Potential Theory

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Abstract. We give a summary of classical potential theory.

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Contents

1. About the Document

This is a summary of classical potential theory. It only contains standard material that may be found in a number of textbooks. The ones it is mainly based on are [3], [1], [4] and to some extent also [2]. One difference is that we rely more on the language of distributions in some parts, and assume that this theory, as-well as basic measure and integration theory are well known by the reader. Especially the section about the PWB-method relies heavily on the corresponding material in [3], and the last two chapters about the fine topology and the Martin boundary on [1], and to some extent also on [3].

I have not had as much time to spend on checking this appendix for errors as I would have wanted, so the reader is warned that it is not as reliable as for instance those books mentioned above.

2. Preliminaries

2.1. Notation

We let $\mathbb{R}$ denote the real numbers, and $\mathbb{N}$ the natural numbers. By $\mathbb{R}^N$ we denote $N-$dimensional Euclidean space ($N \geq 2$), and we also put

$$\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{\infty\},$$

$$\overline{\mathbb{R}}^N = \mathbb{R}^N \cup \{\infty\}.$$

For $\Omega \subset \mathbb{R}^N$ we let $\partial\Omega$ denote the boundary of Ω w.r.t $\overline{\mathbb{R}}^N$.

We use $\nearrow$ to denote non-decreasing convergence:

- for $\overline{\mathbb{R}}-$valued functions we mean this pointwise in the usual order of $\overline{\mathbb{R}}$.
- for sets we mean this w.r.t. set inclusion.

In the same way we use $\searrow$ for non-increasing convergence. Also, we denote by $E_n \nearrow E$ (c.c.) that $E_n \nearrow E$ and $\overline{E}_n \subset E$ for every n, where the closure is (as it usually is unless otherwise is stated or obvious from the context) w.r.t $\overline{\mathbb{R}}^N$ (or sometimes w.r.t. some other topological space, for instance another compactification of an open set in $\mathbb{R}^N$, that should be clear from the context).

A word of warning is that $C(A)$ will be used both to denote continuous real-valued functions on A, and the Green-capacity of A. But since the former is a function space and the latter a number there should be no risk of confusion. We will also use the following radial mollifying kernel on $\mathbb{R}$:

$$\eta(x) := \begin{cases} C\exp(1/(|x|^2-1)) & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$

where C is chosen so that $\int \eta(|x|)dm = 1$ on $\mathbb{R}^N$ (so C depends on N, but η is regarded as a function on $\mathbb{R}$), where m is Lebesgue-measure. Then we put on $\mathbb{R}^N$

$$\eta_\varepsilon(x) := \frac{1}{\varepsilon^N}\eta\left(\frac{|x|}{\varepsilon}\right).$$

Also for $\Omega \subset \mathbb{R}^N$ open, we let Ω_ε denote the subset of Ω where the points are at-least a distance ε away from $\partial\Omega$.

For topological spaces S, T we let

$$C(S,T) := \{f : S \to T : f \text{ is continuous.}\}.$$

When $T = \mathbb{R}$ we also write

$$C(S) := C(S,\mathbb{R}).$$

The bounded real-valued functions on S will be denoted by

$$B(S).$$

We also use superpositioning in the sense that

$$CB(S) := C(S) \cap B(S),$$

for instance. On $B(S)$ we define the supremum norm $||\cdot||_s$ by

$$||f||_s := \sup_{x \in S} |f(x)|.$$

If S is a locally compact Hausdorff space, then

$$C_c(S)$$

denotes the set of continuous real-valued functions with compact support in S.

Measures will play a fundamental role in this paper, so we recall some basic terminology from measure theory. Suppose S is a locally compact Hausdorff space. We will by G_δ denote the sets in S which are countable intersections of open sets, and F_σ those which are countable unions of closed sets. By K_σ we will denote those sets in S which are countable unions of compact sets in G_δ. The Baire $\sigma-$algebra is defined as the $\sigma-$algebra generated by K_σ, or equivalently, the smallest $\sigma-$algebra w.r.t. which all continuous real valued functions with compact support are measurable. The $\sigma-$algebra generated by all open sets is called the Borel $\sigma-$algebra, and it by definition contains the Baire $\sigma-$algebra. If S is metrizable, then the Baire and Borel $\sigma-$algebras are the same.

If μ is a measure defined on a $\sigma-$algebra Σ in S containing the Baire (Borel) sets, then μ is called a Baire (Borel) measure. A Borel measure μ is called inner regular if for every measurable set A we have

$$\mu(A) = \sup\{\mu(K) : K \subset A,\ K \text{ closed.}\},$$

and outer regular if

$$\mu(A) = \inf\{\mu(O) : A \subset O,\ O \text{ open.}\}.$$

We will sometimes use the notation $\ll$ to denote absolute continuity, and $\perp$ to denote mutual singularity of measures. A Radon measure on S is a Borel measure which is finite on the compact sets.

Definition 2.1. *For any measure μ, by a carrier for μ is meant a measurable set B such that $\mu(B^c) = 0$. If μ is a Baire measure on some locally compact Hausdorff space S then we define*

$$\text{supp}(\mu) := \left\{ x \in S : \forall f \in C_c(S), f \geq 0, f(x) > 0 \Rightarrow \int f d\mu > 0 \right\}.$$

Note that $\text{supp}(\mu)$ is always closed, but not necessarily $\mu-$measurable.

2.2. Calculus

We will need some basic integration and distribution theory. The notation here is more or less standard now, so we don't make any comments about it now. One remark however regarding measures is that we will by support of a measure μ (written $\text{supp}(\mu)$) mean the closed support, and by measure we mean unless otherwise stated a Radon measure on $\mathbb{R}^N$. By a carrier for a measure μ we simply mean a measurable set where the complement has measure zero. Also we denote Lebesgue measure by m and surface measure by σ.

Theorem 2.2 (Divergence theorem). *Suppose Ω is a bounded domain in $\mathbb{R}^N$ ($N \geq 2$) and $\partial\Omega$ is C^1. If $u \in C^1(\overline{\Omega})$, then*

$$\int_\Omega \frac{\partial u}{\partial x_i} dm = \int_{\partial\Omega} u \cdot n_i d\sigma \quad (i = 1, \ldots, N).$$

Here $\hat{n} = (n_1, \ldots, n_N)$ is the outward pointing unit normal.

Corollary 2.3 (Green's formulas). *Let $u, v \in C^2(\Omega)$ where Ω is as above, then:*

$$(a) \quad \int_\Omega \Delta u dm = \int_{\partial\Omega} \frac{\partial u}{\partial \hat{n}} d\sigma$$

$$(b) \quad \int_\Omega \nabla u \cdot \nabla v dm = -\int_\Omega u \Delta v dm + \int_{\partial\Omega} u \frac{\partial v}{\partial \hat{n}} d\sigma$$

$$(c) \quad \int_\Omega (u\Delta v - v\Delta u)\, dm = \int_\Omega \left(u\frac{\partial v}{\partial \hat{n}} - v\frac{\partial u}{\partial \hat{n}} \right) d\sigma.$$

2.3. Topology

Let X be a Hausdorff-space, and $A \subset X$. We call $y \in X$ a limit point of A if

$$y \in \overline{A \setminus \{y\}} \quad (\text{ closure in } X).$$

If $f : A \to \overline{\mathbb{R}}$, and $y \in X$ is a limit point of A, then we define

$$(*) \ \limsup_{x \to y, x \in A} f(x) := \inf_{U \in N_y} \left(\sup_{x \in (U \cap A) \setminus \{y\}} f(x) \right),$$

where N_y is the set of open neighborhoods of y. (It is easy to see that we may replace N_y with any neighborhood base around y in the definition.) We call f as above upper semicontinuous (u.s.c.) on A if for each $a \in \mathbb{R}$, the set

$$\{x \in A : f(x) < a\}$$

is relatively open in A. If $-f$ is u.s.c. f is said to be lower semicontinuous (l.s.c.).

Lemma 2.4. *If $f : A \to \overline{R}$, then f is u.s.c. if and only if*

$$(**)\ f(y) \geq \limsup_{x \to y, x \in A} f(x),$$

for every limit point y of A, $y \in A$.

Proof. If f is u.s.c. and $y \in A$ is a limit point of A:
1. If $f(y) = +\infty$ then $(**)$ holds trivially.
2. If $f(y) = -\infty$, then for each $n \in \mathbb{N}$ the sets

$$A_n = \{x \in A : f(x) < -n\}$$

are open in A, and contains y, so

$$\inf_{U \in N_y} \left(\sup_{x \in (U \cap A) \setminus \{y\}} f(x) \right) \leq \inf_{n \in \mathbb{N}} \left(\sup_{x \in A_n \setminus \{y\}} f(x) \right) = -\infty.$$

3. $-\infty < f(y) < \infty$. For every $\varepsilon > 0$ the set

$$\{x \in A : f(x) < f(y) + \varepsilon\}$$

is open and contains y, so this case follows in the same way as the case above.

Conversely, suppose $(**)$ holds. If

$$A_a := \{x \in A : f(x) \geq a\},$$

suppose $y \in A$ is a limit point of A_a, then every neighborhood of U of y intersects A_a, so since

$$f(y) \geq \limsup_{x \to y, x \in A} f(x)$$

we obviously have $f(y) \geq a$ as-well. Hence A_a is closed (relative to A), and this proves the lemma. □

• It is obvious that if A is compact, and $f : A \to [-\infty, \infty)$ is u.s.c. then f is bounded from above.

• It is also obvious that if $\{f_i\}_{i \in I}$ is any family of u.s.c. functions, then $f = \inf_{i \in I} f_i$ is u.s.c. because $(f < c) = \bigcup_{i \in I} (f_i < c)$.

Lemma 2.5. *If X is a metric space, $A \subset X$, and $f : A \to [-\infty, \infty)$ is bounded from above. Then f is u.s.c. if and only if*

$$f(x) = \inf\{g(x) : g \in C(A), g \geq f \text{ on } A\} \quad \forall x \in A.$$

Proof. The if part is clear. Let

$$C_f := \{g \in C(A) : g \geq f \text{ on } A\},$$

then $C_f \neq \emptyset$ by assumption, and $l := \inf C_f$ is u.s.c. Also, given $a \in A$, $\varepsilon > 0$, the set

$$A_\varepsilon^a = \{x \in A : f(x) < f(a) + \varepsilon\}$$

is open and contains a. Let $B(a, r) \subset B(a, 2r) \subset\subset A_\varepsilon^a$, and put

$$g_\varepsilon^a = \begin{cases} \sup_{x \in A} f(x) & x \in A \setminus B(a, 2r) \\ f(a) + \varepsilon & x \in \overline{B(a, r)} \end{cases},$$

and extend it to all of A s.t. $g_\varepsilon^a \geq f(a) + \varepsilon$ on $B(a, 2r) \setminus \overline{B(a,r)}$ in a continuous way. Then obviously $g_\varepsilon^a \in C_f$, so $l = f$. □

Theorem 2.6. *Suppose X is a second countable Hausdorff-space. For any family $\{u_i\}_{i\in I}$ of u.s.c. functions on X, put*

$$u^I := \inf_{i\in I} u_i.$$

Then there is a countable set $J \subset I$ such that $u^J = u^I$.

Proof. Since X is second countable, and since

$$(u^I < c) = \bigcup_{i\in I} (u_i < c)$$

there is a countable set $J \subset I$ such that

$$(u^I < c) = \bigcup_{j\in J} (u_j < c)$$

for each rational $c \in \mathbb{R}$. But then this equality holds for any $c \in \mathbb{R}$ since

$$(u^I < c) = \bigcup_{d<c, d \text{ rational}} (u^I < d),$$

holds trivially, and in the same way for u^J.
Obviously $u^J \geq u^I$, and

$$\begin{aligned}(u^J < c) = (u^I < c)\, \forall c \in \mathbb{R} \quad &\Rightarrow \quad (u^J \leq c) = (u^I \leq c)\, \forall c \in \mathbb{R} \\ &\Rightarrow \quad (u^J = c) = (u^I = c)\, \forall c \in \mathbb{R},\end{aligned}$$

so $u^J = u^I$. □

For any function $f : A \to \overline{\mathbb{R}}$ we define the u.s.c. regularization of f by:

$$\check{f}(y) := \max\left\{ f(y), \limsup_{x\to y, x\in A} f(y) \right\} = \inf_{U\in N_y} \left(\sup_{x\in A\cap U} f(x) \right).$$

Theorem 2.7 (Choquet's lemma). *Let $\{f_i\}_{i\in I}$ be a family of functions from X to $\overline{\mathbb{R}}$, where X is a second countable Hausdorff-space, and let $f := \sup_{i\in I} f_i$. Then there is a countable set $J \subset I$ such that*

$$\check{g} = \check{f},$$

where

$$g = \sup_{j\in J} f_j.$$

Proof. Let (B_n) be a sequence of open sets such that $\{B_n : n \in \mathbb{N}\}$ is a base for the topology on X, and also such that each element occurs infinitely often. For each n, choose $x_n \in B_n$ such that

$$f(x_n) > \sup\{f(x) : x \in B_n\} - n^{-1},$$

and then choose $i_n \in I$ such that

$$f_{i_n}(x_n) \geq f(x_n) - n^{-1}.$$

Let $J := \{i_n : n \in \mathbb{N}\}$, and g be as in the statement. Then

$$\sup\{g(x) : x \in B_n\} \geq \sup\{f(x) : x \in B_n\} - 2n^{-1} \quad (n \in \mathbb{N}).$$

Since each B_n occurs infinitely often, we have

$$\check{g}(x) = \inf_{n:x\in B_n} (\sup\{g(y) : y \in B_n\}) \geq \inf_{n:x\in B_n} (\sup\{f(y) : y \in B_n\}) = \check{f}(x),$$

so $\check{g} \geq \check{f}$. But the reverse holds trivially, so $\check{g} = \check{f}$. □

2.4. The Compactification Lattice

Let $\Omega \subset \mathbb{R}^N$ be open. A couple $(\partial\Omega, \tau)$, where $\partial\Omega$ is a set with $\partial\Omega \cap \Omega = \emptyset$, and τ is a Hausdorff topology on $\Omega \cup \partial\Omega$ such that

1) $\Omega \cup \partial\Omega$ is compact w.r.t. τ,
2) Ω is dense in $\Omega \cup \partial\Omega$,
3) the topology induced on Ω by τ is the Euclidean topology,

will be said to compactify Ω. The space $\Omega \cup \partial\Omega$ with the toplogy τ is called a compactification of Ω. Usually we will not specify τ, but it should be clear from the context what it is. We will denote the set of boundaries compactifying Ω by

$$K(\Omega).$$

Let us start by proving that Ω is automatically open in $\Omega \cup \partial\Omega$. To do this, let $B \subset\subset \Omega$ be an (Euclidean) open ball in Ω. Then by compactness the Euclidean closure and the τ closure of B coincide, and we will denote it $\overline{B}$. If $O \in \tau$ and $O \cap \Omega = B$, then we have $O = B$, because if there was an $x \in \partial\Omega$ such that $x \in O$, then it must belong to $\overline{B}$, since Ω is dense in $\Omega \cup \partial\Omega$. This proves the statement.

We will use $\approxeq$ to denote homeomorphism between topological spaces. A fundamental concept for us will be the natural order that $K(\Omega)$ carries. It is defined as follows: For two elements $\partial^1\Omega, \partial^2\Omega$ we define

$$\partial^1\Omega \prec \partial^2\Omega$$

to mean that there is a continuous mapping

$$P : \Omega \cup \partial^2\Omega \to \Omega \cup \partial^1\Omega$$

such that

$$P(x) = x \text{ for every } x \in \Omega.$$

(This order is rather between compactifications, but it is for notational convenience that we let $K(\Omega)$ denote the boundaries instead of the actual compactifications. We will often also say that P goes from $\partial^2\Omega$ to $\partial^1\Omega$ and it is to be understood that this actually means that it is also defined as the identity on Ω.) Note that both $\partial^1\Omega \prec \partial^2\Omega$ and $\partial^2\Omega \prec \partial^1\Omega$ if and only if $\Omega\cup\partial^1\Omega \approxeq \Omega\cup\partial^2\Omega$. We will usually consider homeomorphic compactifications as identical.

The most important tool for studying the space $K(\Omega)$ is the method of constructing compactifications by making embeddings into product-spaces. The idea goes back to Tychonoff, but the theorem on existence and uniqueness of so called $Q-$compactifications given below is due to Constantinescu and Cornea.

Definition 2.8. *For $Q \subset C(\Omega, \overline{\mathbb{R}})$ we say that a compactification $\Omega \cup \partial\Omega$ is a $Q-$compactification of Ω if*

$$1) \quad \forall f \in Q \ \exists \hat{f} \in C(\Omega \cup \partial\Omega, \overline{\mathbb{R}}) \text{ such that } \hat{f}\lfloor\Omega = f,$$

$$2) \quad \text{the functions } \{\hat{f} : f \in Q\} \text{ separates the points of } \partial\Omega.$$

We must first realize that each compactification is a $Q-$compactification for suitable Q. In fact, if we have a compactification $\Omega \cup \partial\Omega$ of Ω, and let $Q \subset CB(\Omega)$ be dense (with respect to supremum norm) in the set

$$\{f\lfloor\Omega : f \in C(\Omega \cup \partial\Omega)\},$$

then $\Omega \cup \partial\Omega$ is a $Q-$compactification, because it is well known that the functions in $C(\Omega \cup \partial\Omega)$ separates the points of $\partial\Omega$, since it is a compact Hausdorff space. (The reason that we do not just take the whole of $C(\Omega \cup \partial\Omega)$ above will be seen when we characterize the metrizable compactifications.)

Now we give the existence and uniqueness theorem. Also we may remark that any element in $C_c(\Omega)$ extends as zero on every compactification, so we may always add these to Q without changing anything.

Theorem 2.9. *For $Q \subset C(\Omega, \overline{\mathbb{R}})$ there is a $Q-$compactification of Ω, and it is unique up to homeomorphism. If we denote this boundary by $\partial^Q\Omega$ and $Q_1 \subset Q$, then $\partial^{Q_1}\Omega \prec \partial^Q\Omega$.*

Proof. We start by proving existence and uniqueness.

Let $Q' = Q \cup C_c(\Omega)$ (and in the same way for Q_i later), and $I_f := [-\infty, \infty]$ for every f. Look at

$$\psi : \Omega \to \Pi_{f \in Q'} I_f$$

defined by

$$\psi(x) := \{f(x)\}_{f \in Q'}.$$

We give $\Pi_{f \in Q'} I_f$ the product topology, which makes it into a compact Hausdorff space, and we let π_f denote the projection onto the f :th coordinate. Since $\pi_f \circ \psi = f$ for each f it follows that ψ is continuous, and since Q' separates the points on Ω it is injective as-well. Let us put

$$R := \psi(\Omega).$$

Let $x \in B \subset\subset \Omega$, where B is open, and choose $f \in C_c(\Omega)$ with $\mathrm{supp}(f) \subset B$ and $f(x) \neq 0$. If we put

$$V := \{y \in \overline{R} : \pi_f(y) \neq 0\},$$

then V is open, and since $(V \setminus \psi(\overline{B})) \cap R = \emptyset$ we must have that $V \setminus \psi(\overline{B}) = \emptyset$, because by compactness of $\psi(\overline{B})$ the set in question is relatively open in $\overline{R}$, and since R is dense in this space it must either intersect R or be empty. Hence

$V \subset \psi(\overline{B}) \subset R$, and since $f = 0$ on ∂B we see that we get $V \subset \psi(B)$. This proves that ψ is an open mapping, and together with continuity and bijectivity of $\psi : \Omega \to R$ we see that it is a homeomorphism. We now identify Ω with R and let $\partial^Q \Omega := \overline{R} \setminus R$, where the closure still is with respect to $\Pi_{f \in Q'} I_f$. We recall that this identifies f with π_f so it is clear that this is a $Q-$compactification of Ω.

We now prove uniqueness up to homeomorphism, and assume that $\Omega \cup \partial\Omega$ is another $Q-$compactification of Ω. Let us now define

$$\phi : \Omega \cup \partial\Omega \to \Pi_{f \in Q'} I_f,$$

by

$$\phi(x) := \{f(x)\}_{f \in Q'}$$

where we now consider the functions in Q' as extended to $\Omega \cup \partial\Omega$. By the same argument as above ϕ is continuous and injective, and we see that ϕ is a homeomorphism of $\Omega \cup \partial\Omega$ onto $\phi(\Omega \cup \partial\Omega)$ by compactness. But we also have $\phi\lfloor\Omega = \psi$, so this proves uniqueness up to homeomorphism.

To prove the last part, we note that this is an immediate consequence of the fact that the mapping between $\Pi_{f \in Q'} I_f$ to $\Pi_{f \in Q'_1} I_f$ defined by

$$\{x_f\}_{f \in Q'} \to \{x_f\}_{f \in Q'_1}$$

is continuous. □

The theorem above is a very convenient tool for introducing the lattice structure on $K(\Omega)$. For a family

$$\{\partial^i \Omega\}_{i \in I} \subset K(\Omega)$$

we first put

$$Q_i := \{f\lfloor\Omega : f \in C(\Omega \cup \partial^i \Omega)\},$$

and then note that we have already proved that

$$\Omega \cup \partial^i \Omega \approxeq \Omega \cup \partial^{Q_i} \Omega.$$

To intoduce the least upper bound with respect to $\prec$ of the family we put

$$Q := \bigcup_{i \in I} Q_i.$$

It is easy to see that $\partial^Q \Omega$ is the least upper bound in the order $\prec$, because any boundary strictly larger then each $\partial^i \Omega$ must have by definition that each element in Q has a continuous extension to the boundary. We denote the least upper bound by

$$\curlyvee_{i \in I} \partial^i \Omega := \partial^Q \Omega.$$

If we instead look at

$$Q := \bigcap_{i \in I} Q_i,$$

then it is equally easy to see that $\partial^Q \Omega$ is the greatest lower bound of the family w.r.t. $\prec$, and we denote it

$$\curlywedge_{i \in I} \partial^i \Omega := \partial^Q \Omega.$$

(The least upper bound and greatest lower bound are only defined up to homeomorphism, and the above construction gives a canonical representative.) Usually when we have for instance two boundaries $\partial^1\Omega$ and $\partial^2\Omega$ we write $\partial^1\Omega \curlyvee \partial^2\Omega$ for the least upper bound and similarly for the greatest lower bound. Let us just remark that from the proof of the theorem above it is easy to see that if $P_i : \partial^1\Omega \curlyvee \partial^2\Omega \to \partial^i\Omega$, denotes the projections, then for any set $B \subset \partial^1\Omega \curlyvee \partial^2\Omega$, we have

$$B = P_1^{-1}(P_1(B)) \cap P_2^{-1}(P_2(B)).$$

This formula could, at-least for metric compactifications, be used in the next section to prove that if the two original boundaries are $h-$resolutive (to be defined later), then so is their least upper bound, but we will see that there is a more efficient method.

We will end this section by characterizing the metrizable compactifications.

Theorem 2.10. *For a compactification $\Omega \cup \partial\Omega$ of Ω the following are equivalent:*

a) $\Omega \cup \partial\Omega$ *is second countable,*

b) $C(\Omega \cup \partial\Omega)$ *is second countable,*

c) $\Omega \cup \partial\Omega$ *is metrizable.*

Proof. *a)* $\Rightarrow$ *b)* : If $\Omega \cup \partial\Omega$ is second countable, then we choose a countable base $\{B_n\}_{n=1}^{\infty}$ for the topology on $\Omega \cup \partial\Omega$. If we for each pair B_n, B_m such that $\overline{B_n} \cap \overline{B_m} = \emptyset$ choose a function $f_{n,m} \in C(\Omega \cup \partial\Omega)$ which is 1 on B_n and 0 on B_m (which is possible by Tietze's extension theorem) we see that there is a countable subset $A \subset C(\Omega \cup \partial\Omega)$ which contains the constant function 1 and separates the points of $\Omega \cup \partial\Omega$. Now let Q denote the set of all functions of the form

$$\prod_{i=1}^{N} \left(\sum_{j=1}^{M_N} a_j^i f_j^i \right),$$

where the a_j^i are rational and f_j^i in A. Then we see that Q is countable, and $\overline{Q}$ is a closed algebra of continuous functions which contains the constant functions and separates the points on $\Omega \cup \partial\Omega$. Hence $\overline{Q} = C(\Omega \cup \partial\Omega)$ by the Stone-Weierstrass theorem. This proves the first part, because we have proved that $C(\Omega\cup\partial\Omega)$ contains a countable dense subset, and since $C(\Omega \cup \partial\Omega)$ is a metric space this is equivalent to second countability.

b) $\Rightarrow$ *c)* : Let $Q' \subset C(\Omega \cup \partial\Omega)$ be countable and dense in $C(\Omega \cup \partial\Omega)$, and define

$$Q := \{f\lfloor\Omega : f \in Q'\}.$$

As we have seen earlier we know that

$$\Omega \cup \partial\Omega \approxeq \Omega \cup \partial^Q\Omega.$$

And since $\partial^Q\Omega$ is defined by embedding Ω in $\prod_{f\in Q} I_f$ and then taking the closure in this space, and this product space is metrizable for Q countable, it follows that $\Omega \cup \partial^Q\Omega$, and hence also $\Omega \cup \partial\Omega$, is metrizable.

$c) \Rightarrow a)$: This is immediate from the fact that Ω contains a countable dense subset, and a metric space is as we stated above second countable if and only if it contains a countable dense subset. $\square$

In particular we note that if both $C(\Omega \cup \partial^1\Omega)$ and $C(\Omega \cup \partial^2\Omega)$ are second countable, then the same is true of both their union and intersection (seen as restricted to Ω). So both $\partial^1\Omega \curlyvee \partial^2\Omega$ and $\partial^1\Omega \curlywedge \partial^2\Omega$ are metrizable if this is true for $\partial^1\Omega$ and $\partial^2\Omega$.

3. Harmonic, sub- and superharmonic functions

Let $\Omega \subset \mathbb{R}^N$ be open. A function $u : \Omega \to (-\infty, \infty]$ such that u is l.s.c., not identically ∞ on any component of Ω and for each $B(a,r) \subset\subset \Omega$ we have

$$u(a) \geq \frac{1}{\sigma(\partial B(a,r))} \int_{\partial B(a,r)} u d\sigma,$$

is called superharmonic in Ω.
u is called subharmonic if $-u$ is superharmonic, and harmonic if it is both sub- and superharmonic.

Theorem 3.1. *If u is continuous on Ω and satisfies*

$$u(a) = \frac{1}{\sigma(\partial B(a,r))} \int_{\partial B(a,r)} u d\sigma \quad \forall B(a,r) \subset\subset \Omega,$$

then $u \in C^\infty(\Omega)$.

Proof. Let η_ε be standard smooth radial mollifiers, and $B(a,\varepsilon) \subset\subset \Omega$, then

$$\begin{aligned} u^\varepsilon(a) : &= \int_\Omega \eta_\varepsilon(a-y)u(y)dm(y) = \frac{1}{\varepsilon^N} \int_{B(0,\varepsilon)} \eta\left(\frac{|a-y|}{\varepsilon}\right) u(y)dm(y) = \\ &= \frac{1}{\varepsilon^N} \int_0^\varepsilon \eta\left(\frac{r}{\varepsilon}\right) \left(\int_{\partial B(a,r)} u d\sigma\right) dr = \\ &= \frac{1}{\varepsilon^N} u(a) \int_0^\varepsilon \eta\left(\frac{r}{\varepsilon}\right) \sigma(\partial B(a,r))dr = u(a), \end{aligned}$$

so in Ω_ε we have $u^\varepsilon \equiv u$. $\square$

In particular, if u is harmonic, it is C^∞.
We now define the Poisson kernel for the ball $B = B(0,1)$:

$$P(x,y) := P_0^1(x,y) := \frac{1}{\sigma(\partial B)} \frac{1-|x|^2}{|y-x|^N}.$$

By a simple Euclidean transformation we get

$$P_a^r(x,y) = \frac{r^{N-2}}{\sigma(\partial B(a,r))} \frac{r^2-|x-a|^2}{|y-x|^N} \text{ for } B(a,r).$$

Theorem 3.2. *Suppose $f \in L^1(\partial B)$, then*

$$u := P[f] = \int_{\partial B} f(y)P(\cdot, y)d\sigma(y)$$

is harmonic in B and satisfies:

$$\liminf_{x\to y, x\in\partial B} f(x) \leq \liminf_{x\to y, x\in B} u(x) \leq \limsup_{x\to y, x\in B} u(x) \leq \limsup_{x\to y, x\in\partial B} f(x)$$

for every $y \in \partial B$.

Proof. That u is harmonic follows from the fact that $P(\cdot, y)$ is harmonic for each $y \in \partial B$, and Fubini's theorem. Now

$$\limsup_{x\to y, x\in B} \frac{1}{\sigma(\partial B)} \left(\int_{\partial B\cap B(y,\varepsilon)} \frac{1-|x|^2}{|t-x|^N} f(t)d\sigma(t) + \int_{\partial B\setminus B(y,\varepsilon)} \frac{1-|x|^2}{|t-x|^N} f(t)d\sigma(t) \right) =$$
$$= \limsup_{x\to y, x\in B} \frac{1}{\sigma(\partial B)} \int_{\partial B\cap B(y,\varepsilon)} \frac{1-|x|^2}{|t-x|^N} f(t)d\sigma(t) \leq$$
$$\leq \sup_{t\in B(y,\varepsilon\cap\partial B} f(t) \limsup_{x\to y, x\in B} \frac{1}{\sigma(\partial B)} \int_{\partial B\cap B(y,\varepsilon)} \frac{1-|x|^2}{|t-x|^N} d\sigma(t) \leq \sup_{t\in B(y,\varepsilon)\cap\partial B} f(t).$$

Since this holds for every $\varepsilon > 0$, the last inequality follows, and the first follows from this one by replacing f with $-f$ in the last one. ☐

Corollary 3.3. *If $f \in C(\partial B)$, then $u = P[f]$ solves Dirichlet's problem:*

$$\begin{cases} -\Delta u = 0 & \text{in } B \\ u = f & \text{on } \partial B. \end{cases}$$

Theorem 3.4. *If $u : \Omega \to (-\infty, \infty]$ is l.s.c. then the following are equivalent:*

(a) *u is superharmonic*

(b) *$\forall B(a,r) \subset\subset \Omega$ we have $u(a) \geq \dfrac{1}{m(B(a,r))} \displaystyle\int_{B(a,r)} udm$*

(c) *for every compact $K \subset \Omega$, and each $h \in C(K)$ harmonic in $\operatorname{int}(K)$ we have that if $u \geq h$ on ∂K, then $u \geq h$ on K.*

Proof. $(a) \Rightarrow (b)$: Let $B(a,r) \subset\subset \Omega$, then by definition:

$$u(a) \geq \frac{1}{\sigma(\partial B(a,r))} \int_{\partial B(a,r)} ud\sigma,$$

so

$$\frac{1}{m(B(a,r))} \int_{B(a,r)} udm = \frac{1}{m(B(a,r))} \int_0^r \left(\int_{\partial B(a,t)} ud\sigma \right) dt \leq$$
$$\leq \frac{1}{m(B(a,r))} \int_0^r u(a)\sigma(\partial B(a,t))dt = u(a).$$

$(b) \Rightarrow (c)$: Put $v = u - h$ on K, then v is l.s.c. on K, so v takes its minimum somewhere on K, say at a. If $a \in \partial K$ we have $v \geq 0$ on K by assumption. If $a \in \text{int}(K)$, choose $r = d(a, \partial K)$, and notice that $\forall r' < r$ we have

$$v(a) \geq \frac{1}{m(B(a,r))} \int_{B(a,r)} v dm \geq v(a),$$

so $v \equiv v(a)$ in $B(a, r')$, and hence on $B(a, r)$, by the continuity of the integral. So $v(a) = v(x)$ for some $x \in \partial K$ and we are done.

