Potential theory in Denjoy domains

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Abstract. This paper presents an account of Denjoy domains in relation to minimal harmonic functions, the boundary behaviour of the Green function, and to their usefulness as a source of counterexamples in potential theory. The discussion begins with an exposition of key work of Ancona and Benedicks and then moves on to describe several very recent results.

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1. Introduction

A result of Denjoy [16], dating from 1909, says that a domain of the form $\mathbb{C}\backslash E$, where $E \subset \mathbb{R}$, supports nonconstant bounded analytic functions if and only if E has positive Lebesgue measure. Domains $\Omega \subset \mathbb{C}$ for which $\mathbb{C}\backslash\Omega$ is contained in a line, or, more generally, domains $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ for which $\mathbb{R}^n\backslash\Omega$ is contained in a hyperplane, have subsequently come to be known as *Denjoy domains.* The purpose of this article is to describe how the special geometry of Denjoy domains has led to some very precise and illuminating potential theoretic results. We will not attempt to provide a comprehensive survey of the subject, but will present some of the key theory and also outline several recent developments.

Denjoy domains sometimes arise naturally in the study of certain domain properties. To take a simple example, let $\mathcal{H}_c(\Omega)$ denote the collection of harmonic functions on a domain Ω that have a finite-valued continuous extension to compactified space $\mathbb{R}^n \cup \{\infty\}$. It is easy to see, by consideration of Poisson integral representations in halfspaces, that $\mathcal{H}_c(\Omega)$ does not separate the points of Ω if Ω is a Denjoy domain. In fact, essentially only Denjoy domains have this non-separation property. More precisely, when $n \geq 3$, domains Ω with this property must be of the

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form $\omega\backslash F$, where ω is a Denjoy domain and F is a relatively closed polar subset of ω; and, when $n = 2$, the characterization is similar, except that ω can be the image of a Denjoy domain under a linear fractional transformation (for this and related results, see [9]).

Denjoy domains also arise naturally in the study of null quadrature domains, that is, domains Ω on which every integrable harmonic function has integral 0. A simple example of a domain with this property is a halfspace; for, if h is an integrable harmonic function on $\omega = \mathbb{R}^{n-1} \times (0, \infty)$, then the function

$$t \mapsto \int_{\mathbb{R}} ... \int_{\mathbb{R}} h(x_1, ..., x_{n-1}, t)\ dx_1...dx_{n-1} \qquad (t > 0)$$

is of the form $at + b$ for some $a, b \in \mathbb{R}$ (by Theorem 1.5.12 in [8]), so $a = 0 = b$ and hence $\int_\omega h = 0$. It follows that any Denjoy domain also has this property, but it seems that very few other domains have it. In fact, when $n = 2$, Sakai [23] has shown that the only other possibilities are complementary to closed elliptic or parabolic regions, and something similar has been conjectured to be true in higher dimensions (see [22]).

However, while such examples may be of some interest, the main potential theoretic motivation for studying Denjoy domains is the desire to gain insight into how the geometry of a domain affects subtle potential theoretic phenomena. One such topic is the Martin boundary (see Chapter 8 of the book [8]), which provides an integral representation for positive harmonic functions on a Greenian domain in the spirit of the Riesz-Herglotz representation in a ball. Since the relationship between the Euclidean and Martin boundaries can be quite complicated, several authors, beginning with Ancona [3] and Benedicks [10], have analysed what can occur in the case of Denjoy domains. Another topic where the case of Denjoy domains has provided insight is the relationship between the geometry of a domain and the boundary behaviour of its Green function (see Carleson and Totik [14] and Carroll and Gardiner [12]). As we will explain later, these two apparently distinct questions are, in fact, intimately related.

Finally, Denjoy domains, and variants of them, can be a fruitful source of counterexamples. In the final section of this article we will illustrate this with reference to recent work of Sjödin [26] concerning integrability and positive harmonic functions, and the solution by Gardiner and Hansen [21] of a long-standing open question about the Riesz decomposition in fine potential theory. We note, in passing, that Denjoy domains have also arisen recently as natural counterexamples in connection with the study of minimal harmonic functions on John domains [2].

Other papers where potential theoretic aspects of Denjoy-type domains have been considered include [1], [4], [5], [6], [7], [13], [15], [20], [24], [25] and [27].

2. Notation and preliminary material

Here we set out our notation and recall some of the potential theoretic facts we need, a convenient reference for which is the book [8].

We will work in Euclidean space $\mathbb{R}^n$ $(n \geq 2)$, using the notation

$$x = (x_1, x_2, \ldots, x_n) = (x', x_n) \text{ and } \widetilde{x} = (x', -x_n), \text{ where } x' \in \mathbb{R}^{n-1},$$

and writing $|x|$ for the Euclidean norm of x. We define

$$H_+ = \{x \in \mathbb{R}^n : x_n > 0\},\ H_- = \{x \in \mathbb{R}^n : x_n < 0\},\ L = \{x \in \mathbb{R}^n : x_n = 0\},$$

$$a_t = (0', t) \quad (t > 0) \quad \text{and} \quad T_t = \{a_s : 0 < s < t\} \quad (0 < t \leq \infty),$$

and write $B(x, r)$ for the open ball of centre x and radius r.

To avoid confusion, sequences in $\mathbb{R}^n$ will be denoted $x^{(k)}$ and not x_k (since this is used for the kth coordinate of x). Sequences in $\mathbb{R}$ and sequences of sets will, however, be represented using the familiar subscript notation.

If η is a measure, then

$$\mathcal{L}^p(\eta) := \begin{cases} \{f : |f|^p \text{ is } \eta - \text{integrable}\} & (0 < p < \infty) \\ \{f : |f| \text{ is essentially bounded}\} & (p = \infty) \end{cases}.$$

By supp(η) we will always denote the closed support of a measure η.

