\lambda - v$ by the definition of UB_μ^λ, whence $U\mu - \overline{u} \leq UB_\mu^\lambda$ on W_+ and so $u \leq \overline{u}$ there. It follows that $\partial u/\partial x_N \leq 0$ on H.

Let $\Omega_+ = \Omega(\mu) \cap W_+$. Since $u = 0$ on ω^c (where ω is given by (3)), and so on $\Omega(\mu)^c$, and since every point of $\partial\Omega(\mu)$ is the limit of some sequence of points of Lebesgue density of $\Omega(\mu)^c$, we see that $|\nabla u| = 0$ on $\partial\Omega_+ \cap W_+$. We note from (8) that Δu is constant in Ω_+, so the function $\partial u/\partial x_N$ is harmonic there, and hence $\partial u/\partial x_N \leq 0$ on Ω_+, by the maximum principle. Further, since u is nonconstant in each component of Ω_+, and $u = 0$ on $W_+ \backslash \Omega_+$, we actually have $\partial u/\partial x_N < 0$ on Ω_+. We now define

$$g(x') = \sup\{t > 0 : (x', t) \in \Omega_+\} \qquad (x' \in A).$$

Clearly $\Omega(\mu) \cap W_+$ lies under the graph of g. Conversely, if (x', x_N) lies under the graph of g and $x_N > 0$, then $u(x', x_N) > 0$ and so $(x', x_N) \in \omega(\mu) \subseteq \Omega(\mu)$. Thus (9) holds.

It remains to check that g is continuous and vanishes at ∂A. In fact, since $\Omega(\mu)$ is open and

$$\{x' : g(x') > c\} = \{x' : (x', c) \in \Omega(\mu)\} \qquad (c > 0),$$

it is clear that g is lower semicontinuous. On the other hand, if we apply the result of the previous paragraph with hyperplanes of varying orientation, we see that each point of $\partial\Omega_+ \cap W_+$ is the vertex of a vertical cone lying in $\Omega(\mu)^c$, and so g is also upper semicontinuous. In fact g vanishes continuously at ∂A, since we can apply the preceding reasoning with H replaced by a slightly lower hyperplane. □

We now proceed to the proof of Theorem 14. Let η be the Riesz measure associated with $\min\{U^{\chi_{\Omega_1}}, U^{\chi_{\Omega_2}}\}$, and $\Omega_0 = \Omega(\eta)$; thus $a_N\eta = -\Delta\min\{U^{\chi_{\Omega_1}}, U^{\chi_{\Omega_2}}\}$, where a_N is a dimensional constant. The first step is to prove a simple lemma about Ω_0.

Lemma 16. *With the above notation,* $B_\eta^\lambda = \lambda|_{\Omega_0}$ *and* $\Omega_1 \cap \Omega_2 \subset \Omega_0$.

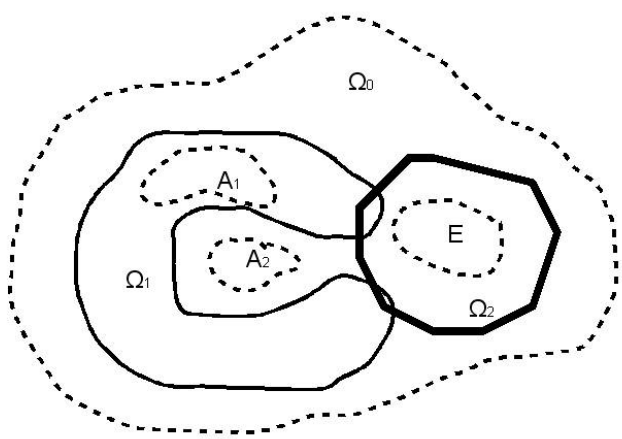
Proof. An application of Theorem 9 shows that the set of points

$$\{x : U^{\chi_{\Omega_1}}(x) = U^{\chi_{\Omega_2}}(x)\} \cap [(\Omega_1 \setminus \Omega_2) \cup (\Omega_2 \setminus \Omega_1)]$$

has Lebesgue measure zero. Also, we can apply Kato's inequality, also known as Grishin's lemma (see [4, 5]), to $U^{\chi_{\Omega_i}} - U\eta$ $(i = 1, 2)$ to see that $\eta \geq \lambda|_{\Omega_1\cap\Omega_2}$. Hence η is of the form $\eta|_D + \eta_1$, where D is an open set, $\eta \geq \lambda|_D$ and η_1 is singular with respect to Lebesgue measure. Since $B_\eta^\lambda = \lambda|_{\Omega_0} + \eta|_{\Omega_0^c}$ we see that $D \subset \Omega_0$. It follows that $\eta|_{\Omega_0^c} = \eta_1|_{\Omega_0^c} = 0$, since $B_\eta^\lambda \leq \lambda$. □

It is easy to see that $\lambda(\Omega_1) = \lambda(\Omega_2) = \lambda(\Omega_0)$ and, by construction, we have $U^{\chi_{\Omega_0}} \leq U^{\chi_{\Omega_i}}$ $(i = 1, 2)$. Further, $\partial\Omega_i \subset \overline{\Omega}_0$ $(i = 1, 2)$, because if there were an open ball B in $\mathbb{R}^N \setminus \overline{\Omega}_0$ with center in $\partial\Omega_i$, then we would arrive at the impossible situation that $U^{\chi_{\Omega_i}} - U^{\chi_{\Omega_0}}$ is non-negative and superharmonic on B and vanishes on $B \setminus \Omega_i$ but not on all of B.