$(c) \Rightarrow (a)$: Let $B(a, r) \subset\subset \Omega$, and choose $f_n \in C(\partial B(a, r))$ with $f_n \nearrow u$ on $\partial B(a, r)$ (by l.s.c.).
Let $h_n := P_a^r[f_n]$, then if c holds we must have $u \geq h_n$ in $B(a, r)$, so

$$u(a) \geq \frac{1}{\sigma(\partial B(a,r))} \int_{\partial B(a,r)} f_n d\sigma \nearrow \frac{1}{\sigma(\partial B(a,r))} \int_{\partial B(a,r)} u d\sigma.$$

□

Remark: *It is clear from the above that if u is superharmonic on an open set Ω, then the set $(u = \infty)$ has Lebesgue-measure zero, because for any ball B in Ω we get that if $m(B \cap (u = \infty)) > 0$, then by the above $u \equiv \infty$ on B, and from this the statement easily follows. Since on any ball B as above, for the same reason, $\sigma((u = \infty) \cap \partial B) = 0$ we have reason to believe that it infact must by much smaller than just have Lebesgue-measure zero, and this question will be addressed later when we introduce the concept of polar sets (which is just such sets as above).*

Theorem 3.5. *(a) If u, v is superharmonic on Ω then so is $\min\{u, v\}$.*
(b) If u is bounded and superharmonic, and $\phi : \mathbb{R} \to \mathbb{R}$ is concave, then $\phi \circ u$ is superharmonic.

Proof. (a): That $\min\{u, v\}$ is l.s.c is obvious, and

$$u(a) \geq \frac{1}{m(B(a,r))} \int_{B(a,r)} u dm \geq \frac{1}{m(B(a,r))} \int_{B(a,r)} \min\{u, v\} dm,$$

and the same way for v gives the result.

(b): This is an immediate consequence of Jensen's inequality and the definition. □

Lemma 3.6. *If u is superharmonic in Ω, then*

$$u(x) = \lim_{r \searrow 0} \frac{1}{\sigma(\partial B(x,r))} \int_{\partial B(x,r)} u d\sigma = \lim_{r \searrow 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} u dm$$

for every $x \in \Omega$ (the existence of the limits are part of the conclusion).

Proof. By l.s.c. we have that given $\varepsilon > 0$ there is a $\delta > 0$ such that $r \leq \delta$ implies that $\overline{B(x,r)} \subset \{y \in \Omega : u(y) > u(x) - \varepsilon\}$. From this we immediately get that

$$u(x) \leq \liminf_{r \searrow 0} \frac{1}{\sigma(\partial B(x,r))} \int_{\partial B(x,r)} u d\sigma = \liminf_{r \searrow 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} u dm.$$

But that

$$u(x) \geq \limsup_{r \searrow 0} \frac{1}{\sigma(\partial B(x,r))} \int_{\partial B(x,r)} u d\sigma = \limsup_{r \searrow 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} u dm$$

follows immediately from the defintion of superharmonicity (and theorem 3.4), and hence the proof is done. □

Theorem 3.7 (Minimum principle). *If u is superharmonic in a domain Ω and u attains its infimum at some point $a \in \Omega$, then u is constant in Ω (the connectedness of Ω is obviously necessary).*

Proof.

$$u(a) \geq \frac{1}{m(B(a,r))} \int_{B(a,r)} u dm \geq u(a),$$

so $u \equiv u(a)$ in $B(a,r)$, and therefore the set

$$\{x \in \Omega : u(x) = u(a)\}$$

is both open and closed in Ω. □

Theorem 3.8. *If $u \in C^2(\Omega)$, then u is superharmonic in Ω if and only if $-\Delta u \geq 0$ in Ω.*

Proof. Assume that $-\Delta u \geq 0$ and let

$$F_a(r) := \frac{1}{\sigma(\partial B(a,r))} \int_{\partial B(a,r)} u d\sigma.$$

Then

$$\begin{aligned} F_a'(r) &= \lim_{h \searrow 0} \frac{F_a(r+h) - F_a(r)}{h} = \\ &= \lim_{h \searrow 0} \frac{1}{h} \left(\frac{1}{\sigma(\partial B(a,r+h))} \int_{\partial B(a,r+h)} u d\sigma - \frac{1}{\sigma(\partial B(a,r))} \int_{\partial B(a,r)} u d\sigma \right) = \\ &= \frac{1}{\sigma(\partial B(a,r))} \int_{\partial B(a,r)} \nabla u \cdot \hat{n} d\sigma = \frac{1}{\sigma(\partial B(a,r))} \int_{B(a,r)} \Delta u dm \leq 0. \end{aligned}$$

Hence

$$F_a(R) - u(a) = \int_0^R F_a'(r) dr \leq 0,$$

and therefore

$$F_a(R) \leq u(a),$$

which proves that u is superharmonic.

In the other direction, the assumption that $F_a(R) \leq u(a)$ implies

$$0 \geq \lim_{r \searrow 0} \frac{1}{m(B(a,r))} \int_{\partial B(a,r)} \frac{\partial u}{\partial r} d\sigma = \lim_{r \searrow 0} \frac{1}{m(B(a,r))} \Delta u dm = \Delta u(a).$$

□

Theorem 3.9 (Liouville). *If u is harmonic on $\mathbb{R}^N$, and $|u| \leq C$ for some constant C, then u is constant.*

Proof. Fix $a \in \mathbb{R}^N$, and take $r > 0$, then

$$\begin{aligned} |u(a) - u(0)| &= \frac{1}{m(B(a,r))} \left| \int_{B(a,r)} udm - \int_{B(0,r)} udm \right| \leq \\ &\leq \frac{1}{m(B(0,r))} \int_{B(a,r) \cap B(0,r)} |u| dm \to 0 \text{ as } r \to \infty. \end{aligned}$$

□

Theorem 3.10 (Minimum principle, second version). *Suppose u is superharmonic and v is subharmonic in Ω. Also assume that for every sequence (x_n) in Ω without limit point in Ω there is a subsequence (which we still denote (x_n)) such that*

$$\liminf_{n \to \infty}((u - v)(x_n)) \geq 0,$$

then $u \geq v$ on Ω

Proof. WLOG $v = 0$, and Ω connected. Suppose that

$$A_\varepsilon := \{x \in \Omega : u(x) < -\varepsilon\} \neq \emptyset$$

for some $\varepsilon > 0$. We can't have $\overline{A_\varepsilon} \subset \Omega$ (where the closure is with respect to $\mathbb{R}^N$ of course), because then u would attain its infimum in some point of Ω, and hence would be constant < 0. So there must be a sequence (x_n) in A_ε without a limit point in Ω, which gives a contradiction. □

Theorem 3.11. *If u is a superharmonic function on Ω, and $r \leq R$, $B(a, R) \subset\subset \Omega$, then*

$$\frac{1}{\sigma(\partial B(a,R))} \int_{\partial B(a,R)} ud\sigma \leq \frac{1}{\sigma(\partial B(a,r))} \int_{\partial B(a,r)} ud\sigma.$$

Proof. Since $P_a^R[u] \leq u$ on $B(a, R)$ it follows that $P_a^r[u] \geq P_a^R[u]$ on $B(a, r)$, so since $P_a^r(a, y) = \frac{1}{\sigma(\partial B(a,r))}$ for every $y \in \partial B(a, r)$ the theorem follows. □

We now introduce some notation:

$$\begin{aligned} H(\Omega) &= \text{harmonic functions on } \Omega \\ U(\Omega) &= \text{superharmonic functions on } \Omega \\ S(\Omega) &= \text{subharmonic functions on } \Omega \\ P(\Omega) &= \text{non-negative functions on } \Omega. \end{aligned}$$

We also use superpositioning: $HP(\Omega) = H(\Omega) \cap P(\Omega)$ for instance.

$$\begin{aligned} \eta_\varepsilon &= \text{smooth radial mollifiers} \\ l_\varepsilon &= \frac{1}{m(B(0,\varepsilon))} \chi_{B(0,\varepsilon)}. \end{aligned}$$

Corollary 3.12. *If* $u \in U(\Omega)$*, then*

$$u * \eta_\varepsilon \nearrow u,\ u * l_\varepsilon \nearrow u \ \text{as}\ \varepsilon \searrow 0.$$

Proof. We prove that $u * \eta_\varepsilon \nearrow u$, the proof for the $l_\varepsilon-$case is similar, and we also only prove it at 0 WLOG.

$$\frac{1}{\varepsilon^N}\int_{B(0,\varepsilon)} \eta_\varepsilon(x)u(x)dm(x) = \frac{1}{\varepsilon^N}\int_{B(0,\varepsilon)} \eta\left(\frac{|x|}{\varepsilon}\right)u(x)dm(x) =$$

$$= \{x := \varepsilon y\} = \int_{B(0,1)} \eta(|y|)u(\varepsilon y)dm(y) = \int_0^1 \eta(r)\left(\int_{\partial B(0,r)} u(\varepsilon y)d\sigma(y)\right)dr =$$

$$= \int_0^1 \eta(r)\sigma(\partial B(0,r))f(\varepsilon r)dr,$$

where

$$\begin{aligned} f(\varepsilon r) &= \frac{1}{\sigma(\partial B(0,r))}\int_{\partial B(0,r)} u(\varepsilon y)d\sigma(y) = \\ &= \{t := \varepsilon y\} = \frac{1}{\sigma(\partial B(0,r))}\int_{\partial B(0,\varepsilon r)} u(t)\frac{1}{\varepsilon^{N-1}}d\sigma(t) = \\ &= \frac{1}{\sigma(\partial B(0,\varepsilon r))}\int_{\partial B(o,\varepsilon r)} u(t)d\sigma(t) \nearrow u(0) \text{ as } \varepsilon \searrow 0. \end{aligned}$$

So by monotone convergence:

$$\int_0^1 \eta(r)\sigma(\partial B(0,r))f(\varepsilon r)dr \nearrow u(0)\int_0^1 \eta(r)\sigma(\partial B(0,r))dr = u(0).$$

□

Definition 3.13. *The function*

$$\Phi(x) := \begin{cases} \frac{-1}{2\pi}\log|x| & N = 2 \\ C_N|x|^{2-N} & N \geq 3 \end{cases}$$

where $C_N = (\sigma(\partial B(0,1))(N-2))^{-1}$*, is called the Newtonian kernel on* $\mathbb{R}^N$*. (Usually in two dimensions it is called the logarithmic kernel, but we will not make this distinction).*

Theorem 3.14. *(a)* Φ *is a fundamental solution to* $-\Delta$*, that is* $-\Delta\Phi = \delta_0$ *in the distribution sense.*

(b) If μ *is a Radon measure with compact support in* $\mathbb{R}^N$*, then if we put*

$$U^\mu = \Phi * \mu,$$

the Newtonian potential of μ*, then* U^μ *and* ∇U^μ *are in* $L^1_{loc}(m)$*, and* $-\Delta U^\mu = \mu$ *in the distribution sense. Since* Φ *is radial we often also write* $\Phi(r)$ *for* r *real, meaning that we substitute* $|x|$ *by* r *in the formula.*

Proof. (b) follows from the fact that Φ and $\nabla\Phi$ are in L^1_{loc} and Fubini's theorem. The second part is a direct consequence of (a).
To prove (a), let $\phi \in C_c^\infty(\mathbb{R}^N)$, then

$$\begin{aligned}
&-\int \Phi(x-y)\Delta\phi(x)dm(x) = -\int \Phi(z)\Delta\phi(y+z)dm(z) = \\
= &-\int_{(|z|<\varepsilon)} \Phi(z)\Delta\phi(y+z)dm(z) - \int_{(|z|\geq\varepsilon)} \Phi(z)\Delta\phi(y+z)dm(z) = \\
= &-\int_{(|z|<\varepsilon)} \Phi(z)\Delta\phi(y+z)dm(z) - \int_{(|z|>\varepsilon)} \phi(y+z)\Delta\Phi(z)dm(z) - \\
- &\int_{(|z|=\varepsilon)} \Phi(z)\frac{\partial\phi}{\partial\hat{n}}(y+z)d\sigma(z) + \int_{(|z|=\varepsilon)} \phi(y+z)\frac{\partial\Phi}{\partial\hat{n}}d\sigma(z).
\end{aligned}$$

We now treat each integral in the last expression one by one:

$$-\int_{(|z|<\varepsilon)} \Phi(z)\Delta\phi(y+z)dm(z) \to 0 \text{ as } \varepsilon \searrow 0, \text{ since } \Phi \in L^1_{loc}.$$

$$\int_{(|z|>\varepsilon)} \phi(y+z)\Delta\Phi(z)dm(z) = 0, \text{ since } \Delta\Phi = 0 \text{ here.}$$

$$\left|\int_{(|z|=\varepsilon)} \Phi(z)\frac{\partial\phi}{\partial\hat{n}}(y+z)d\sigma(z)\right| \leq \sup_{\mathbb{R}^N} |\nabla\phi|\Phi(\varepsilon)\varepsilon^{N-1}\sigma(\partial B(0,1)) \to 0 \text{ as } \varepsilon \searrow 0.$$

$$\int_{(|z|=\varepsilon)} \phi(y+z)\frac{\partial\Phi}{\partial\hat{n}}d\sigma(z) = \left(\frac{d\Phi(r)}{dr}\lfloor r=\varepsilon\right)\int_{(|z|=\varepsilon)} \phi(y+z)d\sigma(z) \to \phi(y) \text{ as } \varepsilon \searrow 0.$$

□

Note that $\operatorname{singsupp}\{\Phi\} = \{0\}$. If $u \in U(\Omega)$, then we know that $u \geq u^\varepsilon = u * \eta_\varepsilon$ on Ω_ε and $u^\varepsilon \nearrow u$ on Ω. Also, for any $B(a,r) \subset\subset \Omega$ we have

$$\begin{aligned}
&\frac{1}{m(B(a,r))}\int_{B(a,r)} u^\varepsilon(x)dm(x) = \\
= \;&\frac{1}{m(B(a,r))}\int_{B(a,r)} \left(\int_{B(y,\varepsilon)} \eta_\varepsilon(y)u(x-y)dm(y)\right) dm(x) = \\
= \;&\int_{B(y,\varepsilon)} \left(\frac{1}{m(B(a,r))}\int_{B(a,r)} u(x-y)dm(x)\right)\eta_\varepsilon(y)dm(y) \leq \\
\leq \;&\int_{B(y,\varepsilon)} \eta_\varepsilon(y)u(a-y)dm(y) = u^\varepsilon(a),
\end{aligned}$$

so u^ε is superharmonic in Ω_ε. This implies that $-\Delta u^\varepsilon \geq 0$, so $-\Delta u \geq 0$ in the distribution sense.

Conversely, suppose $u \in \mathcal{D}'(\Omega)$ has $-\Delta u \geq 0$ in the distribution sense. Let $\Omega' \subset\subset \Omega$, and let $\mu' := -\Delta u\lfloor\Omega'$. We know that μ' is a Radon-measure, and if we look at

$$v := u - U^{\mu'} \text{ on } \Omega',$$

then $-\Delta v = 0$ here. Hence v has a representative as a harmonic function in $C^\infty(\Omega')$, and therefore, with $u = U^{\mu'} + v$ this gives a representation of u as a l.s.c. function which is superharmonic on Ω'. Since Ω' was arbitrary it is easy to see that the same holds for Ω (since any such representation given on two different domains must coincide on their intersection). That is $u \in \mathcal{D}'(\Omega)$ has a representative as a l.s.c. function in $U(\Omega)$ if and only if $-\Delta u \geq 0$ in the distribution sense.

Note that for $\mu \geq 0$ U^μ is l.s.c. by Fatou's lemma, and superharmonic by Fubini's theorem, because Φ is (this was used above). Also note that the statement $u \geq h$ on $\partial K \Rightarrow u \geq h$ on $\mathrm{int}(K)$ for h harmonic, as was one of the ways to characterize superharmonicity, is actually a local statement, so being superharmonic is a local property.

Theorem 3.15 (Harnack's inequality). *If $\Omega \subset \mathbb{R}^N$ is open, and $K \subset \Omega$ is compact, then there is a constant $C = C(K)$ such that for every $h \in HP(\Omega)$ we have for every component ω of Ω:*

$$\sup_{K\cap\omega} h \leq C \inf_{K\cap\omega} h.$$

Proof. WLOG Ω is assumed to be connected. Let $r = \frac{1}{4}d(K, \partial\Omega)$. For $x, y \in K$ with $y \in B(x, r)$ we have

$$\begin{aligned} h(x) &= \frac{1}{m(B(x,2r))}\int_{B(x,2r)} hdm \geq \frac{1}{m(B(x,2r))}\int_{B(y,r)} hdm = \\ &= \frac{m(B(x,r))}{m(B(x,2r))}\frac{1}{m(B(y,r))}\int_{B(y,r)} hdm = \frac{1}{2^N}h(y). \end{aligned}$$

Note that if $y \in B(x, r)$, then $x \in B(y, r)$, so we get $h(y) \geq \frac{1}{2^N}h(x)$, and this gives that if $y_1, y_2 \in B(x, r)$, then

$$h(y_1) \geq \frac{1}{2^{2N}}h(y_2).$$

Now we simply cover K by a finite number, say k, such balls $B_1, \ldots, B_k$ with $B_i \cap B_{i+1} \neq \emptyset$ to conclude that

$$h(x) \leq \frac{1}{2^{2Nk}}h(y) \quad \forall x, y \in K.$$

□

Theorem 3.16 (Harnack's convergence theorem). *Let $\{h_i\}_{i\in I} \subset HP(\Omega)$, where $\Omega \subset \mathbb{R}^N$ is an open domain. Suppose there is a point $a \in \Omega$ such that*

$$\sup_{i\in I} h_i(a) < \infty,$$

then the family is locally uniformly bounded and equicontinuous on Ω. (So by the Ascoli-Arzela theorem it is a compact subset in $C(\Omega)$ w.r.t. uniform convergence on compact subsets of Ω.)

Proof. That the family is locally uniformly bounded is a direct consequence of Harnack's inequality. Now, if $B(a,r) \subset B(a,2r) \subset\subset \Omega$, then

$$h_i(x) = \int_{\partial B(a,2r)} h_i(y) P_a^{2r}(x,y) d\sigma(y),$$

and therefore

$$|h_i(x_1) - h_i(x_2)| \leq M \int_{\partial B(a,2r)} |P_a^{2r}(x_1,y) - P_a^{2r}(x_2,y)| d\sigma(y),$$

where M is an upper bound for the family on $\partial B(a,2r)$. It is therefore easy to get equicontinuity on $B(a,r)$ from the above estimate, and a simple covering argument gives it on any compact $K \subset \Omega$. □

We remark that the Poisson kernel is real-analytic, so each harmonic function is real-analytic. We also note that a real-analytic function vanishing on a set of positive Lebesgue-measure on a domain Ω of its definition vanishes identically on Ω, and hence:

- If $h \in H(\Omega)$, $\Omega \subset \mathbb{R}^N$ is a domain, and $m(\{x \in \Omega : h(x) = 0\}) > 0$, then $h \equiv 0$ on Ω.

Theorem 3.17. *Suppose $\{u_a\}_{a \in A} \subset U(\Omega)$ is an upward-directed net of superharmonic functions, then*

$$u := \sup_{a \in A} u_a$$

is either superharmonic or $\equiv \infty$ on each component of Ω.

Proof. Suppose Ω is connected WLOG. Since u is l.s.c. we know that there is a countable subset $A' \subset A$ such that

$$u = \sup_{a \in A'} u_a.$$

So this reduces the problem to the one where we have an increasing sequence $u_n \in U(\Omega)$, and $u_n \nearrow u$, and we then get:

$$u(a) \geq u_n(a) \geq \frac{1}{m(B(a,r))} \int_{B(a,r)} u_n dm \nearrow \frac{1}{m(B(a,r))} \int_{B(a,r)} u dm.$$

□

Theorem 3.18 (Weak fundamental convergence theorem). *Let $\{u_a\}_{a \in A} \subset U(\Omega)$ be locally bounded from below on Ω. Let*

$$u := \inf_{a \in A} u_a,$$

then

(1) $\hat{u} = u \;\; m - a.e.$

(2) $\hat{u}$ *is superharmonic in* Ω.

(3) $\hat{u}(y) = \liminf_{x \to y} u(x) \;\; \forall y \in \Omega.$

(Here $\hat{u}$ denotes the l.s.c. regularization of u).

Proof. We know by Choquet's lemma that there is a countable subset $A' \subset A$ such that

$$v = \inf_{a \in A'} u_a$$

has

$$\hat{v} = \hat{u}.$$

Now for every $a \in A'$ and $x \in \Omega$ we have ($B(x,r) \subset\subset \Omega$):

$$\frac{1}{m(B(x,r))} \int_{B(x,r)} v dm \leq \frac{1}{m(B(x,r))} \int_{B(x,r)} u_a dm \leq u_a(x),$$

so

$$F(x) := v * l_r(x) \leq v(x)$$

and $F \in C(\Omega_r)$, so

$$\hat{v} * l_r(x) \leq v * l_r(x) \leq \hat{v}(x) \leq v(x) \text{ on } \Omega_r.$$

Let $r \searrow 0$ and use that

$$\lim_{r \searrow 0} f * l_r(x) = f(x) \;\; m - \text{a.e. for } f \in L^1_{loc}(m)$$

to get

$$\hat{v} \leq v \leq \hat{v} \leq v \;\; m - \text{a.e.}$$

which immediately gives

$$\hat{v} = \hat{u} \leq u \leq v = \hat{v} = \hat{u} \;\; m - \text{a.e.}$$

To conclude the proof we note that by definition

$$\hat{u}(y) = \min\{u(y), \liminf_{x \to y, x \in \Omega} u(x)\}.$$

But since

$$u_a(y) = \liminf_{x \to y, x \in \Omega} u_a(x) \quad \forall y \in \Omega$$

by the super-mean-value inequality and l.s.c. we get

$$\liminf_{x \to y, x \in \Omega} u(x) = \liminf_{x \to y, x \in \Omega} \left(\inf_{a \in A} u_a(x) \right) \leq$$

$$\leq \inf_{a \in A} \left(\liminf_{x \to y, x \in \Omega} u_a(x) \right) = \inf_{a \in A} u_a(y) = u(y).$$

So

$$\min\{u(y), \liminf_{x \to y, x \in \Omega} u(x)\} = \liminf_{x \to y, x \in \Omega} u(x).$$

□

Definition 3.19. *A nonempty family F of superharmonic functions on an open set $\Omega \subset \mathbb{R}^N$ is called saturated if*

$$(a) \quad \min\{u, v\} \in F \ \ \forall u, v \in F,$$

$$(b) \quad \widetilde{u} := \begin{cases} P_a^r[u] & \text{on } B(a,r) \\ u(x) & \text{on } \Omega \setminus B(a,r) \end{cases} \quad \text{belongs to } F \ \ \forall u \in F.$$

Note that $\widetilde{u} \in U(\Omega)$ for $u \in U(\Omega)$, which may be seen by the fact that for $y \in \partial B(a,r)$ we have

$$\liminf_{x \to y, x \in B(a,r)} P_a^r[u](x) \geq \liminf_{x \to y, x \in \partial B(a,r)} u(x) \geq u(y),$$

which gives l.s.c. Also $\widetilde{u} \leq u$ on $B(a,r)$ gives the super-mean-value inequality on $\partial B(a,r)$ trivially.

Theorem 3.20. *If F is a saturated family of superharmonic functions on the open set $\Omega \subset \mathbb{R}^N$, and*

$$h := \inf F \ \textit{(pointwise infimum)},$$

then for each component of Ω either $h \equiv -\infty$, or h is harmonic.

Proof. WLOG assume Ω connected, and $h \neq -\infty$. Since for each $u \in F$ $\widetilde{u} \leq u$ ($B(a,r) \subset\subset \Omega$ fixed) we get that

$$h = \inf\{\widetilde{u} : u \in F\},$$

so h must be harmonic on $B(a,r)$, by Harnack's theorem, because we have that the family is uniformly bounded from below on $B(a,r)$. Since $B(a,r)$ was arbitrary the theorem follows. □

Corollary 3.21. *If $s \in S(\Omega)$, and*

$$F := \{u \in U(\Omega) : u \geq s\} \neq \emptyset,$$

then F is saturated and contains a smallest element, $h = \inf F$ which is harmonic. We call h the smallest superharmonic majorant to s.

4. Green functions and Potentials

An open set $\Omega \subset \mathbb{R}^N$ is called Greenian if the Newtonian kernel has a subharmonic minorant on Ω, that is for every $y \in \Omega$ the function $\Phi(\cdot - y)$ has a subharmonic minorant. Trivially any open set in $\mathbb{R}^N$ for $N \geq 3$ and any bounded open subset of R^2 is Greenian. For $N = 2$, suppose u is a subharmonic minorant to $\Phi(\cdot - z)$ ($z \in \Omega \subset \mathbb{R}^2$), and $y \in \Omega$. If $x \in \Omega \setminus B(z,r)$, then (we skip the $\frac{1}{2\pi}$ constant on the logarithmic kernel in the next calculations for simplicity)

$$\Phi(x-y) - u(x) \geq \Phi(x-y) - \Phi(x-z) \geq \log\left(\frac{|z-x|}{|y-z|+|z-x|}\right) \geq$$

$$\geq \quad -\frac{|y-z|}{|z-x|} \geq \frac{-|y-z|}{r}.$$

Since $\Phi(\cdot - y) - u$ is bounded from below on $B(z,r)$ it follows that it is enough that there is a subharmonic minorant at one point in each component of Ω. Also, if $\overline{\Omega}^c$ is nonempty, say contains an open ball $B = B(y,r)$, then we may choose nonnegative constants c_1, c_2 with $c_1 \leq 1$ such that

$$c_1 \log|\cdot - y| - c_2 \leq \log|\cdot - z| \text{ on } \partial B,$$

for some $z \in \Omega$ fixed. So by the minimum principle the left hand side is a harmonic minorant to $\log|\cdot - z|$ on Ω, so it is Greenian.

We now let $h_\Omega(\cdot, y)$ be the largest subharmonic minorant to $\Phi(\cdot - y)$, if Ω is Greenian, and put

$$G_\Omega(x,y) = \Phi(x-y) - h_\Omega(x,y),$$

called the Greens function of Ω.
For $N \geq 3$ it is obvious that the Newtonian kernel gives the Green's-function for $\mathbb{R}^N$:

$$G_{\mathbb{R}^N}(x,y) = \Phi(x-y).$$

Theorem 4.1. *For the ball $B(a,r)$ Green's function G is given by:*
(1) If $N = 2$:

$$G(x,y) = \begin{cases} -\frac{1}{2\pi}\log\frac{r|x-y|}{|y-a||x-y^*|} & \textit{for } y \in B(a,r)\setminus\{a\}, x \in B(a,r)\setminus\{a,y\} \\ -\frac{1}{2\pi}\log\frac{|x-y|}{r} & \textit{for } y = a, x \in B(a,r)\setminus\{y\} \\ \infty & \textit{for } x = y \end{cases}$$

(2) If $N \geq 3$:

$$G(x,y) = \begin{cases} C_N(|x-y|^{2-N} - \frac{|y-a|^{2-N}}{r^{2-N}}|x-y^*|^{2-N}) & \textit{for } y \in B(a,r)\setminus\{a\}, \\ & x \in B(a,r)\setminus\{a,y\} \\ C_N(|x-y|^{2-N} - r^{2-N}) & \textit{for } y = a, x \in B(a,r)\setminus\{y\} \\ \infty & \textit{for } x = y \end{cases}$$

where $y^* = a + \frac{y-a}{|y-a|^2}$.

Proof. ($N \geq 3$, the proof for $N = 2$ is similar).
First note that $-C_N\frac{|y-a|^{2-N}}{r^{2-N}}|x-y^*|^{2-N}$ ($y \neq a$, y fixed) and $-C_N r^{2-N}$ are harmonic. Also note that $G(x,y) = G(y,x)$ if $x = y$ or $y = a$.

We may without loss assume that $a = 0$ and $r = 1$ (just study $t := \frac{x-a}{r}$ and $s := \frac{y-a}{r}$ instead). Then

$$G(x,y) = C_N(|x-y|^{2-N} - |y|^{2-N}|x-y^*|^{2-N})$$

(if $y \neq 0$, $\{x\} \cap \{a,y\} = \emptyset$). We only need to prove that

$$|y|^{2-N}|x-y^*|^{2-N} = |x|^{2-N}|y-x^*|^{2-N}$$

to see that $G(x,y) = G(y,x)$. We have

$$\begin{aligned} |y|^{2-N}|x-y^*|^{2-N} &= ||y|x - \frac{y}{|y|}|^{2-N} = ||y||x|e_x - \frac{|y|e_y}{|y|}|^{2-N} \\ &= ||x|y - \frac{x}{|x|}|^{2-N} = |x|^{2-N}|y-x^*|^{2-N}. \end{aligned}$$

Since

$$C_N(|x-y|^{2-N} - |y-a|^{2-N}|\frac{x-a|}{r} - r(y-a)^*|^{2-N})$$

has a continuous extension to $\partial B(a,r)$ when y is fixed inside $B(a,r)$ the theorem follows. (Note that $C_N(|x-y|^{2-N} - \frac{|y-a|^{2-N}}{r^{2-N}}|x-y^*|^{2-N}) = 0$ when $y \in \partial B(a,r)$). □

Theorem 4.2. *Let $\Omega \subset \mathbb{R}^N$ be Greenian, then:*

$$\begin{aligned} &(a) \quad G_\Omega(x,y) = G_\Omega(y,x) \quad \forall x,y \in \Omega \\ &(b) \quad G_\Omega : \Omega \times \Omega \to \overline{\mathbb{R}} \textit{ is continuous.} \\ &(c) \quad G_\Omega \textit{ is superharmonic on } \Omega \times \Omega. \\ &(d) \quad \textit{If } \Omega' \subset \Omega, \textit{ then } \Omega' \textit{ is Greenian, and} \\ &\qquad G_{\Omega'} \le G_\Omega \textit{ on } \Omega' \times \Omega'. \\ &(e) \quad \textit{If } \Omega_n \nearrow \Omega, \textit{ then } G_{\Omega_n} \nearrow G_\Omega. \end{aligned}$$

Proof. (b),(d) and (e) are trivial, and once we have proved (a), (c) follows since we then get in the distribution sense ($G = G_\Omega$)

$$\begin{aligned} \langle -\Delta G(x,y), \phi(x,y)\rangle &= \langle -\Delta_x G(x,y), \phi(x,y)\rangle + \langle -\Delta_y G(x,y), \phi(x,y)\rangle \\ &= -\int \left(\int G(x,y)\Delta_x \phi(x,y) dm(x) \right) dm(y) \\ &\quad - \int \left(\int G(x,y)\Delta_y \phi(x,y) dm(y) \right) dm(x) = \\ &= \int \phi(x,y) dm(y) + \int \phi(x,y) dm(x) \ge 0 \quad \forall \phi \in C_c^\infty P(\Omega). \end{aligned}$$

To prove (a), let $B_0 \subset\subset \Omega$ be a ball, and let (B_n) $(n \ge 1)$ be a sequence of balls compactly contained in Ω such that each element occurs infinitely often, and $\Omega = \bigcup_{n=0}^\infty B_n$ with $\partial B_n \cap \overline{B}_0 = \emptyset$ for every $n \ge 1$. Let $B_n = B(z_n, r_n)$, and let P_n be the Poisson-kernel of the ball B_n. Now define

$$u_1(x,y) := \begin{cases} P_1[\Phi(\cdot - y)](x) & x \in B_1 \\ \Phi(x-y) & x \in \Omega \setminus B_1 \end{cases}$$

and inductively

$$u_{k+1}(x,y) := \begin{cases} P_{k+1}[u_k(\cdot, y)](x) & x \in B_{k+1} \\ u_k(x,y) & x \in \Omega \setminus B_{k+1} \end{cases}$$

Then each u_n is continuous on $\Omega\times\overline{B}_0$, and $u_n(x,\cdot)\in H(B_0)$ for every $x\in\Omega$. Also each $u_n(\cdot,y)$ is superharmonic on Ω, with $u_n(\cdot,y)\leq\Phi(\cdot,y)$, and form a decreasing sequence. Also $h_\Omega(\cdot,y)\leq u_n(\cdot,y)$ for each n, and if we let

$$u(\cdot,y):=\lim_{n\to\infty}u_n(\cdot,y)$$

for each $y\in B_0$, then $u(\cdot,y)$ is harmonic , because if we on B_n had $u>P_n[u(\cdot,y)]$, then for some $x\in B_n$, $K\in\mathbb{N}$ we would have $u(x)>P_n[u(\cdot,y)]+\varepsilon$, and $u_k(x)\geq P_n[u_k(\cdot,y)]+\varepsilon\ \forall k\geq K$, since $u_k\searrow u$ and $P_n[u_k(\cdot,y)]\searrow P_n[u(\cdot,y)]$, which gives a contradiction. We conclude that $u(x,y)=h_\Omega(x,y)$. Since on B_0 we have, say

$$u_1(x,y)=\int_{\partial B_1}\Phi(y-t)P_1(x,t)d\sigma(t),$$

so that $u_1(x,\cdot)$ is harmonic on B_0 for $x\in\Omega$ fixed. The same applies to each u_k inductively, so $h_\Omega(x,\cdot)\in H(\Omega)$ (because of the arbitrary nature of B_0). Since

$$h_\Omega(x,y)\leq\Phi(x-y)=\Phi(y-x)$$

we get

$$h_\Omega(x,y)\leq h_\Omega(y,x)$$

and this clearly finishes the proof. □

Let μ be a Radon-measure on a Greenian open set Ω. Then we define

$$G_\Omega^\mu(x):=\int G_\Omega(x,y)d\mu(y)\text{ on }\Omega,$$

and if $G_\Omega^\mu\not\equiv\infty$ on every component of Ω we say that G_Ω^μ is a potential. Also note that if $G_\Omega^\mu(x_0)<\infty$ and $x_0\in K\subset\omega$, where K is a compact subset of the component ω of Ω, then we may choose $K\subset D\subset\subset\omega$, and by Harnack's inequality there is a real constant C such that

$$G_\Omega(x,y)\leq CG_\Omega(x_0,y)\quad\forall x\in K,y\in D^c.$$

Hence

$$\begin{aligned}&\int\left(\int_K G_\Omega(x,y)dm(x)\right)d\mu(y)=\\&=\int_D\left(\int_K G_\Omega(x,y)dm(x)\right)d\mu(y)+\int_{D^c}\left(\int_K G_\Omega(x,y)dm(x)\right)d\mu(y)\leq\\&\leq\int_D\left(\int_K G_\Omega(x,y)dm(x)\right)d\mu(y)+Cm(K)G_\Omega^\mu(x_0)<\infty.\end{aligned}$$

So in particular we have that if G_Ω^μ is a potential, then it is locally integrable on Ω.