For an open set $D \subset \mathbb{R}^n$ we introduce the following classes of real-valued functions on D.

$\mathcal{P}(D)$: the non-negative functions on D,

$\mathcal{L}^p(D)$: the pth power Lebesgue integrable functions on D,

$\mathcal{H}(D)$: the harmonic functions on D.

We also use superpositioning so that, for instance, $\mathcal{HP}(D) = \mathcal{H}(D) \cap \mathcal{P}(D)$. If $D \subset \mathbb{R}^n$ is Greenian, then $G_D(\cdot, \cdot)$ denotes its Green function, and if μ is a measure on D we denote its Green potential by $G_D\mu$.

For a set $A \subset \mathbb{R}^n$ we denote by ∂A its boundary in $\mathbb{R}^n$, and by $\partial^\infty A$ its boundary in $\mathbb{R}^n \cup \{\infty\}$. The harmonic measure on $\partial^\infty D$ of a Greenian open set D with respect to a point $x \in D$ is denoted by λ_x^D.

We recall that, if D is a Greenian domain D, then the Martin kernel normalized at some reference point x_0 is defined on $D \times (D \setminus \{x_0\})$ by

$$M_D(x, y) := \frac{G_D(x, y)}{G_D(x_0, y)}.$$

The Martin compactification $\widehat{D}$ is the smallest compactification of D such that $M_D(x, \cdot)$ can be continuously extended to the boundary $\partial^M D = \widehat{D} \setminus D$ for each x. (We still use $M_D(\cdot, \cdot)$ for the extended function.) For each $h \in \mathcal{HP}(D)$ there is (at least) one representing measure η on $\partial^M D$ for h, that is, for which

$$h(x) = \int M_D(x, y) d\eta(y). \quad (\text{Clearly } \eta(\partial^M D) = h(x_0)).$$

In general, such measures are not unique, but if we require them to be carried by the minimal Martin boundary $\partial_1^M D$, then there is a unique measure, which we denote by ν_h.

We will also need the concept of minimal thinness (see Chapter 9 of [8]). If $y \in \partial_1^M D$ and $A \subset D$, then we recall that A is said to be minimally thin at y (with respect to D) if

$$M_D(\cdot, y) \not\equiv \widehat{R}^A_{M_D(\cdot,y)},$$

where $\widehat{R}^A_{M_D(\cdot,y)}$ denotes the regularized reduction of $M_D(\cdot, y)$ over the set A with respect to positive superharmonic functions on D.

3. Basic Theory

3.1. Martin boundary points

We will follow the approach of Ancona [3] to establish a kind of boundary Harnack principle for Denjoy domains, and then use minimal thinness arguments to prove the basic properties of the set of Martin points associated with a Euclidean boundary point for these domains.

Theorem 1. *Suppose that $D \subset \mathbb{R}^n$ is a Greenian domain which contains H_+. There is a constant C_1, depending only on n, such that*

$$G_D(x, y) \le C_1 G_D(x, a_{|y|}) \quad (x \in T_\infty; y \in D \setminus \{0\}). \tag{1}$$

Proof. (I) *For all $x \in H_+$ and $y \in D \cap H_-$ we have $G_D(x, y) \le G_D(x, \tilde{y})$.* This holds because the function

$$u(y) := G_D(x, \tilde{y}) - G_D(x, y)$$

is superharmonic and lower bounded in $D \cap H_-$ and has non-negative lower limit quasi-everywhere on $\partial^\infty(D \cap H_-)$. Hence $u \ge 0$ in $D \cap H_-$ by the minimum principle.

We now identify $\mathbb{R}^n$ with $\mathbb{R}^{n-2} \times \mathbb{C}$ by letting

$$(x_1, \dots, x_{n-2}, x_{n-1}, x_n) = (x_1, \dots, x_{n-2}, x_{n-1} + i x_n) = (x'', z),$$

and use $\arg z$ to denote the value of the argument in $(-\pi, \pi]$.

(II) *There is a constant d_1, depending only on n, such that every positive harmonic function h on the set*

$$\{(x'', z) : 0 < \arg z < \pi/2\}$$

satisfies

$$h(x'', te^{i\theta}) \le d_1 h(x'', te^{i\varphi}) \quad (\theta, \varphi \in [\pi/8, 3\pi/8]; t > 0).$$

This is an easy consequence of Harnack's inequalities.

Let s denote the reflection map with respect to the hyperplane

$$\{(w'', z) \in \mathbb{R}^n : \arg z \in \{-7\pi/8, \pi/8\}\}.$$

(III) *For all $x \in T_\infty$ and $y = (y'', z)$ with $\arg z = -\pi/8$ we have*

$$G_D(x, y) \le d_1 G_D(x, s(y)).$$

This follows by combining (I) and (II).

Now, for fixed $x \in T_\infty$, we apply the minimum principle to the function

$$w(y) := d_1 G_D(x, s(y)) - G_D(x, y)$$

in the set

$$D \cap \{(y'', z) \in \mathbb{R}^n : -\tfrac{\pi}{8} < \arg z < \tfrac{\pi}{8}\}$$

to see from (III) that

$$\sup\left\{G_D(x, (y'', z)) : y'' \in \mathbb{R}^{n-2}, \arg z \in \left[-\tfrac{\pi}{8}, \tfrac{\pi}{8}\right]\right\} \le$$
$$d_1 \sup\left\{G_D(x, (y'', z)) : y'' \in \mathbb{R}^{n-2}, \arg z \in \left[\tfrac{\pi}{8}, \tfrac{3\pi}{8}\right]\right\}.$$