Let $\mu = \lambda|_{\Omega_2} + \lambda|_E$, where $E = \Omega_2 \setminus \Omega_0$. Also let $D = \Omega_0 \cup E = \Omega_0 \cup \Omega_2$. The next step in the proof is to show that $B_\mu^\lambda = \lambda|_D$ and hence $D \subset \Omega(\mu)$. From this we will be able to obtain a contradiction if $\Omega_1 \neq \Omega_2$ by an application of the moving plane method. What follows is a sketch of the remaining proof; full details may be found in [6].



On the set $A_2 = \mathbb{R}^N \setminus \overline{(\Omega_0 \cup \Omega_1 \cup \Omega_2)} = \mathbb{R}^N \setminus \overline{(D \cup \Omega_1)}$ we have by assumption $U\eta = U^{\chi_{\Omega_0}} = U^{\chi_{\Omega_1}} = U^{\chi_{\Omega_2}}$, so $U\mu = U^{\chi_D}$ here. But on the set $A_1 = \Omega_1 \setminus \overline{\Omega}_0 =$

$\Omega_1 \setminus \overline{D}$ it is also clear that $U\eta = U^{\chi_{\Omega_2}}$, since $-\Delta U^{\chi_{\Omega_1}} = a_N\lambda$ on A_1. Hence $U\mu = U^{\chi_D}$ on $\mathbb{R}^N \setminus (\partial\Omega_1 \cup \overline{D})$. It follows by continuity that $U\mu = U^{\chi_D}$ on $\mathbb{R}^N \setminus \overline{D}^\circ$, and we also have $U\mu \geq U^{\chi_D}$ everywhere. Hence $\Omega(\mu) \subset \overline{D}^\circ$. But $\lambda(\Omega(\mu)) = \lambda(D)$, so $D \subset \Omega(\mu)$, and these sets differ by at most a Lebesgue null set. Thus $B_\mu^\lambda = \lambda|_D$. Since $\lambda(\partial\Omega_2) = 0$ and the set $\Omega(\mu) \setminus (E \cup \partial\Omega_2)$ is open, the latter is a subset of Ω_0 by construction (it differs from $D \setminus E = \Omega_0$ by at most a Lebesgue null set). Therefore $D \subset \Omega(\mu) \subset D \cup \partial\Omega_2$.

According to the moving plane method it is not possible for $\Omega(\mu)$ to have any holes outside the set $\overline{\Omega}_2$ (like A_1 or A_2 in the figure). Hence $\Omega_1 \subset \Omega(\mu)$, since $\partial\Omega_1 \subset \overline{\Omega}_0$, so $\Omega_1 \subset \Omega_0$, and using the fact that Ω_1 and Ω_0 have equal Lebesgue measure, and are both saturated with respect to Lebesgue measure, it follows that they are in fact identical.

But this implies that $U^{\chi_{\Omega_1}} \leq U^{\chi_{\Omega_2}}$ everywhere. Since $U^{\chi_{\Omega_2}} - U^{\chi_{\Omega_1}}$ is superharmonic on $\mathbb{R}^N \setminus \overline{\Omega}_1$ (which is connected) and attains the value 0 it follows that $U^{\chi_{\Omega_1}} = U^{\chi_{\Omega_2}}$ on this set. So $\Omega_2 \subset \Omega_1$, and since they are both solid with equal Lebesgue measure it follows that they are also identical.

References

[1] D. H. Armitage and S. J. Gardiner, *Classical Potential Theory*, Springer, London, 2001.

[2] J. Bliedtner and W. Hansen, *Potential theory. An analytic and probabilistic approach to balayage.* Springer, Berlin, 1986.

[3] J. Bourgain, "On the Hausdorff dimension of harmonic measure in higher dimension", *Invent. Math.* 87 (1987), 477–483.

[4] H. Brezis and A.C. Ponce, "Kato's inequality when Δu is a measure", *C.R. Acad. Sci. Paris, Ser. I* 338 (2004), 599-604.

[5] B. Fuglede, "Some properties of the Riesz charge associated with a $\delta-$subharmonic function", *Potential Analysis* 1 (1992), 355-371.

[6] S. J. Gardiner and T. Sjödin, "Convexity and the exterior inverse problem of potential theory", *Proc. Amer. Math. Soc.* 136 (2008), 1699-1703.

[7] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, Berlin, 2001.

[8] B. Gustafsson and M. Sakai, "Properties of some balayage operators, with applications to quadrature domains and moving boundary problems", *Nonlinear Anal.* 22 (1994), 1221–1245.

[9] W. Hansen and H. Hueber, "Singularity of harmonic measure for sub-Laplacians", *Bull. Sci. Math. (2)*, 112 (1988), 53–64.

[10] B. K. Øksendal, "Null sets for measures orthogonal to $R(X)$", *Amer. J. Math.* 94 (1972), 331–342.

[11] M. Sakai, *Quadrature Domains*, Lecture Notes in Math. 934, Springer, Berlin, 1982.

School of Mathematical Sciences
University College Dublin
Belfield, Dublin 4, Ireland.
E-mail address: `stephen.gardiner@ucd.ie`

Department of Mathematics
Linköping University
SE-581 83 Linköping, Sweden.
E-mail address: `tosjo@mai.liu.se`