Theorem 4.3. *Suppose G_Ω^μ is a potential, then*

(a) $-\Delta G_\Omega^\mu = \mu$ *in the distribution sense (so G_Ω^μ is superharmonic on Ω).*

(b) *If $\Omega' \subset\subset \Omega$ is open and $\mu(\Omega') = 0$, then* G_Ω^μ *is harmonic on Ω'.*

(c) *The greatest subharmonic minorant to G_Ω^μ on Ω is* 0.

Proof. (a):

$$\begin{aligned} & \langle -\Delta G_\Omega^\mu(x), \phi(x)\rangle = \langle G_\Omega^\mu(x), -\Delta\phi(x)\rangle = \\ = & \int G_\Omega^\mu(x)(-\Delta\phi(x))dm(x) = \int \left(\int G_\Omega(x,y)d\mu(y)\right)(-\Delta\phi(x))dm(x) = \\ = & \{\text{Fubini}\} = \int \phi(y)d\mu(y) = \langle \mu, \phi\rangle \quad \forall \phi \in C_c^\infty(\Omega). \end{aligned}$$

(b): follows immediately from (a).

(c): If μ has compact support in Ω, this is obvious since then

$$G_\Omega^\mu \leq cG_\Omega(\cdot, y)$$

for some c, y fixed (we assume WLOG that Ω is connected). To treat the general case, let $K_n \nearrow \Omega$ be compact sets increasing to Ω, then

$$G_\Omega^\mu = G_\Omega^{\mu\lfloor K_n} + G_\Omega^{\mu\lfloor \Omega\setminus K_n}.$$

Now the largest subharmonic minorant to a sum of superharmonic functions is the sum of the largest subharmonic minorants to each of them, because suppose $h \leq u$, $h_1 \leq u_1$ and $h_2 \leq u_2$ are "maximal" (where $u = u_1 + u_2$ are the functions in question), then $h - h_1 \leq u_2$, so $h_2 \geq h - h_1$ and hence $h_1 + h_2 \geq h$. But the reverse inequality is obvious. So applied to the above this means that a subharmonic minorant of G_Ω^μ must also be one for $G_\Omega^{\mu\lfloor \Omega\setminus K_n}$ for each n. And since this tends to zero as n tends to infinity the proof is done. □

Remark: *If there is an open set $D \subset\subset \Omega$ such that $G_\Omega^{\mu\lfloor \Omega\setminus D}$ is a potential, then so is G_Ω^μ, because*

$$G_\Omega^\mu = G_\Omega^{\mu\lfloor D} + G_\Omega^{\mu\lfloor \Omega\setminus D},$$

and the first term on the left is finite-valued on $\Omega \setminus D$, and the second one on D.

Theorem 4.4 (Riesz's structure theorem). *Suppose u is superharmonic on Ω, and has a subharmonic minorant on Ω (where Ω is an open subset of $\mathbb{R}^N$). Let $\mu := -\Delta u$ (in the distribution sense). Then μ is a Radon-measure, G_Ω^μ is a potential, and $u - G_\Omega^\mu = h$ is the largest subharmonic minorant to u on Ω (so is harmonic).*

Proof. We already know that μ is a Positive Radon-measure (or rather has such a representation). Now let $\Omega_n \nearrow \Omega$ (c.c.), then with $\mu_n := -\Delta u\lfloor\Omega_n$ we have that $G_\Omega^{\mu_n}$ are potentials, and $G_\Omega^{\mu_n} \nearrow G_\Omega^\mu$ trivially. If h is a harmonic minorant to u, then $u - h \geq 0$ and we claim that $u - h \geq G_\Omega^\mu$ on Ω. On Ω_n we have that $u - G_\Omega^{\mu_n}$ is harmonic, let $h_n := u - G_\Omega^{\mu_n}$. Then we get that $h_n \geq h$ on Ω_n for each n trivially, or equivalently

$$u - h \geq G_\Omega^{\mu_n} \text{ on } \Omega_n \ \forall n.$$

Hence $G_\Omega^{\mu_n}$ does not converge to ∞ identically on any component of Ω, and therefore G_Ω^{μ} is a potential. Since $u - h = G_\Omega^{\mu}$ the largest subharmonic minorant to $u - h$ is 0, and the proof is done. □

Theorem 4.5 (Continuity principle). *Let Ω be Greenian, and u be superharmonic in Ω. Put*

$$\mu := -\Delta u,$$

and

$$E := \operatorname{supp}(\mu) \cap \Omega.$$

Assume that $u\lfloor E$ is continuous at $z \in E$, then u is continuous at z (in the extended sense of-course).

Proof. If $u(z) = \infty$ there is nothing to prove by l.s.c. So assume $u(z) < \infty$. Given $r > 0$ such that $B(z, r) \subset\subset \Omega$ we have that on $B(z, r)$

$$u = U^\eta + h$$

for some positive measure η and harmonic function h. The problem is thus reduced to the case:
Given a Radon-measure μ with compact support, and $U^\mu\lfloor\operatorname{supp}(\mu)$ is continuous at $z \in \operatorname{supp}(\mu)$, then U^μ is continuous at z.

For $r > 0$, let $\mu_r := \mu\lfloor B(z, r)$ and put

$$U^{\mu_r}(z) = \int \Phi(z - y)d\mu_r(y)$$

as usual, and note that we must have $\mu(\{z\}) = 0$ since $u(z) < \infty$. Suppose we can prove that

$$\sup_{x\in B(z,r)} U^{\mu_r}(x) \searrow 0 \text{ as } r \searrow 0.$$

Then, given $\varepsilon > 0$, choose $r > 0$ so small that

$$\sup_{x\in B(z,r)} U^{\mu_r}(x) < \varepsilon/4.$$

Since it is obvious that $U^{\mu-\mu_r}$ is continuous on $B(z, r)$ we may always choose $0 < \delta < r$ such that

$$|U^{\mu-\mu_r}(x) - U^\mu(z)| \leq |U^{\mu-\mu_r}(x) - U^{\mu-\mu_r}(z)| + U^{\mu_r}(x) + U^{\mu_r}(z) < \varepsilon$$

for every $x \in B(z, \delta)$, and we would be done. To prove that

$$\sup_{x\in B(z,r)} U^{\mu_r}(x) \searrow 0 \text{ as } r \searrow 0,$$

we note that if $d(x, a) = d(x, \operatorname{supp}(\mu))$, $a \in \operatorname{supp}(\mu)$, then

$$|a - z| \leq |x - z| + |a - x| \leq 2|x - z|,$$

so for instance if $N \geq 3$ ($N = 2$ is similar) we get

$$\begin{aligned} U^{\mu_r}(x) &= \int_{B(z,r)} \frac{C_N}{|x-y|^{N-2}} d\mu(y) \\ &\leq 2^{N-2} \int_{B(z,r)} \frac{C_N}{|a-y|^{N-2}} d\mu(y) = U^{\mu_r}(a) \cdot 2^{N-2}. \end{aligned}$$

Since $U^{\mu_r}(a)$ converges uniformly to zero as $a \to z$ on $\mathrm{supp}(\mu)$ it follows that U^{μ_r} does so as well, and we are done. □

Corollary 4.6. *If G_Ω^μ is a potential, and such that its restriction to $\Omega \cap \mathrm{supp}(\mu)$ is continuous, then it is continuous on Ω.*

Theorem 4.7 (Domination principle). *Suppose μ is a Radon-measure on a Greenian set Ω, and suppose $G_\Omega^\mu < \infty$ everywhere and*

$$G_\Omega^\mu \leq u \text{ on } \mathrm{supp}(\mu) \cap \Omega$$

where $u \in UP(\Omega)$, then

$$G_\Omega^\mu \leq u \text{ on } \Omega.$$

Proof. Suppose first that μ has compact support in Ω. Let $G_m := \min\{m, G_\Omega\}$, then G_m is superharmonic and continuous, and also

$$G_m^\mu(x) := \int G_m(x,y) d\mu(y) \nearrow G_\Omega^\mu(x)$$

pointwise everywhere on Ω. Now since all functions involved are $\mu - a.e.$ real-valued we may apply Egoroff's theorem and see that there is a compact $K \subset \mathrm{supp}(\mu)$, with $\mu(K^c) < \varepsilon$ and $G_m^\mu \nearrow G_\Omega^\mu$ uniformly on K. Put

$$\mu_1 := \mu \lfloor K,$$

then we get

$$\begin{aligned} |G_\Omega^\mu - G_m^\mu| = \int (G_\Omega(\cdot,y) - G_m(\cdot,y)) d\mu(y) \geq \\ \geq \int (G_\Omega(\cdot,y) - G_m(\cdot,y)) d\mu_1(y), \end{aligned}$$

so

$$G_m^{\mu_1} \nearrow G_\Omega^{\mu_1}$$

uniformly on K. Hence $G_\Omega^{\mu_1}$ is continuous on K, and by the previous result it is therefore continuous on Ω.

On $\Omega \setminus K$ we have that $G_\Omega^{\mu_1}$ is harmonic, and

$$\liminf_{x \to y, x \in \Omega \setminus K} (u - G_\Omega^{\mu_1}) \geq 0 \quad \forall y \in \partial(\Omega \setminus K),$$

so $G_\Omega^{\mu_1} \leq u$ everywhere on Ω by the minimum principle. To conclude the proof for μ with compact support we note that for any $x \notin \mathrm{supp}(\mu)$ we have that

$$d := \sup_{y \in \mathrm{supp}(\mu)} G_\Omega(x,y) < \infty,$$

and

$$G_\Omega^\mu(x) \le G_\Omega^{\mu_1}(x) + \varepsilon d,$$

so $G_\Omega \le u$ everywhere since ε is arbitrarily small.

To treat the general case we let $K_n \nearrow \Omega$ be compacts, and put $\mu_n = \mu\lfloor K_n$, then the above case gives that $G_\Omega^{\mu_n} \le u$ everywhere on Ω for each n. And since $G_\Omega^{\mu_n} \nearrow G_\Omega^\mu$ pointwise the theorem is proved. □

Remark: *As in the proof above, suppose Ω is Greenian, and G_Ω^μ is a finite-valued potential. Then there is an increasing sequence of compacts $K_n \subset \operatorname{supp}(\mu)$ such that $G_\Omega^{\mu_n}$ is continuous and increasing pointwise everywhere to G_Ω^μ, where $\mu_n := \mu\lfloor K_n$. That is, a finite-valued potential may always in a good way be approximated by continuous ones.*

This may also be applied as follows:
suppose we only know that $G_\Omega^\mu \le u$ on some Borel-carrier B for μ in the statement of the Domination principle. Then we may still take such a sequence $K_n \subset B$ such that $G_\Omega^{\mu_n}$ is continuous and increasing to G_Ω^μ, where we used the same notation as above. So we can replace $\operatorname{supp}(\mu) \cap \Omega$ in the Domination principle by any Borel-carrier for μ.

5. Polar sets, Reduced functions and Capacity

5.1. Polar sets

Definition 5.1. *A set $A \subset \mathbb{R}^N$ is called polar if there is a function $u \in U(\Omega)$ such that $u = \infty$ on A, where $A \subset \Omega$.*

Theorem 5.2. *(a) Let $E \subset \Omega$ be polar, $z \in \Omega \setminus E$ and Ω is Greenian. Then there is a potential G_Ω^μ which is ∞ on E, $G_\Omega^\mu(z) < \infty$, and $\mu(\Omega) < \infty$.*

(b) $m(E) = 0$ for E polar.

(c) Countable unions of polar sets are polar.

(d) Any polar set is contained in a polar G_δ.

(e) If u is a locally bounded superharmonic function on Ω and $E \subset \Omega$ is polar, then $(-\Delta u)(E) = 0$.

Proof. (a) Let $E \subset \omega$, ω open and $u : \omega \to (-\infty, \infty]$ be superharmonic with $u = \infty$ on E. WLOG assume $\omega \subset \Omega$, $z = 0 \notin \omega$. Let (B_k) be a sequence of open balls compactly contained in ω, and such that $\bigcup_{k=1}^K B_k \nearrow \omega$ as $K \to \infty$.

$$\nu_k(A) := \frac{(-\Delta u)(A \cap B_k)}{(-\Delta u)(B_k) + 1} \text{ for all Borel sets } A.$$

Now we have that $G_\Omega^{\nu_k}$ is ∞ on $E \cap B_k$, and $G_\Omega^{\nu_k}(0) < \infty$, because on B_k we have that $c = (-\Delta u)(B_k) + 1 < \infty$, and so $cG_\Omega^{\nu_k} - u$ is harmonic on B_k, hence bounded. Now let

$$\mu := \sum_{k=1}^{\infty} 2^{-k} \frac{\nu_k}{1 + u_k(0)}.$$

Then $\mu(\Omega) \leq 1$, so G^μ_Ω is a potential, and $G^\mu_\Omega = \infty$ on E, $G^\mu_\Omega(0) < \infty$, which proves (a).

(b) This is obvious, since superharmonic functions are locally integrable.

(c) If the union is a subset of a Greenian set this is immediate from (a). Otherwise the proof needs a slight modification using logarithmic potentials, but we omit the details.

(d)

$$(u = \infty) = \bigcap_{n=1}^{\infty} (u > n),$$

which is a G_δ.

(e) Since E can be assumed to be a G_δ measurability is not an issue. WLOG assume $E \subset \omega \subset\subset \Omega$, and let $\mu_u := -\Delta u$, and ν be a finite measure with compact support in ω such that $G^\nu_\omega = \infty$ on E. Now

$$u - \inf_\omega u \geq G^{\mu_u}_\omega,$$

so

$$\int_\omega G^\nu_\omega d\mu_u = \int_\omega G^{\mu_u}_\omega d\nu \leq \nu(\omega) \sup_\omega (u - \inf_\omega u) < \infty.$$

But $G^\nu_\omega = \infty$ on E now implies that $\mu_u(\omega \cap E) = 0$, so $\mu_u(E) = 0$. □

A property holding except for a polar set is said to hold quasi everywhere (q.e.).

Theorem 5.3 (Domination principle). *Let Ω be Greenian, and let G^μ_Ω be a finite-valued potential on Ω. If $u \in UP(\Omega)$, and*

$$G^\mu_\Omega \leq u \text{ q.e. on some Borel-carrier } B \text{ for } \mu,$$

then $G^\mu_\Omega \leq u$ on Ω.

Proof. We have already proved this statement if we omit the q.e. (that is if we had the inequality everywhere on some Borel-carrier for μ). But let E be a polar G_δ such that the inequality holds on $B \setminus E$. Then by (e) above $\mu(E) = 0$, and hence $B \setminus E$ is also a Borel-carrier for μ, and we are done.

(Alternatively we may choose $v \geq 0$ superharmonic and ∞ on E, and note that our earlier version of the domination principle assures us that $G^\mu_\Omega \leq u + v/n$ on Ω for every positive n, and this is easy to see implies the result as well). □

Remark: *Note that if μ is carried by a polar set in Ω, and $\mu \not\equiv 0$ and G^μ_Ω is a potential, then $B := (G^\mu_\Omega = \infty)$ is a carrier for μ, because if we look at $\eta_n := \mu\lfloor B_n$, where $B_n := (G^\mu_\Omega < n)$ we see that $G^{\eta_n}_\Omega < n$ on a carrier for η_n, so since it is also carried by a polar set, it must be identically zero for every n. Hence $\mu(B^c) \leq \sum \mu(B_n) = 0$, which proves the claim.*

5.2. Reduced functions

If $u \in UP(\Omega)$ and $E \subset \Omega$ (Ω Greenian), then the reduced function of u relative to E in Ω is

$$R_u^E(x) := \inf\{v(x) : v \in UP(\Omega), v \geq u \text{ on } E\}.$$

We also let

$$\hat{R}_u^E$$

denote the l.s.c. regularization of R_u^E. We already know that

• $\hat{R}_u^E$ is superharmonic on Ω, $\hat{R}_u^E = R_u^E$ $m-$a.e. on Ω

and

• $\hat{R}_u^E(y) = \liminf_{x\to y} R_u^E(x) \quad (\forall y \in \Omega)$.

(The above follows directly from the weak version of the Fundamental convergence theorem.) Obviously $u \geq R_u^E \geq \hat{R}_u^E \geq 0$ on Ω, and $u = R_u^E$ on E, $\hat{R}_u^E = R_u^E$ on $\text{int}(E)$. Also

$$\{u\lfloor\Omega \setminus \overline{E} : v \in UP(\Omega), v \geq u \text{ on } E\}$$

is saturated, so $R_u^E = \hat{R}_u^E$ on $\Omega \setminus \overline{E}$ and is harmonic there.

Lemma 5.4. *The following is equivalent (notation from above):*

(a) E is polar.

(b) $\exists u \in UP(\Omega)$ not identically zero on any component of Ω such that $\hat{R}_u^E \equiv 0$.

(c) $\hat{R}_u^E \equiv 0$ for every $u \in UP(\Omega)$.

Proof. Let E be polar, and $v \in UP(\Omega)$, $v = \infty$ on E, and $v(z) < \infty$ where $z \in \Omega \setminus E$ is an arbitrary fixed point. Then $v/n \geq u$ on E for each n, and this leads to that $R_u^E(z) = 0$ for every z as above. Hence

$$R_u^E(x) = \begin{cases} u(x) & x \in E \\ 0 & x \in \Omega \setminus E \end{cases},$$

so $\hat{R}_u^E \equiv 0$, and this proves that (a)⇒(c).

Trivially (c)⇒(b).

Suppose now that (b) holds, and note that we WLOG may assume that Ω is connected and $\overline{E} \subset \Omega$ (since countable unions of polar sets are polar) . By assumption, for some $x_0 \in \Omega$ fixed $R_u^E(x_0) = 0$. For each n we may therefore choose $v_n \in UP(\Omega)$ such that $v_n \geq u$ on E and $v_n(x_0) \leq 2^{-n}$. Let

$$v := \sum_{n=1}^{\infty} v_n,$$

then $v(x_0) \leq 1$ so $v \in UP(\Omega)$. Since $v = \infty$ on E it follows that E is polar, and hence (b)⇒(a). □

Theorem 5.5. *(All subsets below are subsets of a fixed Greenian set Ω, and the functions are positive and superharmonic.)*

(1) $u \leq v$ on $E \Rightarrow R_u^E \leq R_v^E$, $\hat{R}_u^E \leq \hat{R}_v^E$.

(2) $E \subset F \Rightarrow R_u^E \leq R_u^F$, $\hat{R}_u^E \leq \hat{R}_u^F$.

(3) $R_u^E = \hat{R}_u^E$ *on* $\Omega \setminus \overline{E}$*, and is harmonic there.*
(4) F *polar* $\Rightarrow \hat{R}_u^{E\cup F} = \hat{R}_u^E$.
(5) ω *open* $\Rightarrow \hat{R}_u^\omega = R_u^\omega$.
(6) If $u \in C(\omega)$*,* ω *open,* $E \subset \omega$*, then*

$$R_u^E = \inf\{R_u^A : E \subset A, A \textit{ open}\}.$$

(7) If $\overline{E}$ *is a compact subset of* Ω*, and* $u \in UP(\Omega)$*, then* $\hat{R}_u^E$ *is a potential.*
(8) If (K_n) *is an increasing sequence of compacts with* $\omega = \bigcup_{n=1}^\infty K_n$ *open, then* $\hat{R}_u^{K_n} \nearrow R_u^\omega$.
(9) If $u \in UPC(\Omega)$*, and* (K_n) *is a decreasing sequence of compacts,* $K = \bigcap_{n=1}^\infty K_n$*, then* $R_u^{K_n} \searrow R_u^K$.
(10) If E *is relatively closed in* Ω *and non-polar, then there is a bounded continuous potential* G_Ω^μ*, where* $\mu \not\equiv 0$ *and* $\operatorname{supp}(\mu)$ *is a compact subset of* E.

Proof. (1) and (2) are obvious.
(3) follows from the easy fact that

$$F := \{v\lfloor\Omega \setminus \overline{E} : v \in UP(\Omega), v \geq u \text{ on } E\}$$

is a saturated family of functions on $\Omega \setminus \overline{E}$.
(4): Let $w \in UP(\Omega)$, $w = \infty$ on F, then for $v \in UP(\Omega)$, $v \geq u$ on E it follows that $v + w/n \geq u$ on $E \cup F$. Hence from this we get (since we may for any fixed $z \in \Omega \setminus F$ assume $w(z) < \infty$)

$$R_u^E = R_u^{E\cup F} = R_u^{E\cup F} \text{ on } \Omega \setminus F,$$

so

$$\hat{R}_u^E = \hat{R}_u^{E\cup F}.$$

(5): If ω is open then $\hat{R}_u^\omega = R_u^\omega = u$ on ω by construction, and since $\hat{R}_u^\omega$ is superharmonic we are done.
(6): If $v \in UP(\Omega)$ and $v \geq u$ on E, then $v + 1/n > u$ on some open set A which contains E. So

$$v + 1/n \geq \inf\{R_u^A : E \subset A, A \text{ open }\},$$

and the rest follows directly from this.
(7): WLOG assume Ω connected, and let $\overline{E} \subset \omega \subset\subset \omega' \subset\subset \Omega$. Then, since $\hat{R}_u^{\overline{\omega}} \in H(\Omega \setminus \overline{\omega})$ there is a constant c such that $cG(\cdot, a) \geq \hat{R}_u^{\overline{\omega}}$ ($a \in \Omega$ fixed) on $\partial\omega'$. Hence

$$w(x) := \begin{cases} u(x) & x \in \overline{\omega'} \\ cG(x,a) \wedge u(x) & x \in \Omega \setminus \overline{\omega'} \end{cases}$$

has $w \in UP(\Omega)$, and $w = \hat{R}_u^{\overline{\omega}} = u$ on E, so $\hat{R}_u^E \leq w$. But this implies by the maximum principle that any $h \in H(\Omega)$, $h \leq \hat{R}_u^E$ also has $h \leq cG_\Omega(\cdot, a)$, hence $h \leq 0$, which proves the statement.
(8): $\hat{R}_u^{K_n}$ is increasing to some function v that is superharmonic and satisfies $v \leq R_u^\omega$. Since $\hat{R}_u^{K_n} = R_u^{K_n} = u$ $m-$a.e. on K_n for each n it follows that $v = u$

$m-$a.e. on A, so $v = u$ on ω, and hence $v \geq R_u^\omega$ which finishes the proof of (8).
(9): Obviously

$$\inf\{R_u^{K_n} : n \in \mathbb{N}\} \leq \inf\{R_u^A : K \subset A, A \text{ open }\},$$

so this follows from (6).

(10): WLOG assume E is a compact subset of Ω. Now $\hat{R}_1^E$ is non-zero and bounded by 1, so is a potential, and $-\Delta\hat{R}_1^E$ has support in E. The rest now follows from earlier results. (See the remark after the first version of the domination principle.)

□

• Sometimes we need to specify which set we take reductions with respect to, so when needed we will put ${}^\Omega\!R_u^E$ to specify that the reduction is over Ω.

5.3. Capacity

Let $\Omega \subset \mathbb{R}^N$ be a fixed open set, and let $\mathbb{K}$ denote the set of compact subsets of Ω (denoted $\mathbb{K}(\Omega)$ if necessary). Suppose $\phi : \mathbb{K} \to [0, \infty)$, ϕ increasing (i.e. $\phi(K_1) \leq \phi(K_2)$ if $K_1 \subset K_2$). Let

$$\phi_*(E) := \sup\{\phi(K) : K \subset E, K \in \mathbb{K}\} \quad \forall E \subset \Omega,$$

and

$$\phi^*(E) := \inf\{\phi_*(O) : E \subset O \subset \Omega, O \text{ open}\} \quad \forall E \subset \Omega.$$

We also assume that if $E_n \nearrow E$ then $\phi^*(E_n) \nearrow \phi^*(E)$. If all this is fulfilled, then we call ϕ_* the inner $\phi-$capacity, and ϕ^* the outer $\phi-$capacity. A set E such that $\phi_*(E) = \phi^*(E)$ is called ($\phi-$) capacitable. Trivially all open sets are capacitable. For the rest of this section we regard ϕ as above fixed.

Let $\mathbb{N}^\mathbb{N}$ denote the collection of infinite sequences of natural numbers, and $\bigcup_k \mathbb{N}^k$ the collection of all finite sequences of natural numbers. A set A such that there is a map $K : (\bigcup_k \mathbb{N}^k) \to \mathbb{K}$ with

$$A = \bigcup_{(m_n)\in\mathbb{N}^\mathbb{N}} (K(m_1) \cap K(m_1, m_2) \cap K(m_1, m_2, m_3) \cap, \ldots)$$

is called analytic (on Ω). We denote these by $\mathcal{A}(\Omega)$.

Lemma 5.6. *(1) If $(A_n) \subset \mathcal{A}(\Omega)$, then $\bigcap_{n\in\mathbb{N}} A_n$ and $\bigcup_{n\in\mathbb{N}} A_n$ are also in $\mathcal{A}(\Omega)$.*
(2) Every Borel set is analytic.

Proof. (1): Let

$$A_l := \bigcup_{(m_n)\in\mathbb{N}^\mathbb{N}} (K_l(m_1) \cap K_l(m_1, m_2) \cap \ldots)$$

as in the definition. Also let $n \mapsto (a(n), b(n))$ be bijective between $\mathbb{N}$ and $\mathbb{N}^2$, and let

$$K(m_1, \ldots, m_n) := K_{a(m_1)}(b(m_1), m_2, \ldots, m_n),$$

then

$$A_l = \bigcup_{\{(m_n):a(m_1)=l\}} (K(m_1) \cap K(m_1, m_2) \cap \ldots),$$

so

$$\bigcup_l A_l = \bigcup_{(m_n)\in\mathbb{N}^{\mathbb{N}}} (K(m_1) \cap K(m_1, m_2) \cap \ldots)$$

is analytic. Now for $(m_n) \in \mathbb{N}^{\mathbb{N}}$ fixed, let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be defined as $f(l,p) :=$ $(l,p)-$entry in $(=m_p^{(l)})$

$$(*) \begin{array}{ccccc} m_1 & m_2 & m_4 & m_7 & \ldots \\ m_3 & m_5 & m_8 & \ldots & \ldots \\ m_6 & m_9 & \ldots & \ldots & \ldots \\ m_{10} & \ldots & \ldots & \ldots & \ldots \end{array} =: \begin{array}{ccc} m_1^{(1)} & m_2^{(1)} & \ldots \\ m_1^{(2)} & m_2^{(2)} & \ldots \\ m_1^{(3)} & m_2^{(3)} & \ldots \\ m_1^{(4)} & m_2^{(4)} & \ldots \end{array}$$

And define $K(m_1, \ldots, m_n)$ to be the entry in

$$\begin{array}{lll} K_1(f(1,1)) & K_1(f(1,1), f(1,2)) & \ldots \\ K_2(f(2,1)) & K_2(f(2,1), f(2,2)) & \ldots \\ K_3(f(3,1)) & \ldots\ldots\ldots\ldots\ldots & \ldots \end{array}$$

which corresponds to the position of (m_n) in $(*)$. Now

$$x \in \bigcap_l A_l \Leftrightarrow \forall l \in \mathbb{N}\ \exists (m_n^{(l)}) \in \mathbb{N}^{\mathbb{N}}$$

such that $x \in K_l(m_1^{(l)}) \cap K_l(m_1^{(l)}, m_2^{(l)}) \cap \ldots \Leftrightarrow \exists f : \mathbb{N}^2 \to \mathbb{N}$ with f as above such that $x \in \bigcap_{l\in\mathbb{N}}(K_l(f(l,1)) \cap K_l(f(l,1), f(l,2)) \cap \ldots)$.

Hence

$$\bigcap_l A_l = \bigcup_{(m_n)\in\mathbb{N}^{\mathbb{N}}} (K(m_1) \cap K(m_1, m_2) \cap \ldots),$$

where K is as defined above. This finishes the proof of (1).