By rotating the coordinate system around the x_n-axis and letting $\mathcal{K}$ denote the cone around T_∞ with vertex 0 and half-angle $3\pi/8$ we now obtain:
(IV) *For all $x \in T_\infty$ and $t > 0$ we have*

$$\sup\{G_D(x, y) : |y| = t\} \le d_1 \sup\{G_D(x, y) : |y| = t, y \in \mathcal{K}\}.$$

Applying Harnack's inequality we get:
(V) *There is a constant d_2, depending only on n, such that, for all $t > 0$, $y \in D \cap \partial B(0, t)$, and $x \in T_\infty$ with $|x| \le t/2$ or $|x| \ge 2t$,*

$$G_D(x, y) \le d_2 G_D(x, a_t).$$

To complete the argument let $t = |y|$ and $t/2 < |x| < 2t$. The case $|x| = t$ is trivial, so we assume that $|x| \ne t$. The function $G_D(x, \cdot)$ is harmonic outside $\overline{B}(x, |t - |x|| /2)$ and is majorized on $\partial B(x, |t - |x|| /2)$ by $d_3 G_D(x, a_t)$ for some constant d_3 depending only on n, in view of Harnack's inequalities. Letting $C_1 = \max\{d_2, d_3\}$ we now obtain (1). □

Corollary 2. *Suppose $D \subset \mathbb{R}^n$ is a Greenian domain which contains H_+. There is a constant C_2, depending only on n, such that*

$$G_D(x, y) \le C_2 G_D(x, a_{|y|}) \quad (x \in H_+; y \in D \setminus B(0, 2|x|)). \tag{2}$$

Proof. Let $x = (x', x_n)$ and $a^*_{|y|} = (x', a_{|y-(x',0)|})$. By Theorem 1 we have

$$G_D(x, y) \le C_1 G_D(x, a^*_{|y|}).$$

But the function $G_D(x, \cdot)$ is positive and harmonic on $H_+ \setminus \overline{B(0, |y|/2)}$ and so, by Harnack's inequalities,

$$G_D(x, a^*_{|y|}) \le d_1 G_D(x, a_{|y|})$$

for some constant d_1 depending only on n. Thus (2) holds with $C_2 = C_1 d_1$. □

Theorem 3. *Suppose $D \subset \mathbb{R}^n$ is a Greenian domain which contains H_+. There is a constant C_3, depending only on n, such that, for any $\alpha > 0$,*

$$\frac{G_D(x, a_t)}{G_D(x, a_{2\alpha})} \le C_3 \frac{G_D(y, a_t)}{G_D(y, a_{2\alpha})} \quad (x, y \in H_+ \cap B(0, \alpha); t \ge 10\alpha).$$

Proof. By dilation we may assume, without loss of generality, that $\alpha = 2$. From Corollary 2

$$G_D(x,v) \leq C_2 G_D(x,a_4) \quad (x \in H_+ \cap B(0,2); v \in \partial B(0,4) \cap D).$$

Let $R = D \setminus \overline{B(0,4)}$. Now we have

$$G_D(x,v) \leq C_2 G_D(x,a_4) \lambda_v^R(\partial B(0,4)) \quad (x \in H_+ \cap B(0,2); v \in R). \tag{3}$$

Fix $\phi \in C^\infty(\mathbb{R}^n)$ with support in $\{3 \leq |v| \leq 5\}$ such that $0 \leq \phi \leq 1$, and also $\phi = 1$ on $\partial B(0,4)$. Now

$$\lambda_v^R = \beta_n \Delta G_R(v,\cdot) + \delta_v,$$

where $(\beta_n \max\{1, n-2\})^{-1}$ is the surface area of $\partial B(0,1)$, so

$$\begin{aligned} \lambda_v^R(\partial B(0,4)) &\leq \int \phi(w) d\lambda_v^R(w) \\ &= \int \beta_n G_R(v,w) \Delta\phi(w) dw \quad (v \in D \setminus B(0,5)), \end{aligned}$$

where $G_R = 0$ outside $R \times R$. Hence

$$\lambda_v^R(\partial B(0,4)) \leq \beta_n ||\Delta\phi||_1 \sup\{G_R(v,w) : |w| \leq 5\} \quad (v \in D \setminus B(0,5)). \tag{4}$$

Now suppose that $t > 13$. By Theorem 1 applied to R,

$$\begin{aligned} \sup\{G_R(a_t,w) : |w| \leq 5\} &\leq \sup\{G_R(a_t,w) : |w - a_4| \leq 9\} \\ &\leq C_1 G_R(a_t, a_{13}). \end{aligned}$$

In view of (3) and (4) we now have

$$\begin{aligned} \frac{G_D(x,a_t)}{G_D(x,a_4)} &\leq C_2 \lambda_{a_t}^R(\partial B(0,4)) \\ &\leq \beta_n C_2 ||\Delta\phi||_1 \sup\{G_R(a_t,w) : |w| \leq 5\} \\ &\leq \beta_n C_1 C_2 ||\Delta\phi||_1 G_R(a_t,a_{13}) \\ &= d_1 G_R(a_t,a_{13}), \quad \text{say}, \end{aligned} \tag{5}$$

for all $x \in H_+ \cap B(0,2)$ and all $t \geq 20$, where d_1 depends only on n and our choice of ϕ. However, by Harnack's inequality, there are positive constants d_2, d_3, depending only on n, such that

$$\frac{G_D(y,w)}{G_D(y,a_4)} \geq d_2 > 0 \,\text{and}\, G_R(a_{13},w) \leq d_3 \ (y \in H_+ \cap B(0,2); w \in \partial B(a_{13},2)).$$

By the maximum principle

$$\frac{G_D(y,a_t)}{G_D(y,a_4)} \geq \frac{d_2}{d_3} G_R(a_{13},a_t) \quad (t \geq 20).$$