(2): Trivially $\mathbb{K} \subset \mathcal{A}(\Omega)$, and since $\mathcal{A}(\Omega)$ is closed under countable unions and intersections $\mathcal{A}(\Omega)$ contains all open and closed sets. Let

$$\Phi := \{A \in \mathcal{A}(\Omega) : A^c \in \mathcal{A}(\Omega)\}.$$

Then Φ is closed under countable unions and intersections, since $A = \cup A_n \Rightarrow A^c = \cap A_n^c$, so $\{A_n\}, \{A_n^c\} \subset \mathcal{A}(\Omega) \Rightarrow A, A^c \in \mathcal{A}(\Omega)$. So Φ is a $\sigma-$algebra, and since it contains all open and closed sets it contains all Borel-sets. □

Lemma 5.7. *Suppose*

$$A = \bigcup_{(m_n)\in\mathbb{N}^{\mathbb{N}}} (K(m_1) \cap K(m_1, m_2) \cap \ldots)$$

is analytic and $(k_n) \in \mathbb{N}^{\mathbb{N}}$ *is fixed. If we define*

$$E_l := \bigcup_{\{(m_n)\in\mathbb{N}^{\mathbb{N}} : m_n \le k_n \text{ when } n\le l\}} (K(m_1) \cap K(m_1, m_2) \cap \ldots)\ (l \in \mathbb{N})$$

$$F_l := \bigcup_{\{(m_n)\in\mathbb{N}^l : m_n \le k_n\}} (K(m_1) \cap \ldots \cap K(m_1, \ldots, m_l))\ (l \in \mathbb{N}),$$

and $F := \bigcap_l F_l$, *then*

$$(1)\ E_l \subset A,\ E_l \subset F_l \quad \forall l,\ \textit{and } (E_l) \textit{ is decreasing.}$$

$$(2)\ (F_l) \textit{ is decreasing, each set is compact and } F \subset A.$$

Proof. (1) is obvious and clearly F_l is decreasing. Since each union is finite in the definition of F_l it also follows that they are all compact. If $x \in F$, then for any choice of l there is an $l-$tuple $(m_1^{(l)}, m_2^{(l)}, \ldots, m_l^{(l)})$ such that $m_n^{(l)} \leq k_n$ $\forall n \in \{1, 2, \ldots, l\}$ and

$$x \in K(m_1^{(l)}) \cap K(m_1^{(l)}, m_2^{(l)}) \cap \ldots \cap K(m_1^{(l)}, \ldots, m_l^{(l)}).$$

Since $m_1^{(l)} \in \{1, \ldots, k_1\} \forall l$ we get $\exists m' \in \{1, \ldots, k_n\}$ such that $m_1^{(l)} = m_1'$ for infinitely many l, and similarly $\exists m_2' \in \{1, \ldots, k_2\}$ such that $(m_1^{(l)}, m_2(l)) = (m_1', m_2')$ infinitely often. Hence we obtain a sequence $(m'N)$ such that

$$x \in K(m_!') \cap K(m_1', m_2') \cap \ldots \subset A,$$

so $F \subset A$. □

Theorem 5.8. *If every compact* $K \in \mathbb{K}$ *is* $\phi-$ *capacitable, then so are all analytic sets* $A \in \mathcal{A}(\Omega)$.

Proof. Let

$$A = \bigcup_{(m_n) \in \mathbb{N}^{\mathbb{N}}} (K(m_1) \cap K(m_1, m_2) \cap \ldots)$$

as in the definition of analytic sets. Let $a < \phi^*(A)$. By assumption if we choose k_1 large enough the set E_1 (we use the notation from the previous lemma) satisfies $\phi^*(E_1) > a$ (Because $''E_1(k_1) \nearrow E''$ as $k_1 \nearrow \infty$). Suppose $k_1, k_2, \ldots, k_{n-1}$, has been chosen, such that $\phi^*(E_i) > a$ for every $i = 1, \ldots, n-1$, then we may againchoose k_n such that $\phi^*(E_n) > a$ for the same reason as before.

Now

$$\phi_*(A) \geq \phi_*(F) = \{\text{by assumption}\} = \phi^*(F) = \lim_{n\to\infty} \phi^*(F_n) \geq \lim_{n\to\infty} \phi^*(E_n) \geq a.$$

Above we used that $F_n \searrow F$ for compacts as above implies that $\phi^*(F_n) \searrow \phi^*(F)$. But this holds trivially because for any $\varepsilon > 0$ $\exists O$ open with $F \subset O$ and $\phi^*(F) \geq \phi^*(O) - \varepsilon$, and since $F_n \subset O$ for large n we get the result. □

Greenian capacity

If $K \subset \Omega$ where Ω is Greenian and K is compact, then we know that with $\nu_K := -\Delta \hat{R}_1^K$, then $\hat{R}_1^K = G_\Omega^{\nu_K}$ and also $\text{supp}(\nu_K) \subset \partial K$. We define

$$C(K) := \nu_K(\Omega),$$

called the Green capacity of K with respect to Ω.
Since if $K_1 \subset K_2 \subset \omega \subset\subset \Omega$ gives

$$\begin{aligned} \nu_{K_1}(\Omega) &= \int G_\Omega^{\nu_{\overline{\omega}}} d\nu_{K_1} = \int G_\Omega^{\nu_{K_1}} d\nu_{\overline{\omega}} \leq \\ &\leq \int G_\Omega^{\nu_{K_2}} d\nu_{\overline{\omega}} = \int G_\Omega^{\overline{\omega}} d\nu_{K_2} = \nu_{K_2}(\Omega) \end{aligned}$$

(since $G_\Omega^{\nu_{\overline{\omega}}} = 1$ on ω), so we have

$$C(K_1) \leq C(K_2).$$

We may now therefore define C_*, and C^* as before, called the Green inner and outer capacities. The following theorem shows that it is actually a capacity (i.e. it is continuous w.r.t. increasing convergence) and that also all compact sets are $C-$capacitable, and hence every analytic set is capacitable.

Theorem 5.9. *(a): $C_*(K) = C^*(K)$ for all compact $K \subset \Omega$.*
(b): $E_n \nearrow E \Rightarrow C^*(E_n) \nearrow C^*(E)$. (Since all compacts are capacitable it makes sense to write $C(E)$ for the common value of $C_*(E)$ and $C^*(E)$ for any capacitable set E, since it coincides with the original C on the compacts).

Proof. (a): Let $K \subset \omega_n \subset\subset \Omega$ with $\omega_n \searrow K$. Then

$$\begin{aligned} C_*(K) \leq C^*(K) &\leq C^*(\omega_n) \leq C_*(\overline{\omega}_n) = \\ &= C(\overline{\omega}_n) \searrow C(K) = C_*(K), \end{aligned}$$

because

$$\hat{R}_1^{\overline{\omega}_n} \searrow \hat{R}_1^K.$$

(b): WLOG assume $C^*(E_n) < \infty$ for every n, since otherwise there is nothing to prove. For each n, choose $\omega_n \subset \Omega$ open with $E_n \subset \omega_n$ and

$$C_*(\omega_n) < C^*(E_n) + 2^{-n}\varepsilon,$$

where $\varepsilon > 0$ is fixed. Now

$$C_*(\bigcup_{n=1}^m \omega_n) < C^*(E_m) + (1 - 2^{-m})\varepsilon \quad \forall m \in \mathbb{N},$$

because for $m = 1$ this is by assumption, and if we have it for $m = k$, then

$$C_*(\bigcup_{n=1}^{k+1} \omega_n) + C_*((\bigcup_{n=1}^{k} \omega_n) \cap \omega_{k+1}) \leq C_*(\bigcup_{n=1}^{k} \omega_n) + C_*(\omega_{k+1}),$$

since

$$(*)\ C^*(A \cup B) + C^*(A \cap B) \leq C^*(A) + C^*(B)$$

holds in general. To prove $(*)$ suppose first that A, B are compact, then if $u, v \in UP(\Omega)$ with $u \geq 1$ on A and $v \geq 1$ on B, let

$$w := \hat{R}_1^{A\cup B} + \hat{R}_1^{A\cap B},$$

which is a potential on Ω, and harmonic on $\Omega \setminus (A \cup B)$, then

$$u + v \geq 2 \geq w \text{ on } A \cap B,$$

and on $A \setminus B$ we have $u \geq 1 \geq \hat{R}_1^{A \cup B}$, and $v \geq \hat{R}_1^{A \cap B}$.

So $u + v \geq w$ on $A \setminus B$, and in the same way we get this on $B \setminus A$, so by the domination principle this must hold everywhere on Ω. As before, the claim easily follows by definition for general A, B by approximation with compact sets. (We use $C^*(\omega) = C_*(\omega)$ for open sets, which holds trivially.)

Now

$$C_*(\bigcup_{n=1}^{k} \omega_n) + C_*(\omega_{k+1}) < C^*(E_k) + C^*(E_{k+1}) + (1 - 2^{-k-1})\varepsilon,$$

since

$$E_k \subset (\bigcup_{n=1}^{k} \omega_n) \cap \omega_{k+1}$$

we get the result. Finally

$$\begin{aligned} C^*(E) &= C^*(\bigcup_{n=1}^{\infty} E_n) \leq C_*(\bigcup_{n=1}^{\infty} \omega_n) = \\ &= \lim_{N \to \infty} C_*(\bigcup_{n=1}^{N} \omega_n) \leq \lim_{N \to \infty} C^*(E_N) + \varepsilon, \end{aligned}$$

which proves (b) since ε is arbitrary. Above we used that the statement holds if E_n and E are open, but this is trivial, because given $\varepsilon > 0$ then there is a compact $K \subset E$ such that

$$C^*(E) < C_*(K) + \varepsilon,$$

and since $K \subset E_n$ for large n we get for these n

$$C^*(E_n) \geq C_*(K).$$

□

• The formula

$$C^*(A \cup B) + C^*(A \cap B) \leq C^*(A) + C^*(B) \quad \forall A, B \subset \Omega$$

proved above is worth remembering for its own sake.

Theorem 5.10. *Let $E \subset \Omega$. Then E is polar if and only if $C^*(E) = 0$.*

Note that $C^*(E) = 0 \Rightarrow E$ is capacitable trivially.

Proof. Assume WLOG Ω connected.
If E is polar, and $K \subset E$ is compact, then obviously K is polar, so $\hat{R}_1^K \equiv 0$, and hence $C(K) = 0$. That is $C_*(E) = 0$. But since any polar set is contained in a polar G_δ we may assume that E is a G_δ, and since all Borel-sets are capacitable we get $C^*(E) = 0$.

Conversely, suppose $C^*(E) = 0$. Since if we let $\Omega_n \nearrow \Omega$ (c.c.) and put $E_n = E \cap \Omega_n$, then E is polar if and only if each E_n is polar, and similarly $C^*(E) = 0$ if and only if each $C^*(E_n) = 0$, hence we may WLOG assume $\overline{E} \subset \Omega$. Then given $\varepsilon > 0$ there is an open set $\omega \subset\subset \Omega$ such that $E \subset \omega$, $C^*(\omega) < \varepsilon$. Let $z \in \Omega \setminus \overline{\omega}$ and put $d := d(z, \omega) > 0$. Now for any compact $K \subset \omega$ we have $C^*(K) < \varepsilon$, so

$$\hat{R}_1^K(z) \leq \varepsilon \sup_{y \in \omega} G_\Omega(y, z) \quad \forall K \subset \omega \text{ compact.}$$

Hence

$$\hat{R}_1^E(z) \leq \hat{R}_1^\omega(z) \leq \varepsilon \sup_{y \in \omega} G_\Omega(y, z),$$

and this gives that $\hat{R}_1^E(z) = 0$. By the minimum principle this implies that $\hat{R}_1^E \equiv 0$, and hence E is polar. □

Theorem 5.11 (Fundamental Convergence Theorem). *Let $F \subset U(\Omega)$ be locally uniformly bounded from below, and let $u := \inf F$. Then*

(a) $\hat{u} \in U(\Omega)$.

(b) $\hat{u} = u$ *q.e. on* Ω.

Proof. We already know this with q.e. replaced by $m-$a.e. We know that there is a sequence (u_n) in F such that $\hat{u} = \hat{v}$ where $v := \inf u_n$. Let

$$v_n := \min\{u_1, \ldots, u_n\}.$$

Then $(v_n) \subset U(\Omega)$ is decreasing, with limit v. Note that $\hat{u} \leq u \leq v$, so we only need to prove that $\hat{u} = v$ q.e. on Ω.

It is obviously enough to prove that for each ball $B \subset\subset \Omega$ we have that $v_n \to \hat{u}$ q.e. on B. To do this, let

$$\mu_n := -\Delta v_n \lfloor B,$$

and

$$h_n := v_n - G_B^{\mu_n} \text{ on } B.$$

So $v_n = G_B^{\mu_n} + h_n$ and h_n is harmonic on B. By Harnack's convergence theorem and weak$^*-$compactness of $C(\overline{B})^*$ we may assume WLOG (since v_n is decreasing) that $h_n \to h$ u.c. on B and $\mu_n \rightharpoonup \mu$ where μ has support in $\overline{B}$. Note that if we define $G_B(x, y) = 0$ on $B \times \partial B$ then the functions

$$G_m := \min\{G_B, m\}$$

has $G_m(x, \cdot) \in C(\overline{B})$ for every $x \in B$. So

$$\begin{aligned} \liminf_{n\to\infty} G_B^{\mu_n}(x) &\geq \liminf_{n\to\infty} \int G_m(x, y) d\mu_n(y) = \\ &= \int G_m(x, y) d\mu(y) \nearrow G_B^\mu(x) \text{ as } m \nearrow \infty \ \forall x \in B. \end{aligned}$$

Let

$$E := \left\{ x \in B : \liminf_{n\to\infty} G_B^{\mu_n}(x) > G_B^\mu(x) \right\}.$$

Suppose that there is a compact $K \subset E$ with $C(K) > 0$, then there is a continuous potential G_B^ν on B, where $\nu \neq 0$ and $\operatorname{supp}(\nu) \subset K$. This gives

$$\begin{aligned}\int_K G_B^\mu d\nu &= \int G_B^\nu d\mu = \lim_{n\to\infty} \int G_B^\nu d\mu_n = \lim_{n\to\infty} \int G_B^{\mu_n} d\nu \geq \\ &\geq \int_K \liminf_{n\to\infty} G_B^{\mu_n} d\nu > \int_K G_B^\mu d\nu,\end{aligned}$$

which gives a contradiction. So $v_n \to \hat{u}$ q.e. on B, and the proof is done. □

Corollary 5.12. *If (u_n) is a sequence in $U(\Omega)$ locally uniformly bounded from below and Ω is connected (and Greenian). Let*

$$u := \liminf_{n\to\infty} u_n.$$

Then either $u \equiv \infty$ or $\hat{u} \in U(\Omega)$, $u = \hat{u}$ q.e. on Ω. Also, if $u \geq M$ on Ω, then for any compact $K \subset \Omega$ and $\varepsilon > 0$ there is an n_0 such that $u_n(x) \geq M - \varepsilon$ for all $n \geq n_0$ and $x \in K$.

Proof. Let $v_n := \inf\{u_k : k \geq n\}$. Then $\hat{v}_n \in U(\Omega)$, $\hat{v}_n = v_n$ q.e. on Ω, and $\hat{v}_n \nearrow v$ which is either $\equiv \infty$ or superharmonic on Ω. And $v = u$ q.e. Also $v \leq \hat{u} \leq u$, so $v = \hat{u}$ on Ω. Therefore $\hat{u} = u$ q.e. Now $(M - \hat{v}_n)^+$ is decreasing and u.s.c. with limit 0, so by Dini's theorem the convergence is uniform on K, and this proves the statement. □

Theorem 5.13. *Let Ω be Greenian, and let all reductions below be w.r.t. Ω, all sets are subsets of Ω and all functions in $UP(\Omega)$.*
(1) $\hat{R}_u^E = R_u^E$ q.e. on Ω.
(2) $\hat{R}_u^E = \inf\{v \in UP(\Omega) : v \geq u$ q.e. on $E\}$.
(3) $\hat{R}_u^E = R_u^E$ on $(\Omega \setminus E) \cup \operatorname{int}(E)$.
(4) If $E_n \nearrow E$, then $R_u^{E_n} \nearrow R_u^E$ and $\hat{R}_u^{E_n} \nearrow \hat{R}_u^E$.
(5) If $u_n \nearrow u$, then $\hat{R}_{u_n}^E \nearrow \hat{R}_u^E$.
(6) If $v := \hat{R}_u^E$, then $\hat{R}_v^E = v$.

Proof. (1): is immediate from the fundamental convergence theorem.
(2): $v \geq u$ q.e. on E implies $v \geq \hat{R}_u^E$ q.e. on E, so if F is polar and $v \geq \hat{R}_u^E$ on $E \setminus F$, then $v \geq \hat{R}_u^{E\setminus F} = \hat{R}_u^E$.
(3): We know that $\hat{R}_u^E = R_u^E$ on $\operatorname{int}(E)$. Let F be the polar subset of E where $\hat{R}_u^E < R_u^E$, and choose for $x_0 \in \Omega \setminus E$ $v \in UP(\Omega)$ which is ∞ on F but finite in x_0. Then

$$\hat{R}_u^E + v/n \geq u \text{ on } E,$$

so $\hat{R}_u^E + v/n \geq R_u^E$ for every n, which proves that

$$\hat{R}_u^E(x_0) = R_u^E(x_0).$$

Since x_0 is arbitrary we are done.

(4): $\hat{R}_u^{E_n} \nearrow v \in UP(\Omega)$ trivially, and $v \le \hat{R}_u^E$. Since $v \ge u$ q.e. on E_n for each n it follows that this also holds on E. So $v \ge \hat{R}_u^E$. Also

$$\lim_{n\to\infty} R_u^{E_n}(x) = \begin{cases} u(x) = R_u^E(x) & x \in E \\ \lim_{n\to\infty} \hat{R}_u^{E_n}(x) = \hat{R}_u^E(x) = R_u^E(x) & x \in \Omega \setminus E \end{cases}.$$

(5): $\hat{R}_{u_n}^E \nearrow v \in UP(\Omega)$. Also since $\hat{R}_{u_n}^E = u_n$ q.e. on E, $v = \lim_{n\to\infty} u_n$ q.e. on E. So $\hat{R}_u^E \le v$, and the reverse is obvious.

(6): Since if $w \in UP(\Omega)$ has $w \ge v$ q.e. on E, then since we know that $v \ge u$ q.e. on E we get that $w \ge u$ q.e. on E. So by (2) the statement follows immediately. □

Theorem 5.14. *Let Ω be Greenian, $\omega \subset \Omega$ open. (Reductions w.r.t. Ω).*
(1) If $\operatorname{supp}(\mu) \subset \Omega$, *then* $\limsup_{x\to y, x\in\Omega} G_\Omega^\mu(x) = 0$ *for q.e.* $y \in \partial\Omega \setminus \{\infty\}$.
(2) $G_\omega(x,y) = G_\Omega(x,y) - R_{G_\Omega(\cdot,y)}^{\Omega\setminus\omega}(x)$ *for all* $x, y \in \omega$.

Proof. For each $y \in \partial\Omega$ there is a ball $B(y,\varepsilon)$ such that $\Omega \cup B(y,\varepsilon)$ is Greenian (we don't prove this in general. For $N \ge 3$ this is obvious, and if we assume $\mathbb{R}^2 \setminus \overline{\Omega} \neq \emptyset$ in two dimensions it is also obvious. The general case follows from the fact that a set in $\mathbb{R}^2$ is Greenian if and only if its complement is non-polar). By Harnack's inequality we see that it is enough to prove the statement for $\mu = \delta_a$ for some fixed $a \in \Omega$. So it is enough to prove that (since Green functions are nonnegative we can omit the sup) $G_\omega(\cdot,a)$ has limit 0 q.e. on $\partial\omega \cap \Omega$. Let

$$v_a := \hat{R}_{G_\Omega(\cdot,y)}^{\Omega\setminus\omega}(x),$$

then $G_\Omega(\cdot,a) = \Phi(\cdot - a) - h_a$ and $G_\Omega(\cdot,a) \ge v_a$, so $h_a + v_a$ is a harmonic minorant to $\Phi(\cdot - a)$ on ω. So

$$G_\omega(\cdot,a) \le \Phi(\cdot - a) - (h_a + v_a) = G_\Omega(\cdot,a) - v_a \text{ on } \omega,$$

so

$$0 \le \limsup_{x\to y, x\in\omega} G_\omega(x,a) \le G_\Omega(y,a) - v_a(y),$$

and we know that the last is 0 q.e. on $\Omega \cap \partial\omega$.

To prove (2) we note that if u is a harmonic minorant to $G_\Omega(\cdot,a) - v_a$ on ω, then

$$\limsup_{x\to y, x\in\omega} u(x) \le 0 \text{ q.e. in } \Omega \cap \partial\omega$$

by the above, and by (1) also q.e. on $\partial\Omega \cap \partial\omega$. So $u \le 0$ on ω, and hence

$$G_\omega(\cdot,a) = G_\Omega(\cdot,a) - v_a = G_\Omega(\cdot,a) - \hat{R}_{G_\Omega(\cdot,a)}^{\Omega\setminus\omega}.$$

Since

$$\hat{R}_{G_\Omega(\cdot,a)}^{\Omega\setminus\omega} = R_{G_\Omega(\cdot,a)}^{\Omega\setminus\omega} \text{ on } \omega$$

(2) follows. □

Remark: *It is obvious if $N \ge 3$ that we always have*

$$\lim_{x\to\infty, x\in\Omega} G_\Omega^\mu(x) \le \lim_{x\to\infty} U^\mu(x) = 0.$$

Theorem 5.15. *Let Ω be Greenian, and $E \subset \Omega$, then*

$$(*)\hat{R}^E_{G_\Omega(x,\cdot)}(y) = \hat{R}^E_{G_\Omega(\cdot,y)}(x) \quad \forall x, y \in \Omega.$$

Proof. Assume first that E is relatively closed and $x, y \in \Omega$, $E_n := E\setminus(B(x,1/n)\cup B(y,1/n))$ $\forall n \geq 1$. Then

$$\hat{R}^{E_n}_{G_\Omega(x,\cdot)}(y) = \hat{R}^{E_n}_{G_\Omega(\cdot,y)}(x),$$

because

$$G_{\Omega\setminus E_n}(x,y) = G_\Omega(x,y) - \hat{R}^{E_n}_{G_\Omega(\cdot,y)}(x),$$

and Green's functions are symmetric. As $n \to \infty$ we obtain $(*)$ for E relatively closed.

For E open we can now take $K_n \nearrow E$ where K_n are compact, and get

$$\hat{R}^E_{G_\Omega(x,\cdot)}(y) = \lim_{n\to 0} \hat{R}^{K_n}_{G_\Omega(x,\cdot)}(y) = \lim_{n\to\infty} \hat{R}^{K_n}_{G_\Omega(\cdot,y)}(x) = \hat{R}^E_{G_\Omega(\cdot,y)}(x).$$

For E arbitrary, then for every $\omega \supset E$ open we have $(*)$, so on E_n we get $(*)$ and by letting n go to infinity we get it for E as-well (recall that $R^E_u = \inf\{R^A_u : E \subset A, A \text{ open}\}$ if u is bounded and continuous in a neighborhood of E). ☐

6. Energy

We just introduce the concept of energy and prove some of its simpler properties.

Definition 6.1. *Let Ω be Greenian in $\mathbb{R}^N$ $(N \geq 2)$. Then if μ and ν are positive Radon measures on Ω we define*

$$(\mu,\nu)_e := \int\int G_\Omega(x,y)d\mu(x)d\nu(y)$$

and

$$||\mu||^2_e := (\mu,\mu)_e$$

Now we put $\varepsilon^+(\Omega) := \{\mu : \mu$ is a positive Radon measure with $||\mu||_e < \infty\}$.

Theorem 6.2. *Let Ω be Greenian. Then*
(1) $(\mu,\nu)_e = (\nu,\mu)_e$ $\forall \mu,\nu \in \varepsilon^+(\Omega)$.
(2) $(t\mu,\nu)_e = t(\mu,\nu)_e$ $\forall t \geq 0$, $\forall \mu,\nu \in \varepsilon^+(\Omega)$.
(3) $(\mu,\nu)_e \leq ||\mu||_e||\nu||_e < \infty$ $\forall \mu,\nu \in \varepsilon^+(\Omega)$.
(4) $||\mu||_e \geq 0$ with equality if and only if $\mu = 0$.

Proof. (1) and (2) are obvious ((1) by Fubini and the fact that $G_\Omega(x,y) = G_\Omega(y,x)$). To prove (3), let K_n be a sequence of compact subsets exhausting Ω. Then we can look at the restriction of the measures to K_n and then take the limit as $n \to \infty$. Therefore we can assume without loss of generality that supp(μ) and supp(ν) are contained in a fixed compact subset K of Ω. These assumptions make it possible to define $\lambda = \mu - \nu$ which then becomes a signed measure on Ω. Consider

$$\int\int G_\Omega(x,y)d\lambda(x)d\lambda(y) = \int\int G_\Omega(x,y)d(\mu-\nu)(x)d(\mu-\nu)(y) =$$

$$||\mu||_e^2 + ||\nu||_e^2 - 2(\mu,\nu)_e.$$

We now mollify:

$$\mu_\varepsilon := \mu * \eta_\varepsilon, \ \ \nu_\varepsilon := \nu * \eta_\varepsilon.$$

Then (here it is not important that μ, ν are positive, only that they have compact support in Ω and ε is small)

$$\begin{aligned}\iint G_\Omega(x,y)d\mu_\varepsilon(x)d\nu_\varepsilon(y) &= \int G_\Omega^{\mu_\varepsilon} d\nu_\varepsilon = \\ &= \int G_\Omega^{\mu_\varepsilon}(-\Delta G_\Omega^{\nu_\varepsilon})dm = \int \nabla G_\Omega^{\mu_\varepsilon} \cdot \nabla G_\Omega^{\nu_\varepsilon} dm.\end{aligned}$$

So especially we get $(\lambda_\varepsilon, \lambda_\varepsilon) \geq 0$ by putting $\mu = \nu$ above. But by approximation this must then hold for all Radon measures λ on Ω as above.

We get $||\mu||_e^2 + ||\nu||_e^2 \geq 2(\mu,\nu)_e$, so $(\mu,\nu)_e \leq 1$ if $||\mu||_e$ and $||\nu||_e$ is ≤ 1. Hence

$$(\frac{\mu}{||\mu||_e}, \frac{\nu}{||\nu||_e})_e \leq 1$$

and the theorem follows. □

We now define

$$\varepsilon = \{\lambda : \lambda = \mu - \nu \quad where \quad \mu, \nu \in \varepsilon^+ \quad under \quad equivalence\}.$$

If $\lambda_1, \lambda_2 \in \varepsilon$ define $(\lambda_1, \lambda_2)_e = \iint G_\Omega(x,y)d\lambda_1(x)d\lambda_2(y)$. Now we see that ε is an inner product space, but it is not a Hilbert space, i.e. it is not complete. (Although ε^+ is complete with the induced metric.)

Existence of equilibrium distributions

Let $K \subset \Omega \subset \mathbb{R}^N$ where K is compact and Ω is Greenian. Consider the problem: Minimize $||\mu||_e^2$ over all positive Radon measures μ with $\text{supp}(\mu) \subset K$ and $\mu(K) = 1$. Put $B = \{\mu : \mu$ is a positive Radon measure with support in K , $\mu(K) = 1, ||\mu||_e < \infty\}$.

Theorem 6.3. *With the notation from above we have that either $B = \emptyset$ (which occurs if and only if K has capacity zero), or $B \neq \emptyset$. If the latter holds, then there is a measure $\mu \in B$ such that*

$$||\mu||_e^2 = \inf\{||\mu||_e^2 : \mu \in B\}.$$

Proof. There exists a sequence μ_n such that $||\mu_n||_e^2 \to \inf ||\mu||_e^2$, by well known properties of $\mathbb{R}$. Since the unit ball in $C(K)^*$ is weak$^*-$compact (by Alaoglu's theorem) there is a positive Radon measure with $\text{supp}(\mu) \subset K$, $\mu(K) = 1$ such that $\mu_n \rightharpoonup \mu$ in the weak$^*-$topology. So if we let $G_m(x,y) = \min\{G_\Omega(x,y), m\}$ then we have that for each fixed m $\iint G_m(x,y)d\mu_n(x)d\mu_n(y) \to \iint G_m(x,y)d\mu(x)d\mu(y)$ as $n \to \infty$. If we let $m \to \infty$ we get :

$$\iint G_m(x,y)d\mu(x)d\mu(y) \le \liminf_{n\to\infty} ||\mu_n||_e^2 \quad \forall m,$$

hence

$$||\mu||_e^2 \le \liminf_{n\to\infty} ||\mu_n||_e^2$$

and the assertion follows. □

Now B is a convex set in $\varepsilon^+(\Omega)$, so we have that the μ that solves the problem above fulfills $(\nu-\mu,\mu)_e \ge 0$ $\forall \nu \in B$, i.e., $\int_K G_\Omega^\mu d\mu \le \int_K G_\Omega^\mu d\nu$ $\forall \nu \in B$. Put $\gamma = ||\mu||_e^2 = \int G_\Omega^\mu d\mu$, and for $\alpha \in \mathbb{R}$, $F_\alpha := \{x \in K : G_\Omega^\mu(x) \le \alpha\}$ (here μ is the minimizing measure as above). Since G_Ω^μ is l.s.c. F_α is closed. If $C(F_\alpha) > 0$ (for a fixed α), then we have that there is a $\nu \in B$ with supp$(\nu) \subset F_\alpha$. This gives $\gamma = \int G_\Omega^\mu d\mu \le \int G_\Omega^\mu d\nu \le \alpha$, so $C(F_\alpha) = 0$ $\forall \alpha < \gamma$, especially $\mu(F_\alpha) = 0$ $\forall \alpha < \gamma$.

The conclusion is now that $G_\Omega^\mu \ge \gamma$ on $K \setminus E$ where $E = \bigcup_{\alpha<\gamma} F_\alpha = \bigcup_{n=1}^\infty F_{\gamma-1/n}$. We see that $\mu(E) = 0$, hence $G_\Omega^\mu \ge \gamma$ μ-a.e. on K. But from the definition of γ we have $\int (G_\Omega^\mu - \gamma)d\mu = 0$ so $G_\Omega^\mu = \gamma$ μ-a.e. on K. So $\mu(F_\gamma) = 1$, that is supp$(\mu) \subset F_\gamma$. If we use the domination principle together with the definition of F_γ we have $G_\Omega^\mu \le \gamma$ everywhere in Ω.

To summarize: if $C(K) > 0$ we have found $\mu \in B$ such that $G_\Omega^\mu \le \gamma$ everywhere in Ω, $G_\Omega^\mu = \gamma$ on $K \setminus E$ with E as above (γ is a positive constant). Moreover $\mu(E) = 0$, and E is a countable union of compact sets with capacity zero. Now we come to the point of it all. Recall that we defined the capacity of K by $C(K) = \nu_K(\Omega)$ where $\nu_K = -\Delta u$, $u = \hat{R}_1^K$. Also recall that this means that $G_\Omega^{\nu_K} = 1$ a.e. on K. But the minimization we justified gives us just such a measure times some suitable constant! So we have found another way to determine the capacity of a compact set K.

7. PWB^h-method and the Dirichlet problem

Let $\Omega \subset \mathbb{R}^N$ be open, and suppose $h \in HP(\Omega)$, $h > 0$ everywhere. We then define

$$U_h(\Omega) := \{v/h : v \in U(\Omega)\} = h - \text{superharmonic functions},$$

and similarly for $S_h(\Omega)$ and $H_h(\Omega)$.
For $B(a,r) \subset\subset \Omega$ we let $\eta_{r,a}^h$ be the measure defined by

$$\eta_{r,a}^h(A) := \frac{1}{h(a)m(B(a,r))} \int_{A\cap B(a,r)} h dm.$$

Then

$$\int f d\eta_{r,a}^h = \frac{1}{h(a)m(B(a,r))} \int_{B(a,r)} f h dm.$$

So if $u = v/h \in U_h(\Omega)$ we get

$$\frac{v(a)}{h(a)} \leq \frac{1}{h(a)m(B(a,r))}\int_{B(a,r)} vdm = \frac{1}{h(a)m(B(a,r))}\int_{B(a,r)} uhdm = \int ud\eta^h_{r,a}.$$

And conversely for u l.s.c. on Ω, suppose that

$$(*)\ u(a) \leq \int ud\eta^h_{r,a} \quad \forall B(a,r) \subset\subset \Omega.$$

This is by the above equivalent to

$$u(a) \leq \frac{1}{h(a)m(B(a,r))}\int_{B(a,r)} uhdm \quad \forall B(a,r) \subset\subset \Omega,$$

and this in turn is equivalent to that

$$u(a)h(a) \leq \frac{1}{m(B(a,r))}\int_{B(a,r)} uhdm \quad \forall B(a,r) \subset\subset \Omega,$$

and this is the same as saying that $uh \in U(\Omega)$.
To summarize, $u \in U_h(\Omega)$ if and only if u is l.s.c. and $(*)$ holds. Equivalent statements of-course holds also for $S_h(\Omega)$ and $H_h(\Omega)$. Using the above averaging measures instead of the usual mean-value inequalities (i.e. for $h = 1$) we get in exactly the same way as before:

Theorem 7.1 (Minimum principle). *If $u \in U_h(\Omega)$, Ω connected and u attains its infimum in some point of Ω, then u is constant.*

From now on in this section we will assume that Ω is Greenian and connected, and $(\partial\Omega, \tau)$ compactifies Ω (that is, not necessarily the Euclidean boundary unless so stated). The connectedness is in principle not needed anywhere, but it makes some statements and proofs less tedious.

For $f : \partial\Omega \to \overline{\mathbb{R}}$ we define:

$$UC_h(f) := \left\{ u \in U_h(\Omega) \text{ or } u \equiv \infty : \begin{array}{l} u \text{ is bounded below and} \\ \liminf_{x\to y, x\in\Omega} u(x) \geq f(y)\ \forall y \in \partial\Omega \end{array} \right\}$$

$$LC_h(f) := \left\{ s \in S_h(\Omega) \text{ or } s \equiv -\infty : \begin{array}{l} s \text{ is bounded above and} \\ \limsup_{x\to y, x\in\Omega} s(x) \leq f(y)\ \forall y \in \partial\Omega \end{array} \right\}.$$

Now we introduce the upper and lower PWB^h-functions:

$$\overline{H}^h_f := \inf UC_h(f), \quad \underline{H}^h_f := \sup LC_h(f).$$

Clearly $\overline{H}^h_f \geq \underline{H}^h_f$ and $\overline{H}^h_f$ ($\underline{H}^h_f$) is either $h-$harmonic or $\equiv \pm\infty$ on Ω.

We call f $h-$resolutive if $\overline{H}^h_f = \underline{H}^h_f$ and these are finite-valued. The common value in this case is denoted H^h_f. The set of $h-$resolutive functions is denoted by $R^h(\partial\Omega)$. If $C(\partial\Omega) \subset R^h(\partial\Omega)$, then $\partial\Omega$ is called $h-$resolutive. If for every $h \in H_h L^\infty(\Omega)$ there is $f \in R^h(\partial\Omega)$ such that $h = H^h_f$, then $\partial\Omega$ is called internally $h-$resolutive.

Lemma 7.2. *(a)* $\overline{H}^h_{-f} = -\underline{H}^h_f$, $\overline{H}^h_f, \underline{H}^h_f$ *is increasing with* f.

(b) $\overline{H}^h_{cf} = c\overline{H}^h_f$, $\underline{H}^h_{cf} = c\underline{H}^h_f$ $\forall c \in \mathbb{R}_+$.

(c) $\overline{H}^h_{f+c} = \overline{H}^h_f + c$, $\underline{H}^h_{f+c} = \underline{H}^h_f + c$ $\forall c \in \mathbb{R}$.

(d) $\forall \varepsilon > 0$, *if* $f, g : \partial\Omega \to \mathbb{R}$ *and* $\underline{H}^h_f, \underline{H}^h_g, \overline{H}^h_f, \overline{H}^h_g$ *are finite-valued, if* $|f - g| \leq \varepsilon$, *then*

$$\overline{H}^h_f - \varepsilon = \overline{H}^h_{f-\varepsilon} \leq \overline{H}^h_g \leq \overline{H}^h_{f+\varepsilon} = \overline{H}^h_f + \varepsilon,$$

and

$$\underline{H}^h_f - \varepsilon = \underline{H}^h_{f-\varepsilon} \leq \underline{H}^h_g \leq \underline{H}^h_{f+\varepsilon} = \underline{H}^h_f + \varepsilon.$$

In particular $R^h(\partial\Omega)$ *is closed w.r.t. uniform convergence (and* $H^h_\cdot$ *is continuous w.r.t to uniform convergence on* $R^h(\partial\Omega)$*).*

Proof. (a),(b),(c) are obvious from the definition, and (d) follows directly from (a) and (c). □

One type of argument we will frequently use is the following:
Suppose $u, v, k \in U_h(\Omega)$, and $u \leq v + k/n$ on Ω for ever $n \geq 1$, then we have $u \leq v$ on $(k < \infty)$. So since $(k = \infty)$ is polar this must hold on all of Ω.

Definition 7.3. *A set* $A \subset \partial\Omega$ *such that* $\overline{H}^h_{\chi_A} \equiv 0$ *is called* $h-$*harmonic (measure) null.*

Lemma 7.4. *(a) If* $\overline{H}^h_{\chi_{A_n}} \equiv 0$ *for all* n*, then* $\overline{H}^h_{\chi_A} \equiv 0$ *where* $A = \bigcup_{n=1}^\infty A_n$.

(b) A $h-$*harmonic measure null* $\Leftrightarrow \exists u \in U_h P(\Omega)$ *with limit* ∞ *on* A *(from* Ω*).*

(c) If $f : \partial\Omega \to \overline{R}_+$ *and* $\overline{H}^h_f \equiv 0$*, then* $(f > 0)$ *is* $h-$*harmonic null.*

(d) If $f : \partial\Omega \to \overline{R}$*, and* $\overline{H}^h_f < \infty$*, then* $(f = \infty)$ *is* $h-$*harmonic null.*

Proof. Let $a \in \Omega$ be fixed.
(a): Choose $u_n \in UC_h(\chi_{A_n})$ with $u_n(a) < \varepsilon 2^{-n}$ $(\varepsilon > 0)$, then

$$v := \sum_{n=1}^{\infty} u_n \in UC_h(\chi_A),$$

and hence $\overline{H}^h_{\chi_A} \equiv 0$.
(b): Let $A_n = A$ for every n above, then

$$\liminf_{x \to y, x \in \Omega} v(x) = \infty \quad \forall y \in A.$$

Conversely, the existence of u as in the statement gives that $u/n \in UC_h(\chi_A)$ for every n.
(c): $0 = \overline{H}^h_f \geq (\overline{H}^h_{\chi_{(f>1/n)}})/n$, for $n \geq 1$, so $(f > 1/n)$ is $h-$harmonic null, and the rest follows from (a).
(d): Let $u \in U_h(\Omega) \cap UC_h(f)$, lower bounded by some constant c (by the definition of $UC_h(f)$) with limit ∞ on $(f = \infty)$, then the rest follows from (b),by looking at $u - c$. □

• We use the following lattice notation:

$$f \vee g = \max\{f, g\}, \ f \wedge g = \min\{f, g\}.$$

Let us also intoduce the following lattice notation on $H_hP(\Omega) - H_hP(\Omega)$. For $f, g \in H_hP(\Omega) - H_hP(\Omega)$ we let

$$f \curlywedge g$$

denote the greatest lower bound of all $h-$subharmonic minorants of $f \wedge g$, and

$$f \curlyvee g$$

the least upper bound of all $h-$superharmonic majorants of $f \vee g$. To see that this makes sense let us note that if $f, g \in H_hP(\Omega)$, then $(f - g) \vee 0 \leq f$, which gives a $h-$superharmonic majorant to $(f - g) \vee 0$, and this together with the formula $f \vee g = ((f - g) \vee 0) + g$, which holds for any functions f, g is easily seen to imply that there is a $h-$superharmonic majorant in any case. The case $\wedge$ is treated similarly. It should always be clear from the context w.r.t. which space we take these operations.

Theorem 7.5. *(a) If $f = g$ up to a $h-$harmonic null set, then $\underline{H}^h_f = \underline{H}^h_g$ and $\overline{H}^h_f = \overline{H}^h_g$.*

(b) $\overline{H}^h_f < \infty$ if and only if $\overline{H}^h_{f \vee 0} < \infty$.

(c) If $\overline{H}^h_f < \infty$, $\overline{H}^h_g < \infty$ and if $f + g$ is defined arbitrarily on the $h-$harmonic null set $(f = -\infty,\, g = \infty) \cup (f = \infty, g = -\infty)$, then $\overline{H}^h_{f+g} \leq \overline{H}^h_f + \overline{H}^h_g$.

(d) $R^h(\partial\Omega)$ is a vector-lattice (i.e. is a vector-space over $\mathbb{R}$, and contains $f \vee g$ and $f \wedge g$ for all $f, g \in R^h(\partial\Omega)$). Also, for $f, g \in R^h(\partial\Omega)$ we have:

$H^h_{af+bg} = aH^h_f + bH^h_g \quad \forall a, b \in \mathbb{R}.$

$H^h_{f \vee g} = H^h_f \curlyvee H^h_g.$

$H^h_{f \wedge g} = H^h_f \curlywedge H^h_g.$

(e) If $f_n \nearrow f$ with $\overline{H}^h_{f_1} > -\infty$, then $\overline{H}^h_{f_n} \nearrow \overline{H}^h_f$. If also $f_n \in R^h(\partial\Omega)$, then so does f.

Proof. (a): Let A be the $h-$harmonic null set in question, and note that we only have to prove $\overline{H}^h_f \leq \overline{H}^h_g$ by symmetry. Let $v \in U_hP(\Omega)$ have limit ∞ on A, and let $u \in UC_h(g)$. Then trivially we have $u + v/n \in UC_h(f)$ for all $n \geq 1$, and (a) follows from this and the earlier remark.

(b): If $\overline{H}^h_f < \infty$, then $\exists u \in UC_h(f) \cap U_h(\Omega)$, $\infty > c := \inf u > -\infty$ by assumption. So $u + |c| \in UC_h(f \vee 0) \cap U_h(\Omega)$.

(c): WLOG assume $f, g < \infty$ everywhere (by (a)). If $u \in UC_h(f), v \in UC_h(g)$ then $u + v \in UC_h(f + g)$.

(d): The linearity is obvious from earlier results. Since $f \wedge g = -((-f) \vee (-g))$, $f \vee$

$g = ((f-g)\vee 0)+g$ we only need to treat the $f\vee 0$-case. Let $a\in\Omega$ be fixed, and let $v_n\in UC_h(f)$ with $v_n(a)\le H_f^h(a)+2^{-n}$. If

$$u_0 := \inf\{u : u\in U_h(\Omega), u\ge H_f^h\vee 0\},$$

(note that by (b) the class above is nonempty) then

$$u_0+\sum_{n=k}^{\infty}(v_n-H_f^h)\in UC_h(f\vee 0),$$

because it is positive and $\ge v_k$ for each k. When $k\to\infty$ we get that

$$\overline{H}^h_{f\vee 0}\le u_0,$$

but the reverse inequality is obvious, because $\underline{H}^h_{f\vee 0}\ge H_f^h\vee 0$. which trivially holds since the left hand side is non-negative and larger than or equal to H_f^h. Hence

$$\underline{H}^h_{f\vee 0}\ge u_0\ge \overline{H}^h_{f\vee 0}.$$

(e): Again let $a\in\Omega$ be fixed. If $\overline{H}^h_{f_n}\equiv\infty$ for some n then the result holds trivially. So assume the opposite. Choose $u_m\in UC_h(f_m)$ with $u_m(a)<\overline{H}^h_{f_m}(a)+2^{-m}$. Then

$$\sum_{m=1}^{\infty}(u_m-\overline{H}^h_{f_m})$$

is positive and superharmonic on Ω. Now

$$v_n := \lim_{k\to\infty}\overline{H}^h_{f_k}+\sum_{k=n+1}^{\infty}(u_k-\overline{H}^h_{f_k})\ge u_m \quad \forall m>n.$$

Hence $v_n\in UC_h(f)$, and

$$\lim_{n\to\infty}\overline{H}^h_{f_n}(a)\le\overline{H}^h_f(a)\le\lim_{n\to\infty}v_n(a)=\lim_{n\to\infty}\overline{H}^h_{f_n}(a).$$

Since $\overline{H}^h_f\ge\lim_{n\to\infty}\overline{H}^h_{f_n}$ it follows from the minimum principle that $\overline{H}^h_f\equiv\lim_{n\to\infty}\overline{H}^h_{f_n}$.

If each $f_n\in R^h(\partial\Omega)$ then

$$\overline{H}^h_f\ge\underline{H}^h_f\ge\lim H^h_{f_n}=\overline{H}^h_f.$$

□

Definition 7.6. *Let*

$$\Sigma^h := \{A\subset\partial\Omega : \chi_A\in R^h(\partial\Omega)\}.$$

By the above theorem we see that Σ^h *is a* $\sigma-$*algebra. For* $x\in\Omega$*, let*

$$\nu_x^h(A) := H^h_{\chi_A}(x)\quad\forall A\in\Sigma^h.$$

We call ν_x^h *the* $h-$*harmonic measure w.r.t.* x,Ω *on* $\partial\Omega$*. (If we need to specify* Ω *we will write* $\nu_x^h(\partial\Omega,\cdot)$*. The reason is that sometimes we have different boundaries* *for Ω, and although a set say in $\mathbb{R}^N$ is not determined by its boundary in general, since Ω is in the notation $\partial\Omega$ there should be no risk of confusion).*

Theorem 7.7. *(a) $\nu_x^h \ll \nu_y^h$ $\forall x, y \in \Omega$, and $L^p(\nu_x^h)$ is independent of x.*
(b) $R^h(\partial\Omega) = L^1(\nu_x^h)$, and $H_f^h(x) = \int f d\nu_x^h$ for every $f \in R^h(\partial\Omega)$.
(c) ν_x^h need not be a Baire-measure, but for every $A \in \Sigma^h$ $\exists B_1, B_2 \in \Sigma^h$, B_1 an F_σ and B_2 a G_δ, $B_1 \subset A \subset B_2$ such that $\nu_x^h(B_1) = \nu_x^h(B_2)$. In fact ν_x^h is a Baire measure if and only if $\partial\Omega$ is $h-$resolutive, which in turn is equivalent to that $K \mapsto \overline{H}_{\chi_K}^h$ is additive on the compact G_δ sets $K \subset \partial\Omega$.

Proof. (a): If $\overline{H}_{\chi_A}^h(x) = 0$, then $\overline{H}_{\chi_A}^h(y) = 0$ by the minimum principle. By Harnack's inequality we also have $\frac{d\nu_x^h}{d\nu_y^h} \in L^\infty(\nu_y^h)$.

(b): That $L^1(\nu_x^h) \subset R^h(\partial\Omega)$ is obvious by the definition of Σ^h and $L^1(\nu_x^h)$ and earlier results about $R^h(\partial\Omega)$. Let C_0 be the class of functions on $\mathbb{R}$ which are continuous, with limit 0 at $\pm\infty$, and let $f \in R^h(\partial\Omega)$ be finite-valued. The set

$$\{\phi \in C_0 : \phi \circ f \in R^h(\partial\Omega)\}$$

is a vector-lattice closed under uniform convergence, and contains

$$x \mapsto (1 - |x - n|) \vee 0 \quad \forall n \in \mathbb{N},$$

so is in-fact all of C_0 by Stone's theorem. Also the class of functions $\psi : \mathbb{R} \to \mathbb{R}$ for which $\psi \circ f \in R^h(\partial\Omega)$ is a class closed under bounded monotone convergence, so contains all Borel-measurable bounded functions. If $\psi := \chi_{[a,b]}$ say, then $\psi \circ f = \chi_{(f\in[a,b])}$, which proves that $(f \in [a, b]) \in \Sigma^h$, so f is measurable. It is now trivial to see the stated equality by approximation with step-functions.

(c): Let $A \in \Sigma^h$, and choose $u_n \in UC_h(\chi_A)$, $u_n \searrow H_{\chi_A}^h$, and $s_n \in LC_h(\chi_A)$, $s_n \nearrow H_{\chi_A}^h$. Put

$$g_n := \liminf_{x\to y, x\in\Omega} u_n(x) \ (y \in \partial\Omega),$$

$$f_n := \limsup_{x\to y, x\in\Omega} s_n(x) \ (y \in \partial\Omega).$$

Then $f_n \le \chi_A \le g_n$ everywhere on $\partial\Omega$, $f_n \nearrow f$, $g_n \searrow g$, $f \le \chi_A \le g$, and f, g are Borel-measurable. Now it is easy to see that $\overline{H}_g^h = \overline{H}_{\chi_A}^h$ and $\underline{H}_f^h = \underline{H}_{\chi_A}^h$, so

$$H_{\chi_A}^h = \underline{H}_f^h \le \overline{H}_f^h \le H_{\chi_A}^h \le \underline{H}_g^h \le \overline{H}_g^h = H_{\chi_A}^h,$$

so $f, g \in R^h(\partial\Omega)$. Now let

$$B_1 := \bigcup_{n=1}^{\infty} \{f_n \ge 1/2\},$$

which is an F_σ, since f_n is u.s.c., and

$$B_2 := \bigcap_{n=1}^{\infty} \{g_n > 1/2\},$$

which is a G_δ, since g_n is l.s.c. Now $B_1 \subset A \subset B_2$, and both B_1 and B_2 belongs to Σ^h. We have

$$\{f = 1\} \subset B_1 \subset A \subset \{g \geq 1\} \subset B_2,$$

and since

$$\overline{H}^h_{g-f} = 0,\ \nu^h_x(B_2 \setminus \{g \geq 1\}) = \nu^h_x(\{1/2 \leq g < 1\}) = 0$$

we get

$$\nu^h_x(B_2 \setminus B_1) = 0.$$

That $\partial\Omega$ resolutive implies the additivity on K_σ is also by definition. Now we assume that the additivity holds on the compact G_δ sets. Let K be a compact G_δ, and $K = \bigcap_{n=1}^\infty O_n$, where each O_n is open. Then there are continuous functions $f_n : \partial\Omega \to [0,1]$ with support in O_n and identically 1 on K. We also may define

$$K_n := \{f_n = 0\} = \bigcap_{j=1}^\infty \{f_n < 1/j\},$$

which by the formula above are compact and G_δ. Since we have that

$$K^c = \bigcup_{n=1}^\infty K_n$$

it follows that K^c is a K_σ. Now we get

$$\overline{H}^h_{\chi_K \cup \chi_{K_n}} = \overline{H}^h_{\chi_K} + \overline{H}^h_{\chi_{K_n}},$$

and the left side converges monotonically to 1 and the right side to $\overline{H}^h_{\chi_K} + \overline{H}^h_{\chi_O}$. Since also

$$\underline{H}^h_{\chi_A} = 1 - \overline{H}^h_{\chi_{A^c}}$$

holds in general we see that $K \in \Sigma^h$. To prove the last formula note that if $u \in UC_h(\chi_{A^c})$, then

$$\limsup_{x\to y, x\in\Omega} (1 - u(x)) = 1 - \liminf_{x\to y, x\in\Omega} u(x) \leq 1 - \chi_{A^c} = \chi_A,$$

and similarly in the other direction. □

Regularity: A point $y \in \partial\Omega$ is called $h-$regular if for every $f \in C(\partial\Omega)$ we have

$$\lim_{x\to y} \overline{H}^h_f(x) = \lim_{x\to y} \underline{H}^h_f(x) = f(y).$$

If every $y \in \partial\Omega$ is $h-$regular we call $\partial\Omega$ $h-$regular. Trivially, if $\partial\Omega$ is $h-$regular, it is also $h-$resolutive by the minimum principle.

Definition 7.8. *A function $u \in U_hP(\Omega)$ which is strictly positive on Ω, and such that ($y \in \partial\Omega$ fixed)*

(1) $\lim_{x\to y, x\in\Omega} u(x) = 0$,

(2) $\inf_{\Omega\setminus B(y,\varepsilon)} u > 0\ \forall\varepsilon > 0$.

is called a (strong) $h-$barrier at y. (If u only fulfills (1), then u is called a weak $h-$barrier at y).

Theorem 7.9. *(a) If $D \subset \Omega$ is open, $h_0 = h\lfloor D$, $y \in \partial D$ where $h \in HP(\Omega) \setminus \{0\}$ (∂D is the boundary induced on D by $\Omega \cup \partial\Omega$)), and u_0 is a h_0-barrier at y on D, and there is $B \subset \Omega \cup \partial\Omega$ open with $B \cap D = B \cap \Omega$, and $y \in B$, then there is a $h-$barrier u at y w.r.t. Ω.*

(b) If $f : \partial\Omega \to \mathbb{R}$ is upper bounded, and there is a $h-$barrier u at y on Ω, then

$$\limsup_{x \to y, x \in \Omega} \overline{H}_f^h(x) \le \left(\limsup_{x \to y, x \in \partial\Omega} f(x) \right) \vee f(y).$$

In particular, if f is bounded on $\partial\Omega$ and continuous at y, then

$$\lim_{x \to y, x \in \Omega} \underline{H}_f^h(x) = \lim_{x \to y, x \in \Omega} \overline{H}_f^h(x) = f(y),$$

so y is regular.

Proof. (a): Take B so small that $D \cap \partial B = \Omega \cap \partial B \neq \emptyset$, and put $a := \inf_{D \setminus B} u_0$ which is > 0 by assumption. Then

$$u := \begin{cases} u_0 \wedge a & \text{on } \Omega \cap B \\ a & \text{on } \Omega \setminus B \end{cases},$$

is a $h-$barrier for y on Ω.

(b): The second assertion follows from the first applied to f and $-f$. Let $a > \left(\limsup_{x \to y, x \in \Omega} f(x)\right) \vee f(y)$. Choose a neighborhood B of y so small that $f \le a$ in $B \cap \partial\Omega$, $\delta := \inf_{\Omega \setminus B} u$. Choose n so large that $a + n\delta \ge \sup_{\partial\Omega} f$. Now

$$a + nu \in UC_h(f),$$

and has limit a at y. Hence

$$\limsup_{x \to y, x \in \Omega} \overline{H}_f^h(x) \le a.$$

□

If $\partial\Omega$ is $h-$regular, then letting, for $y \in \partial\Omega$ $f := d(\cdot, y)$, and $u = H_f^h$ on Ω, it is easy to see that this is a $h-$barrier at y on Ω. So any boundary is $h-$regular if and only if there is a strong $h-$barrier at each of its points.

Pointwise regularity is a bit harder to treat, but for the Euclidean boundary it turns out that a (finite) point is regular (read $1-$regular) if and only if there is a weak barrier at the point, and this is of-course equivalent to that the Green's function of the domain has limit zero at the point. So in particular the set of irregular boundary points on the Euclidean boundary is polar.

We now prove that the Euclidean boundary is resolutive and sketch the proof of the above. (When we say regular, resolutive etc. we always mean $1-$regular, $1-$resolutive etc.)

Theorem 7.10. *The Euclidean boundary $\partial\Omega$ of Ω is resolutive.*

Proof. We assume for simplicity that Ω is bounded.

$$\Phi := \{f \in C(\partial\Omega) : \underline{H}_f = \overline{H}_f\}.$$

Obviously constant functions belongs to Φ, and Φ is a vector lattice closed under uniform convergence. Furthermore, if $u \in S(\Omega) \cap C(\overline{\Omega})$, and we put $f = u\lfloor\partial\Omega$, then since $u \in LC(f)$ we get $\underline{H}_f \geq u$, $\underline{H}_f$ is harmonic and obviously satisfies

$$\liminf_{x\to y, x\in\Omega} \underline{H}_f(x) \geq u(y) \quad \forall y \in \partial\Omega.$$

So $\underline{H}_f \in UC(f)$, so $f \in \Phi$. If p is a polynomial, then since $\Delta|\cdot|^2 = 2N$ on $\mathbb{R}^N$ we have that $p + t|\cdot|^2 \in \Phi$ by the above for large t (here we used that Ω is bounded, so that $-\Delta p$ is uniformly bounded on $\overline{\Omega}$). Hence

$$p = p + t|\cdot|^2 - t|\cdot|^2 \in \Phi$$

by the above. Therefore, by Stone's theorem $\Phi = C(\partial\Omega)$, and the proof is done. □

Theorem 7.11. *If there is a weak barrier at $y \in \partial\Omega \setminus \{\infty\}$ on Ω, where $\partial\Omega$ is the Euclidean boundary, then there is also a strong barrier there.*

Proof. (Sketch) Let u be a weak barrier at y. We let $f(x) := |x - y|$ on $\partial\Omega$. Then $f \in R^h(\partial\Omega)$. We aim to show that H_f is a strong barrier at y on Ω. Since $\Delta|\cdot - y| \geq 0$ on Ω it is clear that $\inf_{\Omega\setminus\omega} H_f > 0$ for every neighborhood ω of y in $\Omega \cup \partial\Omega$, because $|\cdot - y|$ itself belongs to $LC(f)$. Then it will be enough to prove that

$$\lim_{x\to y, x\in\Omega} H_f(x) = 0,$$

and to do this it would be enough to prove that for small r such that $\partial B(y, r) \cap \Omega \neq \emptyset$ we have

$$\limsup_{x\to y, x\in\Omega} H_f(x) < 2r.$$

This can be done using some estimates involving the Poisson kernel, but we omit the details. □

Theorem 7.12. *Suppose as before that Ω is Greenian and connected, that $\partial\Omega$ compactifies Ω, and is $h-$resolutive. All sets are subsets of $\Omega \cup \partial\Omega$,and all boundaries are the ones induced by $\Omega \cup \partial\Omega$.*

(a) Suppose $D \subset \Omega$ is open, then ∂D is $h-$resolutive, and for all Borel sets $A \subset \partial D$ we have:

$$\nu_x^h(\partial D, A) = \nu_x^h(\partial\Omega, A \cap \partial\Omega) - \int_{\partial D\cap\Omega} \nu_y^h(\partial\Omega, A \cap \partial\Omega) d\nu_x^h(\partial D, y).$$

(So in particular $\nu_x^h(\partial D, A) \leq \nu_x^h(\partial\Omega, A)$ if $A \subset \partial D \cap \partial\Omega$).

(b) If $D_1 \subset D_2 \subset D_3, \ldots \subset \Omega$ and $\bigcup_{n=1}^{\infty} D_n = \Omega$ are open sets, then

$$\nu_x^h(\partial D_n, \cdot) \rightharpoonup \nu_x^h(\partial\Omega, \cdot).$$

(in the obvious meaning).

Proof. (a): If $A \subset \partial D \cap \Omega$ is compact for instance, then h is lower and upper bounded in a neighborhood of A, and therefore it is obvious that $\overline{H}^h_{\chi_A} = H_{h\chi_A}/h$ (note that $\partial D \cap \Omega$ is the Euclidean boundary, which is known to be resolutive, and that above we only used the obvious fact that in this case we have $u \in UC_h(\chi_A) \Leftrightarrow uh \in UC(h\chi_A)$).

Now if $A \subset \partial D \cap \partial\Omega$ instead, then the function $v - u$, where $v \in UC_h(\chi_A)$ (in Ω), $u \in LC_h(H^{\Omega,h}_{\chi_A} \lfloor \partial D \cap \Omega)$ (in D) is in $UC_h(\chi_A)$ (in D). If we interchange U and L above the same is true. So if $v_n \in UC_h(\chi_A)$ (in Ω), $v_n \searrow H^{\Omega,h}_{\chi_A}$, $u_n \in LC_h(H^{\Omega,h}_{\chi_A} \lfloor \partial D \cap \Omega)$ (in D), $u_n \nearrow H^{D,h}_g$, where $g = H^{\Omega,h}_{\chi_A} \lfloor \partial D \cap \Omega$, then if s_n, t_n is in the same way but for U, L interchanged, then u_n and t_n has the same limits, and so does v_n and s_n. This clearly implies that

$$H^{D,h}_{\chi_A} = H^{\Omega,h}_{\chi_A} - H^{D,h}_g,$$

which is another way to write the formula in (a), and adding all together we see that (a) is proved.

(b): Let $f \in C(\Omega \cup \partial\Omega)$. If $u \in UC_h(f \lfloor \partial\Omega)$ (on Ω), let $u(y) := \liminf_{x \to y, x \in \Omega} u(x)$ on $\partial\Omega$. If $\varepsilon > 0$ we have that $u + \varepsilon - f$ is l.s.c. on $\Omega \cup \partial\Omega$ and strictly positive on $\partial\Omega$. Hence it is strictly positive on ∂D_n for large n. So $u + \varepsilon \in UC_h(f)$ (on D_n) for large n, so

$$\limsup_{n \to \infty} \int f d\nu^h_x(\partial D_n, \cdot) \leq \int f d\nu^h_x(\partial\Omega, \cdot).$$

Together with the corresponding inequality for $-f$ gives the result. □

Extension of reductions to the boundary: As before, let $\Omega \subset \mathbb{R}^N$ be a Greenian domain, and $\partial\Omega$ compactify Ω. We say that a property holds near $A \subset \partial\Omega$ if there is an open subset $D \subset \Omega \cup \partial\Omega$, such that $A \subset D$ and the property holds on $D \cap \Omega$. Now we define for $u \in UP(\Omega)$ the reduction over a set $E \subset \Omega \cup \partial\Omega$ by:

$$R^E_u := \inf\{v : v \in UP(\Omega), v \geq u \text{ on } E \cap \Omega \text{ and near } E \cap \partial\Omega\}.$$

$$\hat{R}^E_u := \text{l.s.c. regularization of } R^E_u.$$

Let us introduce the notation

$$\Lambda_{u,E} := \{v : v \in UP(\Omega), v \geq u \text{ on } E \cap \Omega \text{ and near } E \cap \partial\Omega\}.$$

Sometimes it is also natural to study $h-$reductions where h is a positive harmonic function on Ω: Let $u \in U_hP(\Omega)$, then we define

$${}^h\!R^E_u := \inf\{v : v \in U_hP(\Omega), v \geq u \text{ on } E \cap \Omega \text{ and near } E \cap \partial\Omega\}.$$

Note that

$$\begin{aligned} {}^h\!R^E_u &= \inf\left\{\frac{v}{h} : v \in UP(\Omega), v \geq uh \text{ on } E \cap \Omega \text{ and near } E \cap \partial\Omega\right\} = \\ &= \inf\{v : v \in UP(\Omega), v \geq uh \text{ on } E \cap \Omega \text{ and near } E \cap \partial\Omega\}/h = R^E_{uh}/h. \end{aligned}$$

We will concentrate on usual reductions since properties of $h-$reductions are usually trivial consequences of the above. Note that if $E \subset \partial\Omega$, and if $u \in UC_h(\chi_E)$, then $u + 1/n > 1$ near E for every n. So we have

$$\overline{H}^h_{\chi_E} = {}^h\!R^E_1 = R^E_h/h.$$

Proposition 7.13. *Let $E \subset \Omega \cup \partial\Omega$, $u \in UP(\Omega)$ then:*

(a) If (E_n) is a sequence of subsets of $\Omega \cup \partial\Omega$, then

$$R^{\cup E_n}_u \leq \Sigma R^{E_n}_u.$$

(b) $\hat{R}^E_u = R^E_u$ on $\Omega \setminus (\overline{E \cap \Omega})$, and is harmonic there. In particular, if $E \subset \partial\Omega$, then $\hat{R}^E_u = R^E_u \in H(\Omega)$.