Hence

$$\frac{G_D(x,a_t)}{G_D(x,a_4)} \leq d_1 G_R(a_t,a_{13}) \leq \frac{d_1 d_3}{d_2} \frac{G_D(y,a_t)}{G_D(y,a_4)} \quad (t \geq 20),$$

in view of (5), and the result is now established with $C_3 = d_1 d_3 / d_2$. □

Corollary 4. *Suppose $D \subset \mathbb{R}^n$ is a Greenian domain which contains H_+. Then the closure $\overline{T_1}^M$ of T_1 in $\widehat{D}$ intersects $\partial^M D$ in exactly one point y, and $y \in \partial_1^M D$.*

Proof. Let $M_D(z,y)$ denote the Martin kernel of D with reference point a_{10}. By Theorem 3

$$\frac{1}{C_3} M_D(a_{2\alpha}, x) \leq M_D(a_{2\alpha}, y) \leq C_3 M_D(a_{2\alpha}, x)$$

for all $x, y \in H_+ \cap B(0,\alpha)$ and $\alpha \leq 1$. By continuity this inequality also holds for $x, y \in \overline{H_+ \cap B(0,\alpha)}^M$. Now fix $x \in \cap_{\alpha>0} \overline{H_+ \cap B(0,\alpha)}^M$. If $t < 1$, then by Harnack's inequalities,

$$M_D(z,x) \geq d_1 M_D(z, a_t) \frac{M_D(a_{2t}, x)}{M_D(a_{2t}, a_t)} \qquad (z \in \partial B(a_t, t/2)),$$

where d_1 depends only on n, whence by the minimum principle,

$$M_D(z,x) \geq \frac{d_1}{C_3} M_D(z, a_t) = d_2 M_D(z, a_t), \text{ say} \quad (z \in D \setminus B(a_t, t/2)). \tag{6}$$

If $a_{t_k} \to y \in \overline{T_1}^M \cap \partial^M D$, it follows that

$$M_D(z,x) \geq d_2 M_D(z,y) \qquad (z \in D). \tag{7}$$

Let

$$B_k = B(a_{2^{-k}}, 2^{-k-1}) \text{ and } D_k = \bigcup_{j \geq k} B_j.$$

From (6) we have

$$\begin{aligned} \widehat{R}^{D_k}_{M_D(\cdot,x)}(a_{10}) &\geq \widehat{R}^{B_k}_{M_D(\cdot,x)}(a_{10}) \geq d_2 \widehat{R}^{B_k}_{M_D(\cdot, a_{2^{-k}})}(a_{10}) \\ &\geq d_2 M_D(a_{10}, a_{2^{-k}}) = d_2 \end{aligned}$$

for all k, and so D_1 cannot be minimally thin at all points of $\partial_1^M D$. In particular, $\overline{D_1}^M \cap \partial_1^M D \neq \emptyset$. Further, by (7) and minimality, the sets $\overline{D_1}^M \cap \partial_1^M D$ and $\overline{T_1}^M \cap \partial^M D$ coincide and consist of exactly one point. □

We observe that, in the above proof, we have also shown that

$$\bigcap_{\alpha>0} \overline{H_+ \cap B(0,\alpha)}^M$$

contains exactly one minimal point. (It may also possibly contain non-minimal points.)

By inversion it is easy to see that the preceding results have analogues when a domain satisfies an inner ball condition. In particular, if a Greenian domain $D \subset \mathbb{R}^n$ contains the ball $B(x,r)$ and $y \in \partial D \cap \partial B(x,r)$, then there is exactly one minimal point associated to y that can be reached from the ball $B(x,r)$.

For the remainder of Section 3 we use Ω to denote a Denjoy domain of the form $\mathbb{R}^n \setminus E$, where $E \subset L = \mathbb{R}^{n-1} \times \{0\}$. Further, when $n = 2$, we require that E be non-polar. We define

$$\mathcal{M}_E = \bigcap_{\alpha>0} \overline{(\Omega \cap B(0,\alpha))}^M;$$

that is, $\mathcal{M}_E$ is the set of all Martin boundary points (not necessarily minimal) associated with 0. Further, we let $\mathcal{P}_E$ denote the positive convex cone generated by the functions $\{M_\Omega(\cdot, y) : y \in \mathcal{M}_E\}$. We will say that $\mathcal{P}_E$ is $k-$dimensional if the minimum number of functions in $\mathcal{P}_E$ whose positive linear combinations span $\mathcal{P}_E$ is k.

Remark 5. The functions in $\mathcal{P}_E$ are the same as the set of positive harmonic functions which are bounded outside every neighbourhood of the origin and vanish continuously at every regular boundary point apart from 0. (This will become apparent from the proof of the next theorem.)

The following theorem essentially corresponds to Theorems 2 and 3 in Benedicks [10], but has a somewhat different formulation. Chevallier [13] was the first to exploit minimal thinness in this connection.

Theorem 6. *The set $\mathcal{M}_E \cap \partial_1^M \Omega$ consists of either one or two points, and the corresponding functions span $\mathcal{P}_E$. Further:*

(1) *$\mathcal{P}_E$ is one-dimensional if and only if one of the following equivalent conditions holds:*

(1.a) *all functions in $\mathcal{P}_E$ are symmetric with respect to L;*
(1.b) *there is no point y in $\mathcal{M}_E \cap \partial_1^M \Omega$ such that $\Omega \cap L$ is minimally thin at y.*

(2) *$\mathcal{P}_E$ is two-dimensional if and only if one of the following equivalent conditions holds:*

(2.a) *there is a function in $\mathcal{P}_E$ which is not symmetric with respect to L;*
(2.b) *there is a point y in $\mathcal{M}_E \cap \partial_1^M \Omega$ such that $\Omega \cap L$ is minimally thin at y.*