(c) If E is open in $\Omega \cup \partial\Omega$, then $\hat{R}^E_u = R^E_u = R^{E\cap\Omega}_u$.

(d) If $u_1, u_2 \in UP(\Omega)$, then $R^E_{u_1+u_2} \leq R^E_{u_1} + R^E_{u_2}$.

(e) If $E \subset \partial\Omega$ and $v_n \searrow v$, $v_n \in UP(\Omega)$, then $R^E_{v_n} \searrow R^E_{\hat{v}}$

(f) If $E \subset \partial\Omega$, then $R^E_w = w$, where $w = R^E_u$.

Proof. (a): Let $x \in \Omega$ and suppose that $\Sigma R^{E_n}_u(x) < \infty$, and let $\varepsilon > 0$. For each n, choose $v_n \in \Lambda_{u,E_n}$ such that $v_n(x) < R^{E_n}_u(x) + \varepsilon 2^{-n}$. Then $\Sigma v_n \in \Lambda_{u,\cup E_n}$, and hence

$$R^{\cup E_n}_u(x) \leq \Sigma v_n(x) < \Sigma R^{E_n}_u(x) + \varepsilon.$$

(b): Obvious, since $\{v\lfloor\Omega \setminus (\overline{E \cap \Omega}) : v \in \Lambda_{u,E}\}$ is saturated.

(c): If E is open in $\Omega \cup \partial\Omega$, then $\Lambda_{u,E} = \Lambda_{u,E\cap\Omega}$, so $R^E_u = R^{E\cap\Omega}_u$, and then the statement is reduced to the one for reductions on Ω, where we have already proved this.

(d): $w_j \in \Lambda_{u_j,E} \Rightarrow w_1 + w_2 \in \Lambda_{u_1+u_2,E}$.

(e): $h_1 :=$ greatest harmonic minorant to $\hat{v}$ on Ω, and $h_2 := \lim_{n\to\infty} R^E_{v_n}$. Then $h_1, h_2 \in HP(\Omega)$, and by (d)

$$0 \leq R^E_{v_n} - R^E_{h_1} \leq R^E_{v_n - h_1}.$$

Let $n \to \infty$ to see that $h_2 - R^E_{h_1} \in HP(\Omega)$, and is a minorant of $\lim_{n\to\infty} R^E_{v_n-h_1}$. So therefore also of $\hat{v} - h_1$. Hence $h_2 = R^E_{h_1}$, and since $R^E_{\hat{v}-h_1} \in HP(\Omega)$ is also a minorant of $\hat{v} - h_1$ we have $R^E_{\hat{v}-h_1} = 0$, so

$$R^E_{h_1} \leq R^E_{\hat{v}} \leq R^E_{h_1} = h_2.$$

(f): Take $u_n \in UP(\Omega)$, $u_n \geq u$ near E, and $u_n \searrow w$ on Ω. Let $v_n := \min\{u, u_n\}$, then $v_n \in UP(\Omega)$ and $v_n = u$ near E, so $R^E_{v_n} = R^E_u$ for each n. Also $v_n \to w$ pointwise on Ω. Hence

$$R^E_u = \lim_{n\to\infty} R^E_{v_n} = R^E_w.$$

□

8. Fine Topology

Let $\Omega \subset \mathbb{R}^N$ be open. The smallest topology on Ω such that every $u \in U(\Omega)$ is continuous (in the extended sense) is called the fine topology on Ω. We let F_Ω denote the fine topology, and E_Ω denote the Euclidean topology on Ω. When we write lim, lim sup and so on this is meant w.r.t. E_Ω and $F_\Omega - \lim, F_\Omega - \limsup$ etc. for the fine toplogy. In this section all boundaries are Euclidean. We will need the following:

Proposition 8.1. *Let $B := B(a, r) \subset\subset \Omega$. Given $u \in U(\Omega)$, then there is $v \in U(\mathbb{R}^N)$ such that $v = u$ on B.*

Proof. WLOG assume $B = B(0, 1)$, and let $D := \overline{B}^c$. Look at the functions v_n defined by:

$$v_n(x) := \begin{cases} u(x) & \text{on } \overline{B} \\ \int u d\nu_x(\partial D, \cdot) - n(\Phi(x) - C_N) & \text{on } D \end{cases}$$

(Note that $\Phi(x) - C_N = 0$ on ∂B, for $N = 2$ we replace C_N by 0, and $-n$ by n in the formula).

Now each v_n is l.s.c. and if $B(0, t) \subset\subset \Omega$ for $t > 1$ (which holds for t near 1) we have that v_n converges uniformly to $-\infty$ on $\partial B(0, t)$ as $n \to \infty$. So for n large $v_n \leq u$ on $\partial B(0, t)$ and equal to u on ∂B. Hence by the minimum principle $v_n \leq u$ on $B(0, t)$. It follows directly from this that the super-mean-value inequality holds on ∂B for v_n on small balls with center on ∂B (n large). Hence, since superharmonicity is a local property, we see that $v_n \in U(\mathbb{R}^N)$, and $v_n = u$ on B by construction. □

Proposition 8.2. *(1) $E_\Omega \subset F_\Omega$*

(2) The set $\{\{x \in \Omega : u(x) < a\} : u \in U(\Omega), a \in \mathbb{R}\}$ forms a subbase for F_Ω.

(3) If $y \in \omega \in F_\Omega$, then there is an $E_\Omega-$compact K such that $y \in F_\Omega - \text{int}(K) \subset K \subset \omega$.

(4) A set is $F_\Omega-$compact if and only if it is finite.

(5) $F_\Omega = \{O \cap \Omega : O \in F_{\Omega'}\}$ $\forall \Omega' \supset \Omega$ open.

Proof. (1): $B(y, r) = \{x : \Phi(x - y) > a\}$ for some a.

(2): Since $x \mapsto x_i$ ($x = (x_1, \ldots, x_N)$) is in $U(\Omega)$ it follows that any $E_\Omega-$open cube in Ω may be written as $\bigcap_{i=1}^N \{x \in \Omega : u_i(x) < a\}$. So since the sets

$$\{x \in \Omega : u(x) > a\} \in E_\Omega,$$

these are redundant in the subbase given by all sets of the form $(u < a)$ and $(u > a)$.

(3): WLOG $\omega = \bigcap_{i=1}^m \{x \in \Omega : u_i(x) < a_i\}$. Let

$$u_i(y) < b_i < a_i \text{ for } i = 1, \ldots, m.$$

Then

$$\overline{B}(y,\varepsilon)\cap\left(\bigcap_{i=1}^{m}\{x\in\Omega : u_i(x)\le b_i\}\right)\quad (\overline{B}(y,\varepsilon)\subset\Omega)$$

has nonempty $F_\Omega-$interior which contains y, and it is $E_\Omega-$compact.

(4): Finite sets are always compact,and if K is $F_\Omega-$compact, then K is $E_\Omega-$compact, and therefore bounded. If K is infinite, then there is $y\in K$, $x_n\in K\setminus\{y\}$ distinct, and $x_n\to y$ (in E_Ω). Since $\{x_n : n\in\mathbb{N}\}$ is a polar set, we know that there is a potential $u\in U(\Omega)$ which is ∞ on this set, but finite in y. Now define

$$V_0 := \{x\in\Omega : u(x) < 2u(y)\},$$

and let (V_n) $(n\ge 1)$ be a sequence of balls with centers at x_n, disjoint and neither containing y,

$$V := \Omega\setminus(\{x_n : n\in\mathbb{N}\}\cup\{y\}).$$

Then $\{V_n : n\ge 0\}\cup\{V\}$ is an $F_\Omega-$open cover of K without finite subcover.

(5): WLOG $\Omega'=\mathbb{R}^N$. Let $B_n\subset\overline{B}_n\subset\Omega$ be a sequence of open balls such that $\cup B_n=\Omega$. Let $u\in U(\Omega)$, then for each n there is a $u_n\in U(\mathbb{R}^N)$ such that $u_n=u$ on B_n. So if O is open in $[-\infty,\infty]$:

$$\{x\in\Omega : u(x)\in O\}=\bigcup_n\{x\in B_n : u(x)\in O\}=\bigcup_n\{B_n\cap\{x\in\mathbb{R}^N : u_n(x)\in O\}\},$$

which belongs to $F_{\mathbb{R}^N}$. □

• We introduce $\overline{A}$ for the $E_\Omega-$closure of A, and $\widetilde{A}$ for the $F_\Omega-$closure of A.

Definition 8.3. *A set $E\subset\Omega$ is said to be thin at y if $y\notin\widetilde{E\setminus\{y\}}$, i.e. there is $\omega\in F_\Omega$, $y\in\omega$ and $\omega\cap(E\setminus\{y\})=\emptyset$.*

Theorem 8.4. *Let $y\in\overline{E}\subset\Omega$, then the following are equivalent:*

(1) E is thin at y.

(2) There is a superharmonic function u on a neighborhood of y such that

$$\liminf_{x\to y, x\in E} u(x) > u(y).$$

(3) For every Greenian $\Omega'\subset\Omega$, $y\in\Omega'$ there is $u\in UP(\Omega')$ such that $u(y)<\infty$, and

$$\lim_{x\to y, x\in E} u(x)=\infty.$$

Proof. (3) implies (2) trivially. If (2) holds, then choose $\delta,\varepsilon>0$ such that $u\in U(B(y,\delta))$, $u(x)\ge u(y)+\varepsilon$ on $B(y,\delta)\cap(E\setminus\{y\})$. Then

$$\{x\in B(y,\delta) : u(x) < u(y)+\varepsilon\}\in F_{\mathbb{R}^N},$$

so (1) holds.

If (1) holds, choose $u_1, \ldots, u_m \in U(\mathbb{R}^N)$, and $a_1, \ldots, a_m \in \mathbb{R}$ such that

$$O := \bigcap_{i=1}^{m} \{x \in \Omega : u_i(x) < a_i\}$$

is a fine neighborhood of y, which does not intersect $E \setminus \{y\}$. Now

$$u_n'(x) := \frac{u_n(x) - u_n(y)}{a_n - u_n(y)} \in U(\mathbb{R}^N),$$

$$v := \sum_{n=1}^{m} u_n', \quad w := \min\{u_1', \ldots, u_m'\},$$

then if $x \in E \setminus \{y\}$, then $x \notin O$, so $u_n'(x) \geq 1$ for some n, hence $v \geq 1 + (m-1)w$ on $E \setminus \{y\}$, so

$$\liminf_{x \to y, x \in E} v(x) \geq 1 + (m-1)w(y) = 1 = v(y) + 1.$$

$\mu := -\Delta v$, $\mu_n := \mu \lfloor B(y, r_n)$ where $r_n \searrow 0$ is such that

$$G_{\Omega'}^{\mu_n}(y) < 2^{-n} \text{ (note that } \mu(\{y\}) = 0).$$

Let $u := \sum_{n=1}^{\infty} G_{\Omega'}^{\mu_n}$, then $u \in UP(\Omega')$, and $u(y) < 1$. But $v - G_{\Omega'}^{\mu_n}$ is harmonic on $B(y, r_n)$, so

$$\liminf_{x \to y, x \in E} G_{\Omega'}^{\mu_n}(x) \geq G_{\Omega'}^{\mu_n}(y) + 1 \geq 1.$$

So $u(x) \to \infty$ as $x \to y$ along E (Euclidean limit). □

Corollary 8.5. *If E is thin at y, then there is an open set $V \in E_{\mathbb{R}^N}$ such that $E \setminus \{y\} \subset V$, and V is thin at y.*

Proof. WLOG $y \in \overline{E}$. Then we may choose $u \in U(O)$ for some open (Euclidean) neighborhood of y and $\delta > 0$ such that $u(x) > u(y) + 1$ $\forall x \in (E \cap \overline{B}(y, \delta)) \setminus \{y\}$. Now

$$V := \{x \in O : u(x) > u(y) + 1\} \cup (O \setminus \overline{B}(y, \delta)) \in E_{\mathbb{R}^N}$$

and contains $(E \cap \overline{B}(y, \delta)) \setminus \{y\}$, and is thin at y. □

Theorem 8.6. *If Ω is Greenian, let $E \subset \Omega$, $y \in \Omega$ and $u \in UP(\Omega)$.*

(1) If E is non-thin at y, then

$$\hat{R}_u^{E \cap B(y,r)}(y) = u(y) \quad \forall r > 0.$$

(2) If E is thin at y and u is bounded from above in a neighborhood of y, then

$$\hat{R}_u^{E \cap B(y,r)}(y) \searrow 0 \text{ as } r \searrow 0.$$

Proof. (1): Let $r > 0$, $v := \hat{R}_u^{E \cap B(y,r)}$. Then $v = u$ on $E \cap B(y, r)$ except on a polar set F. If E is non-thin at y, then so is $(E \setminus F) \cap B(y, r)$. (Note that polar sets are everywhere thin). Hence $v(y) = u(y)$, by the fine continuity of u and v.

(2): $y \notin \overline{E \setminus \{y\}} \Rightarrow E \cap B(y, r) \subset \{y\}$ for all small enough r, so $\hat{R}_u^{E \cap B(y,r)} \equiv 0$ for these. If $y \in \overline{E \setminus \{y\}}$ and E is thin at y, then there is $w \in UP(\Omega)$ such that

$0 < w(y) < \infty$ and $w(x) \to \infty$ as $x \to y$ in E (Euclidean limit). If $\varepsilon > 0$, then since u is bounded in a neighborhood of y we may choose r' so small that $w < w(y)u/\varepsilon$ on $(E \cap B(y, r')) \setminus \{y\}$. So

$$\hat{R}_u^{E \cap B(y,r)} \leq \varepsilon w / w(y),$$

which gives

$$\hat{R}_u^{E \cap B(y,r)}(y) \leq \varepsilon$$

when $0 < r < r'$. □

Theorem 8.7. *If Ω is Greenian, and $E \subset \Omega$, $y \in \Omega$, then E is thin at y if and only if*

$$\hat{R}^E_{G_\Omega(\cdot,y)} \neq G_\Omega(\cdot, y).$$

If E is thin at y, then

$$-\Delta \hat{R}^E_{G_\Omega(\cdot,y)}(\{y\}) = 0.$$

Proof. By the above, if we WLOG assume Ω connected, $x_0 \neq y$ in Ω , and choose r_ε such that

$$\hat{R}^{E \cap B(y, r_\varepsilon)}_{G_\Omega(x_0,\cdot)} < \varepsilon G_\Omega(x_0, y).$$

Let

$$u_\varepsilon(x) := \hat{R}^{E \cap B(y,r_\varepsilon)}_{G_\Omega(\cdot,y)}(x), \quad x \in \Omega.$$

Then $u_\varepsilon(x) < \varepsilon G_\Omega(x_0, y)$, because

$$\hat{R}^A_{G_\Omega(x,\cdot)}(y) = \hat{R}^A_{G_\Omega(\cdot,y)}(x).$$

Now $-\Delta u_\varepsilon(\{y\}) < \varepsilon$. If we choose $C_\varepsilon > \sup_{x \in \Omega \setminus B(y, r_\varepsilon)} G_\Omega(x, y)$, then $u_\varepsilon + C_\varepsilon \geq G_\Omega(\cdot, y)$ q.e. on E, so

$$u_\varepsilon(x) + C_\varepsilon \geq \hat{R}^E_{G_\Omega(\cdot,y)}(x) \quad x \in \Omega.$$

It follows immediately that $-\Delta \hat{R}^E_{G_\Omega(\cdot,y)}(\{y\}) = 0$. This proves the second part, and the first part for the thin case. If E is non-thin at y the rest follows by the previous theorem. □

Definition 8.8. *A function $u : \Omega \to [-\infty, \infty]$ is said to peak at y if*

$$\sup\{u(x) : x \in \Omega \setminus B(y, r)\} < u(y) \quad \forall r > 0.$$

Theorem 8.9. *Let Ω be Greenian, $E \subset \Omega$, $u \in UP(\Omega)$. Suppose u peaks at $y \in \Omega$, and $u(y) < \infty$, then E is thin at y if and only if*

$$\hat{R}^E_u(y) < u(y).$$

Proof. If E is non-thin we already know that

$$\hat{R}^E_u(y) = u(y).$$

If E is thin at y choose $r' > 0$ such that

$$\hat{R}_u^{E \cap B(y,r')}(y) < u(y).$$

Next choose $c \in (0,1)$ such that $u(x) < cu(y)$ on $\Omega \setminus B(y, r')$,

$$v(x) := cu(y) + (1-c)\hat{R}_u^{E \cap B(y,r')}(x).$$

Then $v \geq u$ q.e. on E, so $v \geq \hat{R}_u^E$. Since $v(y) < u(y)$ the result follows. □

Theorem 8.10. *Let Ω be Greenian. Then there is a bounded continuous potential $u^{\#}$ on Ω such that a subset $E \subset \Omega$ is thin at y if and only if*

$$\hat{R}_{u^{\#}}^E(y) < u^{\#}(y).$$

Such $u^{\#}$ is said to determine thinness.

Proof. Let (K_n) be a sequence of closed balls in Ω such that $\{\mathrm{int}(K_n) : n \in \mathbb{N}\}$ is a base for E_Ω. For each n, let $u_n := R_1^{K_n}$, $u^{\#} := \Sigma_{n=1}^{\infty} 2^{-n} u_n$.
Then by regularity of ∂K_n and uniform convergence both u_n and $u^{\#}$ are continuous. Suppose $\hat{R}_{u^{\#}}^E(y) < u^{\#}(y)$. Since $\hat{R}_{u^{\#}}^E = u^{\#}$ on $E \setminus F$ for some polar F, and since $\{x \in \Omega : \hat{R}_{u^{\#}}^E(x) < u^{\#}(x)\}$ is a fine neighborhood of y which does not intersect $E \setminus F$, it follows that $E \setminus F$, and hence E, is thin at y. If E is thin at y, $E' := E \setminus \{y\}$. Let (K_{n_k}) be a decreasing subsequence of (K_n) such that

$$\bigcap_k K_{n_k} = \{y\}, \; v_1 := \Sigma_{k=1}^{\infty} 2^{-n_k} u_{n_k}, \; v_2 := u^{\#} - v_1,$$

then

$$\hat{R}_1^E + v_2 = u^{\#} \text{ q.e. on } E,$$

so

$$\hat{R}_{v_1}^E(y) < v_1(y).$$

But v_1 peaks at y, so

$$\hat{R}_{u^{\#}}^E(y) < u^{\#}(y).$$

□

Theorem 8.11. *(1) If $E \subset \mathbb{R}^N$, then $\{x \in E : E$ is thin at $x\}$ is polar.*
(2) A set is polar if and only if it is thin at each of its points.

Proof. (2): follows from (1) and the fact that polar sets are everywhere thin.
(1): WLOG assume E bounded. Let $E \subset \Omega$, Ω Greenian. Let $u^{\#}$ determine thinnes on Ω, then

$$\{x \in E : E \text{ is thin at } x\} = \{x \in E : \hat{R}_{u^{\#}}^E(x) < u^{\#}(x)\}$$

which is polar. □

Theorem 8.12. *(1) For any collection $\{\omega_a\}_{a \in I} \subset F_\Omega$ there is a countable subset $I' \subset I$ such that*

$$\left(\bigcup_{a \in I} \omega_a\right) \setminus \left(\bigcup_{a \in I'} \omega_a\right)$$

is polar.
(2) Any $\omega \in F_\Omega$ may be written as the union of an Euclidean F_σ and a polar set.

Proof. (1): WLOG assume Ω is Greenian, and let $u^{\#}$ determine thinness on Ω. Now let $F_a := \Omega \setminus \omega_a$ for $a \in I$, then we know that there is a countable subset $I' \subset I$ such that

$$\left(\inf_{a\in I} \hat{R}^{F_a}_{u^{\#}}\right)^{\wedge} = \left(\inf_{a\in I'} \hat{R}^{F_a}_{u^{\#}}\right)^{\wedge}.$$

Let $E = \bigcap_{a\in I'} F_a$, $y \in E$. If E is non-thin at y, and $b \in I$, then

$$\begin{aligned}\hat{R}^{F_b}_{u^{\#}}(y) &\geq \left(\inf_{a\in I} \hat{R}^{F_a}_{u^{\#}}\right)^{\wedge}(y) = \\ &= \left(\inf_{a\in I'} \hat{R}^{F_a}_{u^{\#}}\right)^{\wedge}(y) \geq \hat{R}^{E}_{u^{\#}}(y) = u^{\#}(y).\end{aligned}$$

So each F_b is non-thin at y. So

$$E \setminus \bigcap_{a\in I} F_a \subset \{y \in E : E \text{ is thin at } y\}$$

and hence is polar. Now

$$\left(\bigcup_{a\in I} \omega_a\right) \setminus \left(\bigcup_{a\in I'} \omega_a\right) = E \setminus \bigcap_{a\in I} F_a$$

so (1) is done.

(2): Follows from (1), because any set $\omega \in F_\Omega$ can be written as $\bigcup_{y\in\omega} \operatorname{int}(K_y)$, where $y \in K_y \subset \omega$, are Euclidean compacts. □

Lemma 8.13. *Let $y \in \mathbb{R}^N$ and let E_n be a sequence of sets which are thin at y. Then there is a set E which is thin at y and $r_n > 0$ such that $E_n \cap B(y, r_n) \subset E$ for all $n \in \mathbb{N}$.*

Proof. Let $n \in \mathbb{N}$. If y is not an Euclidean limit point of E_n, choose r_n such that $E_n \cap B(y, r_n) \subset \{y\}$, and $u_n := 0$. If y is a limit point of E_n, then take $u_n \in UP(\Omega)$, $r_n \in (0, 1)$ such that $u_n(y) < 2^{-n}$ and $u_n \geq 1$ on $(E \setminus \{y\}) \cap B(y, r_n)$.

$$E := \bigcup_{n\in\mathbb{N}} (E_n \cap B(y, r_n)),\ u := \Sigma_{n\in\mathbb{N}} u_n.$$

Now $u \in UP(B(y, 1))$. If y is a limit point of E, then at-least one $u_n \neq 0$, and

$$\liminf_{x\to y, x\in E} u(x) \geq 1 > u(y),$$

So E is thin at y. □

- $d(s, t) := |\tan^{-1}(s) - \tan^{-1}(t)|$ on $[-\infty, \infty] \times [-\infty, \infty]$.

$(\tan^{-1}(\pm\infty) = \pm\pi/2)$.

Theorem 8.14. *Let $f : O \setminus \{y\} \to [-\infty, \infty]$, where O is a fine neighborhood of y. Then f has fine limit l at y if and only if there is a set E thin at y such that*

$$\lim_{x\to y, x\in O\setminus E} f(x) = l.$$

(Euclidean limit.)

Proof. $\Rightarrow$: $E_n := \{x \in O \setminus \{y\} : d(f(x), l) \geq n^{-1}\}$ for $n \in \mathbb{N}$, and choose E as in the lemma above, then

$$\lim_{x\to y, x\in O\setminus E} f(x) = l.$$

$\Leftarrow$: If

$$\lim_{x\to y, x\in O\setminus E} f(x) = l,$$

then $\forall \varepsilon > 0\ \exists r_\varepsilon > 0$ such that

$$d(f(x), l) < \varepsilon \quad \forall x \in B(y, r_\varepsilon) \setminus (E \cup \{y\}).$$

Since $B(y, r_\varepsilon) \setminus (E \cup \{y\})$ is a fine neighborhood of y, the function f has fine limit l at y. □

Theorem 8.15. *Let Ω be Greenian, $u \in UP(\Omega)$, and $y \in \Omega$, then*

$$F_\Omega - \lim_{x\to y} \frac{u(x)}{G_\Omega(x,y)} = -\Delta u(\{y\}) = \liminf_{x\to y} \frac{u(x)}{G_\Omega(x,y)}.$$

Proof. By subtracting $-\Delta u(\{y\}) G_\Omega(\cdot, y)$ from u we may WLOG assume $-\Delta u(\{y\}) = 0$. Let

$$O_\varepsilon := \{x \in \Omega \setminus \{y\} : u(x)/G_\Omega(x,y) < \varepsilon\} \cup \{y\} \quad (\varepsilon > 0).$$

Let $\varepsilon > 0$ and assume $\Omega \setminus O_\varepsilon$ is non-thin at y. Then

$$u \geq \hat{R}_u^{\Omega\setminus O_\varepsilon} \geq \varepsilon \hat{R}_{G_\Omega(\cdot,y)}^{\Omega\setminus O_\varepsilon} = \varepsilon G_\Omega(\cdot, y).$$

But this would give $-\Delta u(\{y\}) \geq \varepsilon$, and hence a contradiction. Since

$$u \geq -\Delta(\{y\}) G_\Omega(\cdot, y) \text{ on } \Omega$$

the second equality follows as-well. □

Theorem 8.16. *Let y be a limit point of E, then the following are equivalent:*

(1) E is thin at y.

(2) There is a superharmonic function u on a neighborhood of y such that

$$-\Delta u(\{y\}) < \liminf_{x\to y, x\in E} \frac{u(x)}{\Phi(x-y)}.$$

(3) For every Greenian Ω which contains y there is $u \in UP(\Omega)$ such that

$$\lim_{x\to y, x\in E} \frac{u(x)}{G_\Omega(x,y)} = \infty.$$

Proof. Suppose (1) holds. Let $\Omega \ni y$, be Greenian and $x_0 \in \Omega \setminus \{y\}$. (We assume Ω connected WLOG.) Take $r_n \searrow 0$ such that

$$\hat{R}^{E\cap B(y,r_n)}_{G_\Omega(\cdot,x_0)}(y) < 2^{-n},$$

$$v(x) := \Sigma_{n=1}^{\infty} \hat{R}^{E\cap B(y,r_n)}_{G_\Omega(\cdot,y)}(x) \quad (x \in \Omega).$$

Then $v(x_0) < 1$, so $v \in UP(\Omega)$. Also, $v \geq nG_\Omega(\cdot, y)$ on $(E \cap B(y, r_n)) \setminus F_n$, where F_n is some polar set. Let $w \in UP(\Omega)$ where $w = \infty$ on $\bigcup_{n\geq 1} F_n$, and let $u := v+w$, then (3) holds.

Obviously (3) implies (2).

Suppose now that (2) holds. Let B_0 be an open ball with $y \in B_0$, and on which u is defined and bounded below. WLOG assume $u \in UP(B_0)$. Since

$$\frac{\Phi(\cdot - y)}{G_{B_0}(\cdot, y)}$$

has limit 1 at y we can choose positive numbers l and r such that

$$\frac{u(x)}{G_{B_0}(\cdot, y)} > l > -\Delta u(\{y\}) \quad (x \in (E \setminus \{y\}) \cap B(y, r)).$$

So $(E \setminus \{y\}) \cap B(y, r)$, and hence E, is thin at y. □

Theorem 8.17. *Let Ω be Greenian and $y \in \partial\Omega \setminus \{\infty\}$, then y is regular for the Dirichlet problem on Ω if and only if $\mathbb{R}^N \setminus \Omega$ is non-thin at y.*

Proof. If $\mathbb{R}^N \setminus \Omega$ is non-thin at y, let $u(x) := 1 - |x - y|^2$ on $B(y, 1)$, then $-\Delta u = 2N$, so $u \in UP(B(y, 1))$. $v := u - \hat{R}^{B(y,1)\setminus\Omega}_u$ (reduction on $B(y, 1)$), belongs to $UP(B(y, 1) \cap \Omega)$. Note that $v > 0$, because otherwise $v \equiv 0$ on some component of $B(y, 1) \cap \Omega$, which contradicts that $-\Delta v = 2N$ there. Since u peaks at y it follows that

$$\hat{R}^{B(y,1)\setminus\Omega}_u(y) = u(y) = 1,$$

and it easily follows that v is a local weak barrier at y for Ω, so y is regular.

Conversely, if y is regular, choose r' so that $\Omega' = \Omega \cup B(y, r')$ is Greenian (as before, we don't prove that r' exists in general, but remark that it obviously always does for $N \geq 3$, and in the case $N = 2$ if $\mathbb{R}^2 \setminus \overline{\Omega} \neq \emptyset$.) For $r \in (0, r')$ let

$$f_r(x) := \begin{cases} 1 & x \in \partial\Omega \cap B(y, r) \\ 0 & x \in \partial\Omega \setminus B(y, r) \end{cases},$$

then

$$\hat{R}^{B(y,r)\setminus\Omega}_1(x) = R^{B(y,r)\setminus\Omega}_1(x) \geq H^\Omega_{f_r}(x) \to 1 \quad (x \to y, x \in \Omega, 0 < r < r'),$$

(the reductions are on Ω'), by the regularity of y. Since $\hat{R}^{B(y,r)\setminus\Omega}_1 = 1$ on $B(y, r)\setminus\Omega$ except for a polar set we conclude that $\hat{R}^{B(y,r)\setminus\Omega}_1(y) = 1$, so $\mathbb{R}^N \setminus \Omega$ is non-thin at y. □

9. The Martin Boundary

In this section we will let $\Omega \subset \mathbb{R}^N$ be a fixed Greenian domain, and $a \in \Omega$ a fixed point. Let

$$d\mu = \phi dm,$$

where $\phi \in C_c^\infty P(\Omega) \setminus \{0\}$, and define

$$M_\mu(x,y) := \frac{G_\Omega(x,y)}{G_\Omega^\mu(y)} \quad (x,y) \in \Omega \times \Omega.$$

Also let

$$M := \{M_\mu(x,\cdot) : x \in \Omega\} \subset C(\Omega, \overline{\mathbb{R}}).$$

The $M-$compactification

$$\hat{\Omega} := \Omega \cup \partial^M \Omega,$$

of Ω is called the Martin compactification of Ω, and $\partial^M\Omega$ the Martin boundary. If we on $C(\Omega, \overline{\mathbb{R}})$ introduce the metric

$$d(f,g) := || \arctan(f) - \arctan(g)||_s,$$

where arctan is defined as $\pm\pi/2$ at $\pm\infty$, then for A countable and dense in Ω we see that

$$\{M_\mu(x,\cdot) : x \in A\}$$

is countable and dense in M, and hence $\hat{\Omega}$ is metrizable. We now concider $M_\mu(x,\cdot)$ as extended to $\hat{\Omega}$, and use the same notation for the extension. By definition of $\partial^M\Omega$ we have that

$$M_\mu(x,y_1) = M_\mu(x,y_2) \quad \forall x \in \Omega \Leftrightarrow y_1 = y_2,$$

for all $y_1, y_2 \in \partial^M\Omega$, and it is actually easy to see that the same is true for all $y_1, y_2 \in \hat{\Omega}$. By Harnack's inequality we have for $K \subset D \subset\subset \Omega$, where K is compact and D open that there is a constant C such that

$$G_\Omega(x,y) \leq C G_\Omega^\mu(y) \quad \forall x \in K, y \in D^c.$$

If now $y_n \in \Omega$, $y_n \to y \in \partial^M\Omega$, then this gives us that every subsequence has a subsequence such that $M_\mu(\cdot, y_n)$ converges to a positive harmonic function h on Ω (since K, D was arbitrary) with

$$\int h d\mu = 1.$$

Since also $M_\mu(\cdot,y_n) \to M_\mu(\cdot,y)$ it follows that $M_\mu(\cdot,y)$ is a positive harmonic function in Ω with

$$\int M_\mu(\cdot,y) d\mu = 1 \quad \forall y \in \partial^M\Omega.$$

If on the other hand y_n is a sequence in $\hat{\Omega}$ such that $M_\mu(\cdot,y_n)$ converges locally uniformly to a positive harmonic function h, then since for every convergent subsequence $y_{n_j} \to y \in \hat{\Omega}$ we must have $M_\mu(\cdot,y) = h$ it follows also that this y must lie on $\partial^M\Omega$ and is uniquely determined, and hence $y_n \to y$ for the unique $y \in \partial^M\Omega$ such that

$$h \equiv M_\mu(\cdot, y).$$

Hence we may conclude that (the case $y \in \Omega$ is trivial) $y_n \to y$ in $\hat{\Omega}$ if and only if

$$M_\mu(\cdot, y_n) \to M_\mu(\cdot, y)$$

locally uniformly on $\Omega \setminus \{y\}$.

Theorem 9.1. *If η is a positive Radon measure with compact support in Ω, and we define*

$$M_\eta(x,y) := \frac{G_\Omega(x,y)}{G_\Omega^\eta(y)} \quad (x,y) \in \Omega \times (\Omega \setminus \mathrm{supp}(\eta)),$$

then M_η has a unique extension to a continuous function on $\Omega \times (\hat{\Omega} \setminus \mathrm{supp}(\eta))$.

Proof.

$$M_\eta(x,y) = \frac{G_\Omega(x,y)}{G_\Omega^\eta(y)} = \frac{G_\Omega(x,y)}{G_\Omega^\mu(y)} \frac{G_\Omega^\mu(y)}{G_\Omega^\eta(y)} \text{ on } \Omega \times (\Omega \setminus \mathrm{supp}(\eta)).$$

If $y_n \in \Omega$ and $y_n \to y \in \partial^M\Omega$, then

$$\frac{G_\Omega(\cdot, y_n)}{G_\Omega^\mu(y_n)} \to M_\mu(\cdot, y_n)$$

locally uniformly, so

$$\frac{G_\Omega^\eta(y_n)}{G_\Omega^\mu(y_n)} \to \int M_\mu(x,y)d\eta(x) =: C.$$

C is a positive finite constant, which only depends on y and not the particular sequence y_n, hence

$$M_\eta(\cdot, y_n) \to \frac{M_\mu(\cdot, y)}{C}$$

for every sequence y_n in Ω converging to y. Therefore $M_\eta(x,y)$ has a well-defined extension to $\Omega \times (\hat{\Omega} \setminus \mathrm{supp}(\eta))$. It is also clear that it is continuous in each variable separately, but we still need to prove the joint continuity in both variables. So suppose $(x_n, y_n) \to (x,y)$ in $\Omega \times (\hat{\Omega} \setminus \mathrm{supp}(\eta))$. If $x \neq y$, then we know that $M_\eta(\cdot, y_n) \to M_\eta(\cdot, y)$ uniformly on $B(x, \varepsilon)$ for $\varepsilon > 0$ small, so in particular

$$M_\eta(x_n, y_n) \to M_\eta(x,y)$$

in this case. If $x = y$, then they belong to $\Omega \setminus \mathrm{supp}(\eta)$ by assumption, and we may choose $\varepsilon > 0$ such that

$$B := B(y, \varepsilon) \subset\subset \Omega \setminus \mathrm{supp}(\eta),$$

and hence

$$G_\Omega(x,y) \geq \Phi(x-y) - C_0 \text{ on } B \times B$$

for some constant C_0. So given $C > 0$ there is a constant $\delta > 0$ such that for all $x', y' \in B$ with $|x' - y'| < \delta$ we have

$$G_\Omega(x', y') > C,$$

and since G_Ω^η is uniformly bounded on B it is therefore easy to see that this implies that

$$M_\eta(x_n, y_n) \to \infty = M_\eta(y, y),$$

and the proof is done. □

From now on we will forget the measure μ and look at

$$M(x, y) := M_{\delta_a}(x, y) = \frac{G_\Omega(x, y)}{G_\Omega(a, y)},$$

where $a \in \Omega$ is fixed. In the sequel reductions and other similar entities are if otherwise not indicated w.r.t. to $\hat{\Omega}$. Furthermore $\partial\Omega$ denotes the Euclidean boundary if otherwise is not stated.

Lemma 9.2. *If $h \in HP(\Omega)$ and $E \subset \Omega$ is a subset such that $a \notin \partial E$, then there is a measure $\nu_{h,E}$ on $\hat{\Omega}$ with support on ∂E (where the boundary is w.r.t. $\hat{\Omega}$) such that*

$$\hat{R}_h^E(x) = \int_{\hat{\Omega}} M(x, y) d\nu_{h,E}(y) \quad (x \in \Omega),$$

and

$$\nu_{h,E}(\hat{\Omega}) = \hat{R}_h^E(a).$$

Proof. If $\overline{E} \subset \Omega$ is compact, then we know that

$$\hat{R}_h^E(x) = G_\Omega^\mu(x) = \int_{\partial E} G_\Omega(x, y) d\mu(y) = \int_{\partial E} M(x, y) d\nu_{h,E}(y),$$

where μ is some Radon-measure, and we put $d\nu_{h,E} = G_\Omega(a, \cdot) \cdot d\mu$. For the general case we let $\Omega_n \nearrow \Omega$ (c.c.) and such that $a \in \Omega_n$ for every n. Let $E_n := E \cap \Omega_n$, then the above gives a measure ν_{h,E_n} for each E_n instead of E, and weak*-compactness of this class of measures on $\hat{\Omega}$ now gives the result. □

Corollary 9.3. *If $h \in HP(\Omega)$, then there is a Radon-measure μ on $\partial^M\Omega$ such that*

$$h = \int_{\partial^M\Omega} M(\cdot, y) d\mu(y),$$

and $\mu(\partial^M\Omega) = h(a)$.

Definition 9.4. *$h \in HP(\Omega) \setminus \{0\}$ is called minimal if (and only if) for every $h' \in HP(\Omega)$ such that $0 \le h' \le h$ on Ω implies that there is a constant $t \in [0, 1]$ with $h' \equiv th$.*

We define the sets:

$$\Delta_1 := \{y \in \partial^M\Omega : M(\cdot, y) \textit{ is minimal }\},$$

and

$$\Delta_0 := \partial^M\Omega \setminus \Delta_1.$$

Note that if $E \subset \partial^M\Omega$, $h \in HP(\Omega)\setminus\{0\}$ and $R_h^E = ch$, where c is a constant, then

$$ch = R_{ch}^E = cR_h^E = c^2h,$$

so $c \in \{0,1\}$. In particular if h is minimal, then $R_h^E = h$ or $R_h^E = 0$. We therefore say that $y \in \partial^M\Omega$ is a pole of $h \in HP(\Omega)$ if $R_h^{\{y\}} = h$. (This definition can of-course be made for any metric compactification.)

Lemma 9.5. *(1) If $h', h \in HP(\Omega)$ where $0 < h' \leq h$ and y is a pole of h, then y is also a pole of h'.*
(2) A minimal $h \in HP(\Omega)$ has at-least one pole.
(The above is true for any boundary $\partial\Omega$, and the proof does not depend on which it is.)

Proof. (1):

$$R_{h'}^{\{y\}} + R_{h-h'}^{\{y\}} \geq R_h^{\{y\}} = h = h' + h - h' \geq R_{h'}^{\{y\}} + R_{h-h'}^{\{y\}},$$

so $R_{h'}^{\{y\}} = h'$.
(2): $R_h^{\partial^M\Omega} = h$ by the minimum principle. If $B_1, \ldots, B_m$ covers $\partial^M\Omega$ we get

$$0 < h = R_h^{\partial^M\Omega} \leq \Sigma_{j=1}^m R_h^{B_j \cap \partial^M\Omega},$$

and hence some $R_h^{B_j\cap\partial^M\Omega}$ must be non-zero, and by the above remark therefore equal to h since it is assumed to be minimal. Also, by definition, if $R_h^{\{y\}} = 0$, then given $\varepsilon > 0$ there is an open neighborhood B of y in $\hat{\Omega}$ such that $R_h^{B\cap\partial^M\Omega}(a) \leq \varepsilon$, and hence for $\varepsilon < h(a)$, for the same reason as above, it must be 0. So the set $\{y \in \partial^M\Omega : R_h^{\{y\}} = h\}$ is nonempty. Since otherwise we would get a covering as above, by compactness. □

Theorem 9.6. *Let $y \in \partial^M\Omega$ and $h \in HP(\Omega)$, then the following are equivalent:*
(1) y is a pole of h.
(2) $y \in \Delta_1$, and $h \equiv cM(\cdot, y)$ for some constant $c > 0$. In particular h can not have more than one pole on $\partial^M\Omega$.

Proof. Let $E_m := \{x \in \Omega : d(x,y) < 1/m\}$ for large m. Then

$$\hat{R}_h^{E_m}(x) = \int_{\hat{\Omega}} M(x,y)d\nu_{h,E_m} \quad (x \in \Omega).$$

By weak*-compactness some subsequence (we use the same notation for this) has $\nu_{h,E_m} \rightharpoonup \mu$, for some measure μ. But $\text{supp}(\mu) = \{y\}$ trivially, and $\mu(\{y\}) = h(a)$, so $h = h(a)M(\cdot, y)$. Since this holds for all h with poles at y it follows that if $0 < h' \leq h$, then

$$h' = h'(a)M(\cdot,y) = \frac{h'(a)}{h(a)}h,$$

so h is minimal. Hence we have proved that (1) implies (2). Conversely, if $y \in \Delta_1$ then this means by definition that $M(\cdot, y)$ is minimal, and by the above $M(\cdot, y)$

can only have a pole at y (since $M(\cdot,y) \equiv M(\cdot,z) \Rightarrow y = z$), and therefore also y is a pole of $M(\cdot,y)$ (by the previous lemma), and hence of h if it is of the form $cM(\cdot,y)$. □

From the above it is clear that the minimal harmonic functions on Ω has precisely one pole on $\partial^M\Omega$, which also belongs to Δ_1. Furthermore every minimal harmonic function is given by $cM(\cdot,y)$ for some constant c and $y \in \partial^M\Omega$.

Corollary 9.7. *If $y \in \Delta_1$, $E \subset \partial^M\Omega$, then*

$$R^E_{M(\cdot,y)} = \begin{cases} M(\cdot,y) & y \in E \\ 0 & y \notin E \end{cases}$$

Proof. The $y \in E-$case is obvious. Now suppose $K \subset \partial^M\Omega \setminus \{y\}$ is compact, then $R^K_{M(\cdot,y)} = 0$, because we have for each $z \in K$ $R^{\{z\}}_{M(\cdot,y)} = 0$, and as before this implies $R^{B\cap\partial^M\Omega}_{M(\cdot,y)} = 0$ for some open neighborhood of z in $\hat{\Omega}$, so the claim follows by compactness. But since we have (with $h = M(\cdot,y)$ for instance)

$$R^E_h(x)/h(x) = \overline{H}^h_{\chi_E}(x),$$

and we know that $\overline{H}^h_{\chi_{K_n}} \nearrow \overline{H}^h_{\partial^M\Omega\setminus\{y\}}$ if $K_n \nearrow \partial^M\Omega \setminus \{y\}$ are compacts, it follows that

$$0 = R^{K_n}_{M(\cdot,y)} \nearrow R^{\partial^M\Omega\setminus\{y\}}_{M(\cdot,y)} = 0.$$

Hence, if $E \subset \partial^M\Omega \setminus \{y\}$ we get $R^E_{M(\cdot,y)} = 0$. □

Proposition 9.8. *(1) If $h(x) = \int M(x,y)d\mu(y)$ where μ is a Radon-measure with* $\operatorname{supp}(\mu) \subset \partial^M\Omega$, *then for ever $E \subset \partial^M\Omega$ compact we have*

$$R^E_h(x) = \int R^E_{M(\cdot,y)}(x)d\mu(y) \quad \forall x \in \Omega.$$

(2) If $h \in HP(\Omega) \setminus \{0\}$ and $E \subset \partial^M\Omega$ is compact, then there is a measure $\nu_{h,E}$ with support in E such that

$$R^E_h(x) = \int_E M(x,y)d\nu_{h,E}(y) = \int_E R^E_{M(\cdot,y)}(x)d\nu_{h,E}(y) \quad \forall x \in \Omega.$$

(3) For $y \in \partial^M\Omega$: $y \in \Delta_1$ if and only if $R^{\{y\}}_{M(\cdot,y)} = M(\cdot,y)$, and $y \in \Delta_0$ if and only if $R^{\{y\}}_{M(\cdot,y)} = 0$.

Proof. (1): If A is a relatively closed subset of Ω, then

$$\begin{aligned}
&\int_{\partial^M\Omega} R^A_{M(\cdot,y)}(x)d\mu(y) = \int_{\partial^M\Omega}\left(\int_{\Omega\cap\partial A} M(z,y)d\nu_x(\partial(\Omega\setminus A),z)\right)d\mu(y) \\
= &\int_{\Omega\cap\partial A}\left(\int_{\partial^M\Omega} M(z,y)d\mu(y)\right)d\nu_x(\partial(\Omega\setminus A),z) \\
= &\int_{\Omega\cap\partial A} h(z)d\nu_x(\partial(\Omega\setminus A),z) = R^A_h(x),
\end{aligned}$$

where we have used $\nu_x(\partial(\Omega \setminus A), \cdot)$ to denote the harmonic measure on the Euclidean boundary of $\Omega \setminus A$. Now the rest follows by taking $E_k \supset \text{int}(E_k) \supset E$ in $\hat{\Omega}$ to be compacts, $A_k := E_k \cap \Omega$, in such a way that E_k decreases to E. Then $R_h^{A_k} \searrow R_h^E$, and using monotone convergence gives the result.
(2): With the same notation as above we know that

$$\hat{R}_h^{A_k}(x) = \int_{\hat{\Omega}} M(x, y) d\nu_k(y)$$

for some ν_k on ∂A_k (boundary in $\hat{\Omega}$), and

$$\nu_k(\hat{\Omega}) = \hat{R}_h^{A_k}(a) \leq h(a).$$

By weak*-compactness the sequence clusters at some $\nu_{h,E}$ with support on E. Since

$$\hat{R}_h^{A_k} \to R_h^E \text{ on } \Omega \text{ as } k \to \infty,$$

the first equality follows. With $w = R_h^E$ we get

$$R_h^E(x) = R_w^E(x) = \int_E R_{M(\cdot,y)}^E(x) d\nu_{h,E}(y)$$

by (1).
(3): Let $E = \{y\}$ in (2) to get $R_{M(\cdot,y)}^{\{y\}} = cM(\cdot, y)$ for some constant $c \in \{0, 1\}$. If $c = 1$, then y is a pole of $M(\cdot, y)$, so $y \in \Delta_1$, and the converse has already been proved. □

Theorem 9.9. *The set Δ_0 is an F_σ, and $R_h^{\Delta_0} = 0$ $\forall h \in HP(\Omega)$. That is Δ_0 is $h-$harmonic measure null for every such $h > 0$.*

Proof. Let $\{w_k\}_{k=1}^\infty$ be a base for the Martin topology on $\hat{\Omega}$, and for each k, let

$$v_k := \{y \in \partial^M\Omega : R_{M(\cdot,y)}^{w_k}(a) \leq 1/2\}.$$

Let $y_m \in v_k$ converge to y_0 in the Martin-topology, and also let $F_j \nearrow (w_k \cap \Omega) \setminus \{a\}$ be compacts, then

$$R_{M(\cdot,y_0)}^{F_j}(a) = \int_{\Omega \cap \partial F_j} M(z, y_0) d\nu_a(\partial(\Omega \setminus F_j), z) \leq$$

$$\begin{aligned} &\leq \liminf_{m\to\infty} \int_{\Omega \cap \partial F_j} M(z, y_m) d\nu_a(\partial(\Omega \setminus F_j), z) = \liminf_{m\to\infty} R_{M(\cdot,y_m)}^{F_j}(a) \\ &\leq \liminf_{m\to\infty} R_{M(\cdot,y_m)}^{w_k}(a) \leq 1/2, \end{aligned}$$

so as $j \to \infty$ we get $y_0 \in v_k$, so v_k is closed. If $E \subset v_k \cap w_k$ is compact, then

$$R_{M(\cdot,y)}^E(a) \leq R_{M(\cdot,y)}^{w_k}(a) \leq 1/2 \quad \forall y \in E.$$

Also, if $h \in HP(\Omega)$, then

$$R_h^E(a) = \int_E M(a, y) d\nu_{h,E}(y) = \int_E R_{M(\cdot,y)}^E(a) d\nu_{h,E}(y),$$

so $R_h^E(a) = \nu_{h,E}(E) \leq \frac{1}{2}\nu_{h,E}(E)$, hence $R_h^E(a) = 0$, so $R_h^E \equiv 0$. If on the other hand $y \in \Delta_0$, then $R_{M(\cdot,y)}^{\{y\}} = 0$, so $y \in v_k \cap w_k$ for some k, so

$$\Delta_0 = \bigcup_{k=1}^{\infty}(v_k \cap w_k),$$

which is an F_σ. Finally, if $h \in HP(\Omega)$, then

$$R_h^{\Delta_0} \leq \Sigma_{k=1}^{\infty} R_h^{v_k \cap w_k} = 0.$$

□

Note that the above implies that given $h \in HP(\Omega)\setminus\{0\}$ there is a $u \in UP(\Omega)$ such that u/h has limit ∞ on Δ_0. Given such u, let

$$F_n := \left\{ y \in \partial^M\Omega : \liminf_{x \to y, x \in \Omega} u(x)/h(x) \leq n \right\},$$

and

$$F := \bigcup_{n=1}^{\infty} F_n.$$

Then each F_n is compact, and $F_n \subset \Delta_1$. We also have that

$$h \geq R_h^{\Delta_1} \geq R_h^{F_n} \nearrow R_h^F.$$

Furthermore

$$R_u^{\partial^M\Omega\setminus F} \geq nR_h^{\partial^M\Omega\setminus F},$$

so $R_h^{\partial^M\Omega\setminus F} = 0$. Therefore $h = R_h^{\partial^M\Omega} \leq R_h^F + R_h^{\partial^M\Omega\setminus F} = R_h^F \leq h$.

But we also know that for each F_n there is a Radon-measure ν_{h,F_n} supported by F_n such that $R_h^{F_n} = \int_{F_n} M(\cdot,y)d\nu_{h,F_n}$. But this sequence then has a weak*-cluster point, say μ_h, and $\mu_h(\Delta_0) = 0$. But we then have

$$R_h^F = \int M(\cdot,y)d\mu_h(y).$$

But also, for any compact $K \subset \Delta_1$, then

$$\mu_h(K) = \int_K M(a,y)d\mu_h(y) = \int_{\partial^M\Omega} R_{M(\cdot,y)}^K(a)d\mu_h(y) = R_h^K(a).$$

So there can only be one such measure. Finally, before we can state all this as a theorem, we note that

$$\nu_x^h(\partial^M\Omega, A) = R_h^A(x)/h(x) = \int_A \frac{M(x,y)}{h(x)}d\mu_h(y),$$

so

$$d\nu_x^h(\partial^M\Omega,\cdot) = \frac{M(x,\cdot)}{h(x)}d\mu_h.$$

Theorem 9.10 (Martin representation). *If $h \in HP(\Omega)\setminus\{0\}$, then there is a unique Radon-measure μ_h carried by Δ_1 such that*

$$h(x) = \int M(x,y)d\mu_h(y) \quad \forall x \in \Omega.$$

We also have that:
(1) $\nu_x^h(\partial^M\Omega,\cdot) = \frac{M(x,\cdot)}{h(x)}\mu_h$. So in particular (since this is a Radon-measure) $\partial^M\Omega$ is universally resolutive (i.e. $h-$resolutive for every h).
(2) If $k \in H_hL^\infty(\Omega)$, then there is a function $f \in L^\infty(\nu_x^h(\partial^M\Omega,\cdot))$ such that

$$k(x) = \int f(y)d\nu_x^h(\partial^M\Omega, y) \quad \forall x \in \Omega.$$

So, in particular, $\partial^M\Omega$ is universally internally resolutive.

Proof. We have already proved everything before the theorem, except (2). To do this we note that we may WLOG assume that $0 < k < 1$ so $kh \in HP(\Omega)$ and $kh < h$. By uniqueness of representation we have immediately that $\mu_h = \mu_{kh} + \mu_{h-kh}$, so in particular $\mu_{kh} \leq \mu_h$. Now just let

$$f := \frac{d\mu_{kh}}{d\mu_h}$$

be the Radon-Nikodym derivative, then $0 \leq f \leq 1$ μ_h-a.e. and we are done. □

Remark: *The only Radon-measure μ on $\partial^M\Omega$ such that*

$$M(x,y) = \int M(x,z)d\mu(z)$$

for $y \in \Delta_1$ is $\mu = \delta_1$, because if $z \in \operatorname{supp}(\mu)$,let $E_k := \{y \in \partial^M\Omega : d(y,z) < k^{-1}\}$. By minimality we have

$$(\mu(E_k))^{-1}\int_{E_k} M(x,t)d\mu(t) = C(k,z)M(x,y) \quad \forall x \in \Omega.$$

With $x = a$ we get $C(k,z) = 1$ for each k, so as $k \to \infty$ we get $M(x,z) = M(x,y)$, and hence $z = y$.

• We will need that if $u \in UP(\Omega)$, then there is a sequence of potentials $u_n \in C^\infty(\Omega)$ with $u_n \nearrow u$, which is easy to prove by taking reductions and mollification.

Definition 9.11. *If $E \subset \Omega$, let $\nu_y^E := \operatorname{Bal}(\delta_y, E) := -\Delta\hat{R}^E_{G_\Omega(\cdot,y)}$, called the balayage of δ_y onto E.*

Note that if E is relatively closed in Ω, $y \in \omega := \Omega \setminus E$, then

$$(*)\ \hat{R}^E_{G_\Omega(\cdot,y)}(x) = \hat{R}^E_{G_\Omega(x,\cdot)}(y) = \int_{\partial\omega\cap\Omega} G_\Omega(x,z)d\nu_y(\partial\omega, z),$$

where we again have used $\nu_y(\partial\omega,\cdot)$ to denote the harmonic measure on the Euclidean boundary of ω at y.

Theorem 9.12. *If $E \subset \Omega$ and $u \in UP(\Omega)$, then*

$$\hat{R}_u^E(y) = \int u(x)d\nu_y^E(x).$$

Proof. If E is compact and non-thin at each of its points, then we know that

$$\hat{R}^E_{G_\Omega(\cdot,y)} = G_\Omega(\cdot, y) \quad \forall y \in E,$$

so $\nu_y^E = \delta_y$, and if $y \in \Omega \setminus E$ we obtain the result from $(*)$.

If $U \subset \Omega$ is open we can take such $E_k \nearrow U$, and it is easy to see that this gives the result for U. If E is arbitrary and we put

$$v(y) := \int \hat{R}^E_{G_\Omega(x,\cdot)}(y)d\mu(x)$$

(where we WLOG assume $u = G_\Omega^\mu$ is continuous) the function v is superharmonic and non-negative. If E is non-thin at y, then

$$\hat{R}^E_{G_\Omega(x,\cdot)}(y) = G_\Omega(x, y) \quad \forall x \in \Omega,$$

so $v = G_\Omega^\mu$ q.e. on E, and hence $v \geq \hat{R}^E_{G_\Omega^\mu}$. But if $E \subset U \subset \Omega$, U open, then

$$R^U_{G_\Omega^\mu} = \int \hat{R}^U_{G_\Omega(z,\cdot)}(y)d\mu(z) \geq v,$$

so by continuity of G_Ω^μ we get

$$\hat{R}^E_{G_\Omega^\mu}(y) = v(y) = \iint G_\Omega(x,z)d\nu_y^E(z)d\mu(x) = \int G_\Omega^\mu d\nu_y^E.$$

□

Corollary 9.13. *Let $E \subset \Omega$, and $h \in HP(\Omega) \setminus \{0\}$, μ_h be the measure representing h on Δ_1, then*

$$\hat{R}_h^E(x) = \int_{\Delta_1} \hat{R}^E_{M(\cdot,y)}(x)d\mu_h(y) \quad \forall x \in \Omega.$$

Furthermore, if $\hat{R}^E_{M(\cdot,y)}(x)$ is a potential for μ_h-a.e. $y \in \partial^M\Omega$, then $\hat{R}_h^E$ is a potential.

Proof. The first part is immediate from the previous lemma and Fubini's theorem. Let $\Omega_n \nearrow \Omega$ (c.c.), then with

$$v_y(x) := \int M(z,y)d\nu_x^E(z)$$

and

$$v := \hat{R}_h^E,$$

then

$$H^{\Omega_n}_{v_y}(x) = \int \left(\int_{\Delta_1} \hat{R}^E_{M(\cdot,y)}(x)d\mu_h(y) \right) d\nu_x(\partial\Omega_n, \cdot) =$$

$$= \int \left(\int_{\Delta_1} v_y d\mu_h(y) \right) d\nu_x(\partial\Omega_n, \cdot) = \int_{\Delta_1} H^{\Omega_n}_{v_y}(x)d\mu_h(y) \to 0$$

for μ_h−a.e. y, since v_y is a potential μ_h-a.s. (Above we have used $H^{\Omega_n}_{\cdot}$ to denote the PWB-function on Ω_n.) □

Lemma 9.14. *If $y \in \Delta_1$, and w is a Martin topology neighborhood of y, then*

$$\hat{R}^{\Omega\setminus w}_{M(\cdot,y)}$$

is a potential.

Proof. WLOG $a \in \Omega \setminus \overline{w}$. There is a measure ν on $\hat{\Omega} \setminus w$ such that

$$\hat{R}^{\Omega\setminus w}_{M(\cdot,y)}(x) = \int_{\hat{\Omega}\setminus w} M(x,z)d\nu(z) =$$

$$= \int_{\Omega\setminus w} G_\Omega(x,z)\frac{d\nu(z)}{G_\Omega(a,z)} + \int_{\partial^M\Omega\setminus w} M(x,z)d\nu(z).$$

But $\int_{\partial^M\Omega\setminus w} M(x,z)d\nu(z) \leq \hat{R}^{\Omega\setminus w}_{M(\cdot,y)}(x) \leq M(x,y)$ implies that this integral is zero. □

Definition 9.15. *If $E \subset \Omega$, then we may write*

$$\hat{R}^{E}_{M(\cdot,y)} = G^\mu_\Omega + \int_{\Delta_1} M(\cdot,y)d\nu(z),$$

for μ, ν uniquely determined Radon-measures. If $\nu(\{y\}) = 0$, then we say that E is minimally thin at y (w.r.t. Ω).

Note that if $y \in \Delta_1$, then $\operatorname{supp}(\nu) \subset \{y\}$, so E is minimally thin at y if and only if $\hat{R}^{E}_{M(\cdot,y)}$ is a potential in this case.

If on the other hand $y \in \Delta_0$, then $\nu(\{y\}) \leq \nu(\Delta_0) = 0$, so any set E is minimally thin at y, and we therefore get:

$$\Delta_0 = \{y \in \partial^M\Omega : \Omega \text{ is minimally thin at } y\}.$$

Also note that if E is polar, then $\hat{R}^{E}_{M(\cdot,y)} \equiv 0$, so E is minimally thin at every $y \in \partial^M\Omega$.

Lemma 9.16. *Let $E_1, \ldots, E_m \subset \Omega$, and $y \in \Delta_1$.*
(1) If $E_1 \subset E_2$ and E_2 is minimally thin at y, then so is E_1.
(2) If $E_1, \ldots, E_m$ are minimally thin at y, then so is $\bigcup_{i=1}^m E_i$.
(3) If E_1 is minimally thin at y, then there is an open set $W \subset \Omega$ such that $E_1 \subset W$ and W is minimally thin at y.

Proof. (1) is trivial.
(2) follows since if $\hat{R}^{E_k}_{M(\cdot,y)}$ is a potential for each k, and we trivially have $\Sigma_{k=1}^m \hat{R}^{E_k}_{M(\cdot,y)} \geq M(\cdot,y)$ q.e. on $\bigcup_{k=1}^m E_k$, so then $\hat{R}^{\cup E_k}_{M(\cdot,y)}$ is a potential, since it is dominated by the above sum, which is a potential.

(3): If $u := \hat{R}^{E_1}_{M(\cdot,y)}$, then u is a potential, $u \geq M(\cdot,y)$ on $E_1 \setminus Z$ for some polar set Z. Let v be a non-zero potential which is ∞ on Z, and

$$W := \{x \in \Omega : u(x) + v(x) > M(\cdot,y)\}.$$

Then $E_1 \subset W$, and W is open. Furthermore $R^W_{M(\cdot,y)} \leq u + v$, so $R^W_{M(\cdot,y)}$ is a potential. $\square$

Definition 9.17. *The minimal fine topology on $\hat{\Omega}$ is the collection of $W \subset \hat{\Omega}$ such that*
(1) $\Omega \setminus W$ is thin at every point of $W \cap \Omega$.
(2) $\Omega \setminus W$ is minimally thin at every point of $W \cap \partial^M\Omega$.

Note that by definition $\Omega \setminus W$ is thin at $y \in W \cap \Omega$ if y is not a fine limit point of $\Omega \setminus W$, which of-course is the same as saying that $\Omega \cap W$ is open in the fine topology on Ω. Also note that we have already proved that the minimal fine topology is finer than the Martin-topology. Furthermore, if $E \subset \Omega$, and W is a Martin-topology neighborhood of $y \in \partial^M\Omega$, then

$$\hat{R}^E_{M(\cdot,y)} \leq \hat{R}^{E\cap W}_{M(\cdot,y)} + \hat{R}^{\Omega\setminus W}_{M(\cdot,y)},$$

and we know that $\hat{R}^{\Omega\setminus W}_{M(\cdot,y)}$ is a potential. Hence E is minimally thin at y if and only if $E \cap W$ is.

Theorem 9.18. *Let $E \subset \Omega$, $y \in \Delta_1$, then the following are equivalent:*
(1) E is minimally thin at y.
(2) $\hat{R}^E_{M(\cdot,y)} \neq M(\cdot,y)$.
(3) $\inf\left\{\hat{R}^{E\cap W}_{M(\cdot,y)} : W \text{ is a Martin-topology neighborhood of } y\right\} = 0$.

Proof. (2)⇒(1): If $u := \hat{R}^E_{M(\cdot,y)}$,, then $u = v + aM(\cdot,y)$ where v is a potential on Ω, $0 \leq a < 1$. But $u = M(\cdot,a)$ q.e. on E and $\hat{R}^E_v + au = v + aM(\cdot,y)$ q.e. on E. This implies

$$\hat{R}^E_{M(\cdot,y)} = \hat{R}^E_u \leq \hat{R}^E_v + au \leq v + aM(\cdot,y) = \hat{R}^E_{M(\cdot,y)},$$

so

$$a(M(\cdot,y) - u) \equiv 0,$$

hence $a = 0$.
(1)⇒(3): Let (W_m) be a decreasing sequence of Euclidean compacts which (interiors) are Martin-topology neighborhoods of y, $\cap W_m = \{y\}$. Then

$$\hat{R}^{E\cap W_m}_{M(\cdot,y)}$$

is harmonic on $\Omega \setminus W_m$, and decreasing to a limit $h \in HP(\Omega)$. Since $h \leq \hat{R}^E_{M(\cdot,y)}$, which is a potential it follows that h is identically zero.
(3)⇒(2): Obviously there is a W such that

$$\hat{R}^{E\cap W}_{M(\cdot,y)} \neq M(\cdot,y)$$

if (3) holds by assumption. But then $E \cap W$, and hence E, is minimally thin at y, since we already have proved that (2)⇒(1). □

We will for $u \in UP(\Omega)$ denote by μ_u the measure on Δ_1 representing the largest harmonic minorant of u.