Proof. By Corollary 4 and the subsequent remark we know that

$$\bigcap_{\alpha>0} \overline{(H_+ \cap B(0,\alpha))}^M \quad \text{and} \quad \bigcap_{\alpha>0} \overline{(H_- \cap B(0,\alpha))}^M$$

each contain exactly one minimal point. We denote these by y_+ and y_-, respectively. (They may or may not be equal.) It is easy to see that

$$\bigcap_{\alpha>0} \overline{(\Omega \cap B(0,\alpha))}^M = \bigcap_{\alpha>0} \overline{(H_+ \cap B(0,\alpha))}^M \cup \bigcap_{\alpha>0} \overline{(H_- \cap B(0,\alpha))}^M.$$

From this we see that $\mathcal{M}_E \cap \partial_1^M \Omega = \{y_+, y_-\}$. We now wish to prove that any function $M_\Omega(\cdot, y)$, where $y \in \mathcal{M}_E$, can be written in the form

$$M_\Omega(\cdot, y) = c_+ M_\Omega(\cdot, y_+) + c_- M_\Omega(\cdot, y_-),$$

for some constants c_+, c_-. To do this it is enough to prove that

$$\widehat{R}^{\Omega\cap B(0,10\alpha)}_{M_\Omega(\cdot,y)} = M_\Omega(\cdot,y) \quad (\alpha > 0), \tag{8}$$

because every point z in $\partial\Omega$ has at most two minimal points associated with it, and $\Omega\setminus B(z,r)$ is minimally thin at both of these for any $r > 0$ (similar statements hold for ∞, by inversion). From Corollary 2, Theorem 3 and Harnack's inequality we see that there is a constant d (depending on n and α) such that

$$M_\Omega(z,y) \le d(M_\Omega(a_{|z|},a_t) + M_\Omega(-a_{|z|},-a_t))$$

for all $z \in \Omega \setminus B(0,10\alpha)$, $y \in \Omega \cap B(0,\alpha)$ and $0 < t < \alpha$. By continuity this holds also for $y \in \overline{\Omega \cap B(0,\alpha)}^M$ and $\pm a_t$ replaced by $y_\pm$. Suppose now that $y^{(k)}$ is a sequence in Ω converging to 0 in the Euclidean topology, and to some point $y \in \mathcal{M}_E$. By the above estimates (if we assume $y^{(k)} \in B(0,\alpha)$ for all k) we see that $M_\Omega(\cdot, y^{(k)})$ converges to $M_\Omega(\cdot,y)$ with bounded convergence on $\partial B(0,10\alpha)\cap\Omega$, and so

$$\widehat{R}^{\Omega\cap B(0,10\alpha)}_{M_\Omega(\cdot,y^{(k)})} \to \widehat{R}^{\Omega\cap B(0,10\alpha)}_{M_\Omega(\cdot,y)}.$$

Since

$$\widehat{R}^{\Omega\cap B(0,10\alpha)}_{M_\Omega(\cdot,y^{(k)})} = M_\Omega(\cdot,y^{(k)})$$

for each k, we obtain (8), whence the functions corresponding to $\mathcal{M}_E \cap \partial_1^M\Omega$ span $\mathcal{P}_E$.

We can choose our normalization of $M_\Omega(\cdot,\cdot)$ to ensure that

$$M_\Omega(x,y_+) = M_\Omega(\tilde{x},y_-),$$

whence $y_+ \neq y_-$ if and only if $M_\Omega(x,y_+) \neq M_\Omega(\tilde{x},y_+)$ for some $x \in \Omega$.

If $\Omega\cap L$, which is the same as $\Omega\setminus(H_+\cup H_-)$, is minimally thin at y_+, then so is $\Omega\setminus H_+$ (because either $\Omega\setminus H_+$ or $\Omega\setminus H_-$ must be minimally thin at y_+, by Lemma 9.6.1 in [8], and it cannot be the latter). By symmetry, $\Omega\setminus H_-$ is minimally thin at y_-, so $y_+ \neq y_-$.

On the other hand, if $y_+ \neq y_-$, then $y_+ \notin \overline{H_-}^M$, by the observation following Corollary 4, so $\Omega\setminus H_+$ is certainly minimally thin at y_+, and similarly $\Omega\setminus H_-$ is minimally thin at y_-. □

Again there is nothing special about the boundary point 0 in the above result. The same phenomenon holds for all boundary points, including ∞ (as can be seen by means of an inversion). When constructing certain examples we will actually work with ∞ for reasons of notational convenience.

Some simple examples are as follows. Firstly, if E contains $B(0,\varepsilon)\cap L$ for some $\varepsilon > 0$, then it is clear that $\mathcal{P}_E$ is two-dimensional. In fact, $\mathcal{M}_E$ consists of exactly two points in this case, and so, in particular, it is not connected. On the other hand, if there is a sequence of balls $B(x^{(k)},r_k)$ in Ω such that $x^{(k)} \in L$, $x^{(k)} \to 0$ and $r_k \ge c|x^{(k)}|$ for some $c \in (0,1)$, then $\mathcal{P}_E$ is one-dimensional. The reason for this is that $a_{|x^{(k)}|}$ can be connected to $-a_{|x^{(k)}|}$ by a Harnack chain whose length does not depend on k, and so the sequences $\left(\pm a_{|x^{(k)}|}\right)$ converge to the same minimal point.