Theorem 9.19. *Let $E \subset \Omega$, $y \in \Delta_1$ and suppose that y is a Martin-topology limit point of E, then the following are equivalent:*
(1) E is minimally thin at y.
(2) There is a $u \in UP(\Omega)$ such that

$$\liminf_{x \to y, x \in E} \frac{u(x)}{M(x,y)} > \mu_u(\{y\}).$$

(3) There is a potential u on Ω such that

$$\frac{u(x)}{M(x,y)} \to \infty \text{ as } x \to y, x \in E.$$

(In (2) and (3) the limits are in the Martin-topology.)

Proof. (3)⇒(2) is trivial.
(2)⇒(1): If (2) holds there is a $c > \mu_u(\{y\})$ and a Martin-topology neighborhood W of y such that $u \geq cM(\cdot, y)$ on $E \cap W$. If $\hat{R}^{E\cap W}_{M(\cdot,y)} = M(\cdot, y)$, then

$$u \geq \hat{R}^{E\cap W}_u \geq cM(\cdot, y),$$

and this would give

$$\mu_u = c\delta_y + \mu_{u-cM(\cdot,y)} > \mu_u(\{y\})\delta_y,$$

and this gives a contradiction. So $\hat{R}^{E\cap W}_{M(\cdot,y)} \neq M(\cdot, y)$.
(1)⇒(3): Take U open with $E \subset U$, U minimally thin at y. And then Martin-topology open neighborhoods W_n of y such that

$$\hat{R}^{U\cap W_n}_{M(\cdot,y)} \leq 2^{-n}.$$

$$u_1 := \Sigma_n R^{U\cap W_n}_{M(\cdot,y)}$$

is a potential, and since $\hat{R}^{U\cap W_n}_{M(\cdot,y)} \equiv M(\cdot, y)$ on $U \cap W_n$, we get:

$$\frac{u_1(x)}{M(x,y)} \to \infty \text{ as } x \to y, x \in U.$$

□

Theorem 9.20. *Let $E \subset \Omega$, $y \in \Delta_1$, and suppose that y is a Martin-topology limit point of E. Then the following are equivalent:*
(1) E is minimally thin at y.
(2) There is a potential G^μ_Ω such that

$$\liminf_{x \to y, x \in E} \frac{G^\mu_\Omega(x)}{G_\Omega(a,x)} > \int M(z,y)d\mu(z).$$

(3) There is a potential $G_\Omega^{\mu'}$ *such that* $\int M(z,y)d\mu'(z) < \infty$ *and*

$$\frac{G_\Omega^{\mu'}(x)}{G_\Omega(a,x)} \to \infty \text{ as } x \to y, x \in E.$$

Proof. (3)⇒(2) is obvious.

(2)⇒(1): Choose b such that

$$\liminf_{x\to y, x\in E} \frac{G_\Omega^\mu(x)}{G_\Omega(a,x)} > b > \int M(\cdot,y)d\mu.$$

Then $G_\Omega^\mu > bG_\Omega(a,\cdot)$ on $E \cap W$ for some Martin-topology neighborhood W of y. If $\nu := -\Delta \hat{R}^{E\cap W}_{G_\Omega(\cdot,a)}$, then

$$G_\Omega^\mu \geq b\hat{R}^{E\cap W}_{G_\Omega(a,\cdot)} = bG_\Omega^\nu \text{ on } \Omega.$$

Let $K_n \nearrow \Omega$ be compacts, and let $G_\Omega^{\mu_n} := \hat{R}^{K_n}_{M(\cdot,y)}$ then

$$\int \hat{R}^{K_n}_{M(\cdot,y)} d\nu = \int G_\Omega^\nu d\mu_n \leq b^{-1} \int G_\Omega^\mu d\mu_n < 1 = M(a,y).$$

So $E \cap W$, and hence E, is minimally thin at y.

(1)⇒(3): There is an open set $U \subset \Omega$ with $E \subset U$ and U minimally thin at y. Choose W_n Martin-topology neighborhoods of y such that

$$\Sigma_{n=1}^\infty \hat{R}^{U\cap W_n}_{M(\cdot,y)} < \infty.$$

$$\mu' := \Sigma_{n=1}^\infty \nu_n,$$

where

$$\nu_n := -\Delta \hat{R}^{U\cap W_n}_{G_\Omega(\cdot,a)}.$$

Then

$$\int M(z,y)d\mu'(z) = \Sigma_{n=1}^\infty \int M(z,y)d\nu_n(z) = \Sigma_{n=1}^\infty \hat{R}^{U\cap W_n}_{M(\cdot,y)}(a) < \infty,$$

so since

$$G_\Omega^{\nu_n} = \hat{R}^{U\cap W_n}_{G_\Omega(a,\cdot)} = G_\Omega(a,\cdot) \text{ on } U \cap W_n$$

(3) holds. □

Corollary 9.21. *Let* $E \subset \Omega$, $y_0 \in \Omega$, $y \in \Delta_1$, *and suppose that* y *is a Martin-topology limit point of* E. *Then* E *is minimally thin at* y *if and only if there is* $u \in UP(\Omega)$ *such that*

$$\liminf_{x\to y, x\in E} \frac{u(x)}{G_\Omega(y_0,x)} > \liminf_{x\to y, x\in\Omega} \frac{u(x)}{G_\Omega(y_0,x)}.$$

Proof. WLOG $y_0 = a$. Also

$$\liminf_{x\to y, x\in\Omega} \frac{G^\mu_\Omega(x)}{G_\Omega(a,x)} \geq \int M(z,y)d\mu(z),$$

by Fatou's lemma. But the reverse inequality follows since Ω is not minimally thin at y. If E is minimally thin at y, then we may choose b such that

$$\liminf_{x\to y, x\in\Omega} \frac{u(x)}{G_\Omega(a,x)} > b > \liminf_{x\to y, x\in\Omega} \frac{u(x)}{G_\Omega(a,x)}.$$

Let $G^\mu_\Omega := \min\{u, bG^(_\Omega\cdot, a)\}$, then

$$\liminf_{x\to y, x\in E} \frac{G^\mu_\Omega(x)}{G_\Omega(a,x)} = b > \liminf_{x\to y, x\in\Omega} \frac{G^\mu_\Omega(x)}{G_\Omega(a,x)} = \int M(z,y)d\mu(z),$$

so E is minimally thin at y. □

Lemma 9.22. *Let $y \in \Delta_1$, $E_n \subset \Omega$ be thin at y for all $n \in \mathbb{N}$. Then there is a set E minimally thin at y, and a sequence (W_n) of Martin-topology neighborhoods of y such that $E_n \cap W_n \subset E$ $\forall n \in \mathbb{N}$.*

Proof. For every n choose W_n such that $\hat{R}^{E\cap W_n}_{M(\cdot,y)}(a) < 2^{-n-1}$. $E := \bigcup_n (E_n \cap W_n)$, $u := \Sigma_n \hat{R}^{E\cap W_n}_{M(\cdot,y)}$. Since $u \geq M(\cdot, y)$ q.e. on E we have

$$\hat{R}^E_{M(\cdot,y)} \leq u(a) < 1 = M(a,y).$$

So E is minimally thin at y. □

Theorem 9.23. *Let $y \in \Delta_1$, and $f : U \cap \Omega \to [-\infty, \infty]$ where U is a minimal fine neighborhood of y. Then f has minimal fine limit l at y if and only if there is a set $E \subset \Omega$ minimally thin at y such that $f(x) \to l$ in the Martin-topology as $x \to y$ in $U \setminus E$.*

Proof. Follows from the lemma above in the same way as the corresponding result for the fine topology was proved. □

We will write mf lim and so on for minimal fine limits, other limits are w.r.t. the Martin topology. We also remark that when we say for instance that a function has minimal fine limit so and so almost everywhere w.r.t. some measure, we mean that the limit is well defined and has this value almost everywhere. So in particular, to say that the limit has a certain value a.e. on a set of measure zero does not even imply that the limit makes any kind of sense.

Theorem 9.24. *Let $y \in \Delta_1$, and $y_0 \in \Omega$, $u \in UP(\Omega)$, then*

(1) $\mathrm{mf}\lim_{x\to y, x\in\Omega} \frac{u(x)}{M(x,y)} = \mu_u(\{y\}) = \inf_{x\in\Omega} \frac{u(x)}{M(x,y)}$.

(2) $0 < \liminf_{x\to y, x\in\Omega} \frac{u(x)}{G_\Omega(y_0,x)} = \mathrm{mf}\lim_{x\to y, x\in\Omega} \frac{u(x)}{G_\Omega(y_0,x)} \leq \infty$.

Proof. (1): $u \geq \mu_u(\{y\})M(\cdot,y)$, so

$$\inf_{\Omega} \frac{u}{M(\cdot,y)} = \mu_u(\{y\}),$$

and

$$\{x \in \Omega : u(x)/M(x,y) > \mu_u(\{y\}) + \varepsilon\} \quad (\varepsilon > 0)$$

is minimally thin at y.

(2): Let $B_0 := B_\varepsilon(y_0) \subset\subset \Omega$, $b := \inf_{B_0} u/G_\Omega(y_0,\cdot)$,
$l := \liminf_{x\to y, x\in\Omega} u(x)/G_\Omega(y_0,x)$. Then we get

$$u \geq R_u^{B_0} \geq bR^{B_0}_{G_\Omega(y_0,\cdot)} = bG_\Omega(y_0,\cdot) \text{ on } \Omega,$$

so $l \geq b > 0$. If $l = \infty$ we are done, otherwise

$$\{x \in \Omega : u(x)/G_\Omega(y_0,x) > l + \varepsilon\}$$

is minimally thin at y, and we are done. □

Corollary 9.25. *(1) If $y \in \Delta_1$, then*

$$\text{mf}\lim_{x\to y,x\in\Omega} M(x,y) = \sup_{\Omega} M(\cdot,y).$$

(2) If $y \in \Delta_1$ and $y_0 \in \Omega$, and if $G_\Omega(y_0,\cdot)$ has minimal fine limit 0 at y, then $G_\Omega(y_0,\cdot)$ also has Martin-topology limit 0 at y.

Definition 9.26. *For $E \subset \Omega$ we define*

$$E_{mf} := \{y \in \partial^M\Omega : E \textit{ is not minimally thin at } y\}.$$

Lemma 9.27. *Let $E \subset \Omega$ and $f : \Omega \to [-\infty,\infty]$, $h \in HP(\Omega)$, then*
(1) E_{mf} is a Martin-topology G_δ.
(2) $y \mapsto \text{mf}\limsup_{x\to y,x\in\Omega} f(x)$ is Borel-measurable on Δ_1.
(3) The greatest subharmonic minorant to $\hat{R}_h^E$ is

$$\int_{E_{mf}} M(\cdot,y)d\mu_h(y).$$

Proof. (1): Let $\nu_a^E := -\Delta\hat{R}^E_{G_\Omega(\cdot,a)}$. The function $y \mapsto \hat{R}^E_{M(\cdot,y)}(a)$ is l.s.c. on $\partial^M\Omega$ since

$$\liminf_{y\to z,y\in\partial^M\Omega} \hat{R}^E_{M(\cdot,y)} \geq \int \liminf_{y\to z,y\in\partial^M\Omega} M(x,y)d\nu_a^E(x) = \hat{R}^E_{M(\cdot,z)}(a).$$

Let $\{W_n\}$ be a base for the Martin topology induced on $\partial^M\Omega$.

$$A_n := \{y \in W_n \cap \partial^M\Omega : \hat{R}^{E\cap W_n}_{M(\cdot,y)}(a) \leq 1/2\}$$

are closed, and

$$E_{mf} = \partial^M\Omega \setminus \left(\bigcup_n A - n\right),$$

so (1) holds.

(2): For every $b \in \mathbb{R}$ we have

$$\{y \in \Delta_1 : \text{mf}\,\text{limsup}_{x\to y, x\in\Omega} f(x) \geq b\} = \bigcap_{n=1}^{\infty} \{x \in \Omega : f(x) > b - 1/n\}_{mf}.$$

(3):

$$\hat{R}_h^E(x) = \int_{\Delta_1} \hat{R}^E_{M(\cdot,y)}(x) d\mu_h(y) = \int_{E_{mf}} M(x,y) d\mu_h(y) + v(x),$$

where v is a potential. □

Theorem 9.28. *Let $h \in HP(\Omega) \setminus \{0\}$, $u \in UP(\Omega)$ and*

$$A = \left\{ y \in \Delta_1 : \text{mf}\,\text{limsup}_{x\to y, x\in\Omega} \frac{u(x)}{h(x)} \geq b \right\}, \text{ where } b > 0.$$

Then

$$u(x) \geq b \int_A M(x,y) d\mu_h(y) \text{ on } \Omega.$$

Proof. $0 < b' < b$, $E(b') := \{x \in \Omega : u(x)/h(x) > b'\}$ gives that $A \subset E(b')_{mf}$, so

$$u \geq \hat{R}_u^{E(b')} \geq b' \hat{R}_h^{E(b')} \geq b' \int_A M(\cdot, y) d\mu_h(y) \text{ on } \Omega.$$

So the theorem now follows by taking $b' \nearrow b$. □

Corollary 9.29. *If u is a potential and $h \in HP(\Omega) \setminus \{0\}$, then u/h has minimal fine limit 0 μ_h−a.e. on Δ_1.*

Proof. The set

$$\left\{ y \in \Delta_1 : \text{mf}\,\text{limsup}_{x\to y, x\in\Omega} \frac{u(x)}{h(x)} \geq 1/n \right\}$$

has μ_h−measure zero by the theorem above. □

Theorem 9.30. *Let $u, h \in HP(\Omega)$, where $h \neq 0$, and assume that μ_h, μ_u are mutually singular, then $\min\{u,h\}$ is a potential, and u/h has minimal fine limit 0 μ_h-a.e. on Δ_1.*

Proof. Let $\Delta_1 = A \cup C$, $\mu_u(A) = \mu_h(C) = 0$. Let $0 \leq v \leq \min\{u,h\}$, v harmonic, then $\mu_v \leq \mu_u$ and $\mu_v \leq \mu_h$ so $\mu_v \equiv 0$, hence $v \equiv 0$, so $\min\{u,h\}$ is a potential. □

Corollary 9.31. *Let $u, h \in HP(\Omega)$, $h \neq 0$. If $A \subset \partial^M \Omega$ is Borel and $\mu_u(A) = 0$, then u/h has minimal fine limit 0 μ_h−a.e. on A.*

Proof. Assume $\mu_h(A) > 0$ to avoid triviality, and define

$$h_1 := \int_A M(\cdot, y) d\mu_h(y).$$

Then

$$0 \leq u/h \leq u/h_1,$$

and u/h_1 has minimal fine limit 0 μ_h−a.e. on A. □

Corollary 9.32. *Let $h \in HP(\Omega) \setminus \{0\}$, and $A \subset \partial^M\Omega$ Borel. Let*

$$h_1 := \int_A M(\cdot, y) d\mu_h(y).$$

Then h_1/h has minimal fine limit χ_A μ_h-a.e. on $\partial^M\Omega$.

Proof. $h_2 := h - h_1$. We know that h_1/h has minimal fine limit 0 μ_h-a.e. on $\partial^M\Omega \setminus A$, and on A h_2/h has minimal fine limit 0 μ_h-a.e. This clearly implies the theorem. □

- $u_{f,h} := \int f(y)M(\cdot, y)d\mu_h(y)$.

Theorem 9.33. *Let $h \in HP(\Omega) \setminus \{0\}$, $f \in L^1(\mu_h)$. Then $u_{f,h}$ has minimal fine limit f μ_h-a.e. on Δ_1.*

Proof. Assume WLOG that f is positive. If f is a Borel step-function this follows trivially by the above. Otherwise, if f is bounded, let $g \le f \le f + \varepsilon$ (a.e.) where g is such a step-function. Then

$$u_{g,h} \le u_{f,h} \le u_{g,h} + \varepsilon h,$$

so

$$\mu_h\left(\left\{y \in \Delta_1 : \text{mf limsup}_{x\to y, x\in\Omega} \frac{u_{f,h}(x)}{h(x)} - \text{mf liminf}_{x\to y, x\in\Omega} \frac{u_{f,h}(x)}{h(x)} > \varepsilon\right\}\right) = 0.$$

Finally, for general f, let

$$A_n := \{y \in \partial^M\Omega : f(y) < n\}$$

and

$$f_n := f\chi_{A_n}, g_n := f\chi_{\partial^M\Omega\setminus A_n},$$

then $u_{f_n,h}/h$ has minimal fine limit f μ_h-a.e. on A_n, and $u_{g_n,h}$ has minimal fine limit zero μ_h-a.e. here. □

Theorem 9.34 (Fatou-Naim-Doob). *Let $u, v \in UP(\Omega)$, where $v \not\equiv 0$, and let*

$$f := \frac{d\mu_u}{d\mu_v}$$

be the Radon-Nikodym derivative. Then u/v has minimal fine limit f μ_v-a.e. on $\partial^M\Omega$.

Proof. If we let

$$v := w + \int M(\cdot, y)d\mu_v(y),$$

then w is a potential. We put $h = v - w$, so $\mu_v = \mu_h$, and assume (since otherwise there is nothing to prove), that $h \ne 0$. Let ν be the singular component of μ_u w.r.t. μ_v, then let $u_2 := \int M(\cdot, y)d\nu(y)$, and then we have

$$u = u_1 + u_2 + u_{f,h},$$

where u_1 is a potential. Applying the above results we see that u_1/h and u_2/h has minimal fine limit zero μ_v-a.e. And $u_{f,h}/h$ has minimal fine limit f μ_v-a.e. now gives the result, since

$$\frac{u_{f,h}}{h} = \frac{u_{f,h}}{v}\frac{v}{h},$$

and v/h has minimal fine limit 1 μ_h-a.e. on $\partial^M\Omega$, and hence $u_{f,h}/v$ has the same minimal fine limit as $u_{f,h}/h$ μ_h-a.e. on $\partial^M\Omega$. □

Corollary 9.35. *Let $u, v \in UP(\Omega)$, $v \neq 0$. Then the following are equivalent:*

(1) $\min\{u, v\}$ is a potential.

(2) u/v has minimal fine limit 0 μ_v-a.e. on $\partial^M\Omega$.

(3) $\mu_u \perp \mu_v$.

Proof. $u = u_1 + u_2$, $v = v_1 + v_2$ where u_1, v_1 are potentials, and $u_2, v_2 \in HP(\Omega)$. If (3) holds $\min\{u_2, v_2\}$ is a potential, and

$$\min\{u, v\} \leq \min\{u_1, v\} + \min\{u_2, v_2\} + \min\{u, v_1\},$$

so (1) holds. Fatou-Naim-Doob now gives (1)⇒(2) and (2)⇒(3). □

Theorem 9.36. *Suppose $\omega = \Omega \setminus Z$ where Z is a polar set relatively closed in Ω, then we may identify $\hat{\omega} = \hat{\Omega}$ and $\partial^M\omega = \partial^M\Omega \cup Z$ and:*

(1) The minimal points on $\partial^M\omega = \Delta_1 \cup Z$, where Δ_1 are the minimal points on $\partial^M\Omega$.

(2) For $y \in \Delta_1$, $E \subset \omega$ is minimally thin at y w.r.t. ω if and only if it is so w.r.t. Ω.

(3) For $y \in Z$ minimal thinness is the same is thinness.

Proof. Easy, using that

$$G_\omega = G_\Omega \lfloor \omega \times \omega.$$

□

Theorem 9.37. *Suppose $\omega \subset \Omega$ is open, $y \in \Delta_1$ and belongs to the closure of ω in $\hat{\Omega}$. Then $\Omega \setminus \omega$ is minimally thin at y if and only if*

$$(*)\ \limsup_{x \to y, x \in \omega} \frac{G_\omega(a, x)}{G_\Omega(a, x)} > 0.$$

Proof. If $\mu_0 := -\Delta R^{\Omega\setminus\omega}_{G_\Omega(\cdot,a)}$. then $G_\omega(a, \cdot) = G_\Omega(a, \cdot) - G^{\mu_0}_\Omega$ on ω, so

$$\limsup_{x \to y, x \in \omega} \frac{G_\omega(a, x)}{G_\Omega(a, x)} = 1 - \liminf_{x \to y, x \in \omega} \frac{G^{\mu_0}_\Omega(x)}{G_\Omega(a, x)}.$$

If $\Omega \setminus \omega$ is minimally thin at y then ω is not, so

$$\liminf_{x \to y, x \in \omega} \frac{G^{\mu_0}_\Omega(x)}{G_\Omega(a, x)} = \int M(z, y) d\mu_0(z) = \hat{R}^{\Omega\setminus\omega}_{M(\cdot,y)}(a) < 1,$$

so $(*)$ follows.

Conversely, if $(*)$ holds Fatou's lemma gives

$$0 < 1 - \int M(z,y)d\mu_0(z) = 1 - \hat{R}^{\Omega\setminus\omega}_{M(\cdot,y)},$$

so $\Omega\setminus\omega$ is minimally thin at y. $\square$

- If $E \subset \omega$ we will use R^E_u to denote reductions w.r.t. Ω,and ${}^{\omega}\!R^E_u$ w.r.t. ω.

Lemma 9.38. *Let $F \subset \omega$ and $u \in UP(\Omega)$, and define $v = u - R^{\Omega\setminus\omega}_u$ so that $v \in UP(\Omega)$. Then*

$${}^{\omega}\!R^{\omega\setminus F}_u = R^{\Omega\setminus F}_u - R^{\Omega\setminus\omega}_u.$$

Proof. If $w \in UP(\Omega)$, $w \geq u$ on $\Omega\setminus F$, then $w - R^{\Omega\setminus\omega}_u \geq v$ on $\omega\setminus F$, so $w - R^{\Omega\setminus\omega}_u \geq {}^{\omega}\!R^{\omega\setminus F}_u$. Taking the infimum over all such w gives

$$R^{\Omega\setminus F}_u - R^{\Omega\setminus\omega}_u \geq {}^{\omega}\!R^{\omega\setminus F}_u.$$

If $w \in UP(\Omega)$, $w \geq v$ on $\omega\setminus F$, then $w + R^{\Omega\setminus\omega}_u \geq u$ on $\omega\setminus F$.

$$w' := \begin{cases} \min\{u, w + R^{\Omega\setminus\omega}_u\} & \text{on } \omega \\ u & \text{on } \Omega\setminus\omega, \end{cases}$$

and

$$w'' := \hat{W}'.$$

Since

$$\liminf_{x\to y, x\in\Omega} \hat{R}^{\Omega\setminus\omega}_u(x) = u(y) \quad \text{q.e. on } \partial\omega\cap\Omega$$

we get $w'' \in UP(\Omega)$, $w'' = w'$ q.e. and $w'' = u$ q.e. on $\Omega\setminus F$, so

$$w + R^{\Omega\setminus\omega}_u \geq R^{\Omega\setminus F}_u \text{ on } \omega,$$

and hence we have the reverse inequality. $\square$

Theorem 9.39. *Let $y \in \Delta_1$, where y belongs to the closure of ω in $\hat{\Omega}$, and suppose that $\Omega\setminus\omega$ is minimally thin at y, and let*

$$h := M(\cdot,y) - R^{\Omega\setminus\omega}_{M(\cdot,y)} \text{ on } \Omega, \text{ then.}$$

(1) h is minimal on ω.

(2) Let $z_y \in \partial^M\omega$ be the point corresponding to h. Any sequence (x_n) of points in Ω converging to y in $\hat{\Omega}$ and with

$$(**)\ \liminf_{n\to\infty} \frac{G_\omega(a,x_n)}{G_\Omega(a,x_n)} > 0$$

converges to z_y in $\hat{\omega}$.

(3) Let $E \subset \omega$. Then E is minimally thin at z_y w.r.t. $\hat{\omega}$ if and only if E is minimally thin at y w.r.t. $\hat{\Omega}$.

Proof. (1): Let $w \in HP(\Omega)$, where $w \leq h$,

$$w'(x) := \begin{cases} w(x) + R^{\Omega\setminus\omega}_{M(\cdot,y)}(x) & x \in \omega \\ M(x,y) & x \in \Omega \setminus \omega \end{cases}$$

then $w'' := \hat{w}' \in UP(\Omega)$, and $w'' = M(\cdot, y)$ q.e. on $\Omega \setminus \omega$. Since $w'' \leq M(\cdot, y)$ we get that

$$w'' = bM(\cdot, y) + G^{\mu}_{\Omega} \text{ where } 0 \leq b \leq 1,$$

and μ does not charge polar sets. Then $R^{\Omega\setminus\omega}_{G^{\mu}_{\Omega}} = G^{\mu}_{\Omega}$, and since

$$G^{\mu}_{\Omega} = (1 - b)M(\cdot, y),$$

we get

$$w = w'' - R^{\Omega\setminus\omega}_{M(\cdot,y)} = bM(\cdot, y) + (1 - b)R^{\Omega\setminus\omega}_{M(\cdot,y)} - R^{\Omega\setminus\omega}_{M(\cdot,y)} = bh \text{ on } \omega.$$

(2): Let (x_n) be a sequence in ω converging to y and such that $(**)$ holds. Suppose that there is a subsequence (x_{n_k}) such that

$$\frac{G_\omega(\cdot, x_{n_k})}{G_\omega(a, x_{n_k})} \to u \text{ u.c. on } \omega,\ \ u \neq h/h(a).$$

WLOG assume that

$$\frac{G_\omega(a, x_{n_k})}{G_\Omega(a, x_{n_k})} \to b > 0$$

by taking a subsequence if necessary. Then

$$\frac{G_\omega(\cdot, x_{n_k})}{G_\Omega(a, x_{n_k})} \to bu \text{ u.c. on } \omega.$$

Since

$$G_\omega(x, z) = G_\Omega(x, z) - R^{\Omega\setminus\omega}_{G_\Omega(\cdot,z)}(x) \quad \forall x, z \in \omega,$$

Fatou's lemma (together with the fact that $\hat{R}^E_u(y) = \int u(x) d\nu^E_y(x)$, where $\nu^E_y = -\Delta \hat{R}^E_{G_\Omega(\cdot,y)}$) gives that

$$bu(x) = \lim_{k\to\infty} \left\{ M(x, x_{n_k}) - R^{\Omega\setminus\omega}_{M(\cdot,x_{n_k})}(x) \right\} \leq M(x, y) - R^{\Omega\setminus\omega}_{M(\cdot,y)}(x) = h(x).$$

Since $u(a) = 1$, and since (1) gives that u is a multiple of h we get a contradiction.
(3): E is minimally thin at z_y w.r.t. ω if and only if ${}^{\omega}R^E_h \neq h$. Put $F = \omega \setminus E$, $u = M(\cdot, y)$ in the previous lemma to get that this is the same as $R^{(\Omega\setminus\omega)\cup E}_{M(\cdot,y)} \neq M(\cdot, y)$, which in turn is equivalent to that $(\Omega \setminus \omega) \cup E$ is minimally thin at y w.r.t. Ω. This implies that E is minimally thin at y w.r.t. Ω, and we are done. □

Corollary 9.40. *Let $y \in \Omega \cap \partial\omega$, and suppose $\Omega \setminus \omega$ is thin at y, let $h := G_\Omega(\cdot, y) - R^{\Omega\setminus\omega}_{G_\Omega(\cdot,y)}$ on ω.*
(1) The function h is minimal on ω.

(2) Let z_y be the corresponding point on $\partial^M\omega$. Any sequence (x_n) in ω converging to y in the Euclidean topology with

$$\liminf_{n\to\infty} G_\omega(a, x_n) > 0$$

converges to z_y in $\hat{\omega}$.

(3) Let $E \subset \omega$. Then E is minimally thin at z_y w.r.t. ω if and only if E is thin at y.

Proof. Follows easily from our previous results by studying $\Omega \setminus \{y\}$ instead of Ω. □

Lemma 9.41. *Let $\omega \subset \Omega$ be open and $y \in \Delta_1$. If $\Omega \setminus \omega$ is minimally thin at y then there is exactly one component ω' of ω such that $\Omega \setminus \omega'$ is minimally thin at y.*

Proof. Let ω' be a component. If $u \in UP(\Omega)$, $u \geq M(\cdot, y)$ on $\Omega \setminus \omega$, then

$$u' := \begin{cases} \min\{u, M(\cdot, y)\} & \text{on } \omega' \\ M(\cdot, y) & \text{on } \Omega \setminus \omega' \end{cases}$$

belongs to $UP(\Omega)$ and

$$R^{\Omega\setminus\omega'}_{M(\cdot,y)} \leq u' \leq u \text{ on } \omega'.$$

So

$$R^{\Omega\setminus\omega}_{M(\cdot,y)} = R^{\Omega\setminus\omega'}_{M(\cdot,y)} \text{ on } \omega'.$$

If $\Omega \setminus \omega'$ is minimally thin at y, then $R^{\Omega\setminus\omega}_{M(\cdot,y)} \neq M(\cdot, y)$, so there is a component ω' such that this holds for that as-well. If ω'' is another component, then $\omega' \subset \Omega \setminus \omega''$, so this set can't be minimally thin. □

Theorem 9.42. *Let $y \in \Delta_1$ and $y_0 \in \Omega$, and let $\omega \subset \Omega$ be open, $\Omega \setminus \omega$ minimally thin at y. For every $u \in UP(\omega)$ we have:*

(1) $u/M(\cdot, y)$ has a finite minimal fine limit at y w.r.t. Ω.

(2) $u/G_\Omega(y_0, \cdot)$ has a positive (possibly ∞) minimal fine limit at y w.r.t. Ω.

Proof. (1): By above WLOG ω is assumed to be connected. Let h, z_y be as in theorem 9.39. There is $E_1 \subset \omega$ minimally thin at z_y w.r.t ω such that $u(x)/h(x)$ has finite minimal fine limit as $x \to z_y$ along $\omega \setminus E_1$. Also there is $E_2 \subset \Omega$ minimally thin at y w.r.t. Ω such that $\Omega \setminus \omega \subset E_2$ and

$$\frac{h(x)}{M(x,y)} = 1 - \frac{\hat{R}^{\Omega\setminus\omega}_{M(\cdot,y)}(x)}{M(x,y)} \to 1 \quad (x \to y, x \in \Omega \setminus E_2).$$

Now $E := E_1 \cup E_2$ is minimally thin at y w.r.t. Ω and $u(x)/M(x, y)$ has a finite limit as $x \to y$ along $\Omega \setminus E$.

(2): WLOG $y_0 = a \in \omega$. There is $E_1 \subset \omega$ minimally thin at z_y w.r.t. ω such that

$u(x)/G_\omega(a,x)$ has a positive limit as $x \to z_y$ along $\omega \setminus E_1$, and similarly there is $E_2 \supset \Omega \setminus \omega$ such that E_2 is minimally thin at y w.r.t. Ω, and such that

$$\frac{G_\omega(a,x)}{G_\Omega(a,x)} = 1 - \frac{R^{\Omega\setminus\omega}_{G_\Omega(a,\cdot)}(x)}{G_\Omega(a,x)} \to 1 - \liminf_{x\to y, x\in\Omega} \frac{R^{\Omega\setminus\omega}_{G_\Omega(a,\cdot)}(x)}{G_\Omega(a,x)} \quad (x \to y, x \in \Omega \setminus E_2) =$$

$$= \limsup_{x\to y, x\in\Omega} \frac{G_\omega(a,x)}{G_\Omega(a,x)} > 0.$$

So $E := E_1 \cup E_2$ is minimally thin at y w.r.t. Ω and $u(x)/M(x,y)$ has positive limit as $x \to y$ along $\Omega \setminus E$. □

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