Example 7. We now outline a less trivial example, due to Ancona [3], where $\dim \mathcal{P}_E = 2$. As we will work in the plane it is convenient to use complex notation. Let

$$D = \mathbb{C} \setminus \bigcup_{k \in \mathbb{Z}} [k + 1/4, k + 3/4] ,$$

and

$$\Omega = \{z \in \mathbb{C} : 1/z \in D\} = \mathbb{C} \setminus \left(\bigcup_{k \in \mathbb{Z}} \left[\frac{1}{k + 3/4}, \frac{1}{k + 1/4} \right] \cup \{0\} \right).$$

We denote the two (possibly equal) minimal points associated with ∞ on D by w_+ and w_- respectively. We want to prove that $w_+ \neq w_-$. From Theorem 6 we know that $w_+ = w_-$ if and only if the set $D \cap \mathbb{R}$ is not minimally thin at w_+. We also know that $it \to w_+$ as $t \to +\infty$, so it is enough to prove that the unbounded function $M_D(\cdot, w_+)$ is bounded above by some positive constant on $D \cap \mathbb{R}$. To prove this we see from symmetry that

$$G_D(p, it) = G_D(0, p + it) \quad (p \in \mathbb{Z}; t \in (0, \infty)).$$

Hence by Theorem 1 and Harnack's inequalities,

$$\begin{aligned} G_D(p, it) &= G_D(0, p + it) \leq C_1 G_D(0, i\sqrt{p^2 + t^2}) \\ &\leq C_1 d_1 G_D(0, it) \quad (p \in \mathbb{Z}; t \geq 2p), \end{aligned}$$

where d_1 is an absolute constant. Thus

$$M_D(p, w_+) = \lim_{t \to \infty} \frac{G_D(p, it)}{G_D(0, it)} \leq C_1 d_1 \quad (p \in \mathbb{Z}).$$

Harnack's inequalities can be applied to the circles $\partial B(k, 1)$ to see that there is a constant d_2 such that

$$M_D(x, w_+) \leq d_2 \quad (x \in \partial B(k, 1); k \in \mathbb{Z}),$$

and this bound also holds on $B(k, 1) \cap D$ by the maximum principle. In particular, $M_D(\cdot, w_+)$ is bounded on $D \cap \mathbb{R}$, and so $w_+ \neq w_-$.

3.2. A Wiener-type criterion

We have seen that $\mathcal{P}_E$ is either one- or two-dimensional. Benedicks [10] provided an integrated harmonic measure criterion involving "moving cubes" for distinguishing between these two cases. This criterion was recently shown by Carroll and Gardiner [12] to be equivalent to one involving capacity. In this section we will combine ideas from [10] and [12] to give a direct proof of this Wiener-type characterization.

Let $\mathcal{C}(A)$ denote the Newtonian (or logarithmic, if $n = 2$) capacity of a set A. Also, let $\gamma \in (0, 1/3)$ and $D(r) = L \cap \overline{B(0,r)}$ and, for any $k = 0, 1, ...$, let $D_k = D(2^{-k})$ and

$$E_k = (E \cap D_k) \cup D(\gamma 2^{-k}) \cup \overline{D_k \backslash D((1-\gamma)2^{-k})}.$$

In the sequel we will use $C(n, \ldots)$ to denote a constant depending at most on $n, \ldots$; its value may change from line to line.

Theorem 8. *For a Denjoy domain $\Omega = \mathbb{R}^n \backslash E$ with $0 \in E$ the following statements are equivalent:*
(a) $\mathcal{P}_E$ is two-dimensional;
(b) $\begin{cases} \sum 2^{k(n-2)} [\mathcal{C}(D_k) - \mathcal{C}(E_k)] < \infty & (n \geq 3) \\ \sum 2^{k} [\mathcal{C}(D_k) - \mathcal{C}(E_k)] < \infty & (n = 2) \end{cases}$.

In the proof of Theorem 8 we will make use of the following elementary version of the boundary Harnack principle (see Lemma 8.5.1 in [8]).

Lemma 9. *Let $z \in L$ and $0 < \alpha < 1$. If g, h are positive harmonic functions on $B(z,r) \cap H_+$ which continuously vanish on $B(z,r) \cap L$, then g/h has a positive continuous extension to $B(z,r) \cap (H_+ \cup L)$ and*

$$\frac{g(x)}{h(x)} \leq C(n,\alpha) \frac{g(y)}{h(y)} \quad \text{for any} \quad x, y \in B(z, \alpha r) \cap H_+.$$

We will give a proof of Theorem 8 when $n \geq 3$ and leave the adjustments required for the plane case to the reader. For any compact set $K \subset \mathbb{R}^n$, we denote by v_K the capacitary potential of K, and by μ_K the associated Riesz measure. We also write

$$A_k = \{x' : 2^{-k} \leq 3\,|x'| \leq 2^{1-k}\} \quad \text{and} \quad A_k^* = \{x' : \gamma 2^{-k} \leq |x'| \leq (1-\gamma)2^{-k}\}.$$

Suppose firstly that $\mathcal{P}_E$ is two-dimensional. By Theorem 6 there is a function in $\mathcal{P}_E$ which is not symmetric with respect to L. It follows, by consideration of Poisson integral representations in H_+ and in H_-, and by symmetrization, that there is a function u in $\mathcal{P}_E$ satisfying $u \geq |h|$, where $h(x) = x_n\,|x|^{-n}$. Now $1 - v_{E_k}$ vanishes on $D(\gamma 2^{-k})$ and $D_k \backslash D((1-\gamma)2^{-k})$. It thus follows from Lemma 9 and Harnack's inequalities that

$$\frac{1 - v_{E_k}}{u} \leq \frac{1 - v_{E_k}}{|h|} \leq C(n,\gamma) \frac{1 - v_{E_k}(0', 2^{-k})}{h(0', 2^{-k})} \leq C(n,\gamma) 2^{k(1-n)}$$

on $\partial B(0, (1-\gamma/2)2^{-k}) \backslash L$ and on $\partial B(0, \gamma 2^{-k-1}) \backslash L$. We can therefore apply the maximum principle on the open set

$$B(0, (1-\gamma/2)2^{-k}) \backslash [E \cup \overline{B(0, \gamma 2^{-k-1})}]$$

to see that

$$1 - v_{E_k} \leq C(n,\gamma) 2^{k(1-n)} u \quad \text{on} \quad (A_k^* \times \{0\}) \backslash E. \tag{9}$$

Also, $d\mu_{D_0}(x', x_n) = f\,(|x'|)\,dx' d\delta_0$, where δ_0 is the Dirac measure at 0 in $\mathbb{R}$ and $f : [0, 1) \to (0, \infty)$ is continuous. (This can be shown using Green's theorem and the fact that the function $x' \mapsto \lim_{t\to 0+}(1-v_{D_0}(x',t))/t$ is positive and continuous on $\{|x'|<1\}$, by Lemma 9.) Letting $c_1 = \max_{[0,1-\gamma]} f$, we can thus use dilation to see that

$$d\mu_{D_k} \leq 2^k c_1 dx' d\delta_0 \quad \text{on} \quad D((1-\gamma)2^{-k}). \tag{10}$$

Since $v_{D_k} = 1$ on D_k and $E_k \subseteq D_k$, we have

$$\begin{aligned} \mathcal{C}(D_k) - \mathcal{C}(E_k) &= \mathcal{C}(D_k) - \int_{E_k} v_{D_k}(y) d\mu_{E_k}(y) \\ &= \mathcal{C}(D_k) - \int_{E_k}\int_{D_k} \frac{d\mu_{D_k}(x)}{|x-y|^{n-2}} d\mu_{E_k}(y) \\ &= \int_{D_k}\left(1 - \int_{E_k} \frac{d\mu_{E_k}(y)}{|x-y|^{n-2}}\right) d\mu_{D_k}(x) \\ &= \int_{D_k} (1-v_{E_k})\, d\mu_{D_k} \qquad (11) \\ &\leq 2^k c_1 \int_{A_k^*} (1 - v_{E_k}(x',0)) dx' \qquad (12) \\ &\leq C(n,\gamma) 2^{k(2-n)} \int_{A_k^*} u(x',0) dx', \end{aligned}$$

by (10) and then (9). Condition (b) now follows from the integrability of u on D_0.

The elementary lemma given below will be used in the proof of the converse. Let $B = B(0,1)$ and let σ denote surface area measure on ∂B.

Lemma 10. *Let $q \in (0,1)$ be such that $\sigma(S_1) = \sigma(\partial B)/2$, where $S_1 = \{y \in \partial B : |y_n| \geq q\}$, and let $U = B \backslash F$, where F is a closed subset of L and $0 \notin F$. Then $\lambda_0^U(\partial B) \leq 2\lambda_0^U(S_1)$.*

Proof of lemma. Let $S_2 = \partial B \backslash S_1$ and $u_i(x) = \lambda_x^B(S_i)$ $(i = 1,2)$. Clearly $u_1(0) = u_2(0) = \frac{1}{2}$. Further, it follows from the maximum principle and considerations of symmetry that, for any $\varepsilon > 0$, the set $\{u_2 > \frac{1}{2} - \varepsilon\}$ contains $B \cap L$. Hence $u_2 \geq \frac{1}{2} \geq u_1$ on $L \cap B$. Since

$$\lambda_x^U(S_i) = u_i(x) - \int_{B\cap\partial U} u_i d\lambda_x^U \qquad (x \in U)$$

and $B \cap \partial U \subset B \cap L$, we see that

$$\lambda_0^U(S_2) \leq \frac{1}{2} - \int_{B\cap\partial U} u_1 d\lambda_0^U = \lambda_0^U(S_1),$$

and the lemma follows. □

If $x' \in A_k$, we define

$$B_{x'} = B((x',0),(\frac{1}{3}-\gamma)2^{-k}) \quad \text{and} \quad S_{x'} = \{y \in \partial B_{x'} : y_n > q(\frac{1}{3}-\gamma)2^{-k}\}.$$

It follows from Lemma 10, symmetry, dilation and the maximum principle that

$$\begin{aligned}\lambda_{(x',0)}^{B_{x'}\setminus E}(\partial B_{x'}) &\leq 4\lambda_{(x',0)}^{B_{x'}\setminus E}(S_{x'}) \\ &\leq 4\frac{1-v_{E_k}(x',0)}{\min_{S_{x'}}(1-v_{E_k})} \\ &\leq 4\frac{1-v_{E_k}(x',0)}{\min_{S_{x'}}(1-v_{D_k})} \\ &\leq C(n,\gamma)\left(1-v_{E_k}(x',0)\right).\end{aligned}$$

Letting $c_2 = \min_{[0,2/3]} f$, we can now argue as in the first part of the proof to see that

$$\begin{aligned}\int_{A_k} \lambda_{(x',0)}^{B_{x'}\setminus E}(\partial B_{x'})dx' &\leq c_2^{-1}2^{-k}C(n,\gamma)\int_{D_k}(1-v_{E_k})\,d\mu_{D_k} \\ &= C(n,\gamma)2^{-k}\left[\mathcal{C}(D_k)-\mathcal{C}(E_k)\right]. \qquad (13)\end{aligned}$$

Now suppose that $\mathcal{P}_E$ is one-dimensional and let $u \in \mathcal{P}_E$. It follows from Theorem 6 that u is symmetric with respect to L and it is not difficult to see that

$$u(0',t) = C(n)\int_{\mathbb{R}^{n-1}} \frac{tu(x',0)}{\left\{|x'|^2+t^2\right\}^{n/2}}dx' \qquad (t>0). \qquad (14)$$

Since $t^{n-1}u(0',t) \downarrow 0$ as $t \to 0+$, we have

$$t^{n-1}u(0',t) \leq 2^{j(1-n)}u(0',2^{-j}) \qquad (0 < t \leq 2^{-j}). \qquad (15)$$

It follows easily from Theorem 1 that

$$u(x) \leq C(n)u(0,|x|) \qquad (x \in \Omega), \qquad (16)$$

and so, by Harnack's inequalities,

$$u(x',0) = \int u d\lambda_{(x',0)}^{B_{x'}\setminus E} \leq C(n)u(0,|x'|)\lambda_{(x',0)}^{B_{x'}\setminus E}(\partial B_{x'}) \quad (x' \neq 0').$$

Combining this with (14), we obtain

$$\begin{aligned}u(0',2^{-j}) &\leq C(n)\int_{\mathbb{R}^{n-1}} \frac{2^{-j}u(0,|x'|)}{\left\{|x'|^2+2^{-2j}\right\}^{n/2}}\lambda_{(x',0)}^{B_{x'}\setminus E}(\partial B_{x'})dx' \\ &= C(n)\{J_1+J_2+J_3\}, \qquad (17)\end{aligned}$$

where J_1, J_2 and J_3 are integrals of the preceding integrand over

$$\{|x'| \leq 2^{1-j}/3\}, \quad \{2^{1-j}/3 < |x'| \leq 2/3\} \text{ and } \{|x'| > 2/3\},$$

respectively. Using (15), and then (13), we see that

$$\begin{aligned} J_1 &\leq \int_{\{|x'|\leq 2^{1-j}/3\}} \frac{2^{-jn}\,|x'|^{1-n}\,u(0',2^{-j})}{2^{-jn}} \lambda_{(x',0)}^{B_{x'}\backslash E}(\partial B_{x'})dx' \\ &\leq (3/2)^{n-1}u(0',2^{-j})\sum_{k=j}^{\infty}2^{k(n-1)}\int_{A_k}\lambda_{(x',0)}^{B_{x'}\backslash E}(\partial B_{x'})dx' \\ &\leq C(n,\gamma)u(0',2^{-j})\sum_{k=j}^{\infty}2^{k(n-2)}\left[\mathcal{C}(D_k)-\mathcal{C}(E_k)\right]. \end{aligned} \tag{18}$$

Since, by the maximum principle and (16),

$$u \leq \sup_{\partial B(0,r)} u \leq C(n)u(0',r) \quad \text{on} \quad \Omega\backslash\overline{B(0,r)},$$

we can also use (13) and Harnack's inequalities to see that

$$\begin{aligned} J_2 &\leq C(n)\int_{\{2^{1-j}/3<|x'|\leq 2/3\}} \frac{2^{-j}u(0',2^{1-j}/3)}{|x'|^n} \lambda_{(x',0)}^{B_{x'}\backslash E}(\partial B_{x'})dx' \\ &\leq C(n,\gamma)u(0',2^{-j})2^{-j}\sum_{k=1}^{j-1}2^{k(n-1)}\left[\mathcal{C}(D_k)-\mathcal{C}(E_k)\right]. \end{aligned} \tag{19}$$

Also,

$$J_3 \leq C(n)u(0',2^{-j})2^{-j}\int_{\{|x'|>2/3\}}|x'|^{-n}\,dx' = C(n)u(0',2^{-j})2^{-j}. \tag{20}$$

Combining (17) - (20) yields

$$\begin{aligned} 1 \leq\ & C(n,\gamma)\left\{\sum_{k=j}^{\infty}2^{k(n-2)}\left[\mathcal{C}(D_k)-\mathcal{C}(E_k)\right]\right. \\ & \left. +2^{-j}\sum_{k=1}^{j-1}2^{k(n-1)}\left[\mathcal{C}(D_k)-\mathcal{C}(E_k)\right]+2^{-j}\right\} \qquad (j\geq 1), \end{aligned}$$

which is incompatible with convergence of $\sum_k 2^{k(n-2)}\left[\mathcal{C}(D_k)-\mathcal{C}(E_k)\right]$. This completes the proof of Theorem 8.

3.3. Examples

We will now briefly illustrate when the criterion in Theorem 8 holds. When $n=2$, we know that $\mathcal{C}(D_k)=2^{-k-1}$ and $\mathcal{C}(E_k)\geq l_1(E_k)/4$, where l_m denotes m-dimensional measure, so from Theorem 8 it follows that $\mathcal{P}_E$ is two-dimensional if $\sum 2^k l_1(D_k\backslash E_k)<\infty$. However, a sharper result is true.

Corollary 11. *Let $n\geq 2$. If $\sum 2^{nk}\left[l_{n-1}\left(D_k\backslash E_k\right)\right]^{n/(n-1)}<\infty$, then $\mathcal{P}_E$ is two-dimensional.*

The corollary will follow, by dilation, once we establish the following lemma. (We present the details only for the case where $n \geq 3$.)

Lemma 12. *If $F = D(1)\backslash W$, where W is a relatively open subset of $D(1-\gamma)$, then*

$$\mathcal{C}(D(1)) - \mathcal{C}(F) \leq C(n,\gamma)\left[l_{n-1}(W)\right]^{n/(n-1)}.$$

Proof. To see this, we choose ρ such that $l_{n-1}(D(\rho)) = 2l_{n-1}(W)$. Thus $\rho = C(n)\left(l_{n-1}(W)\right)^{1/(n-1)}$. We may assume, without loss of generality, that $\rho < 2^{-3/2}\gamma$. Let

$$U = \{(x', x_n) : |x' - z'| < |x_n| < \rho \text{ for some } (z', 0) \in W\} \cup W,$$

let $h(x) = \lambda_x^{\omega}(\partial B(0,1))$, where $\omega = (B(0,1)\backslash L) \cup W$, and let $m = \sup_W h$. Elementary estimates of harmonic measure show that there is a constant $c_3 \in (0,1)$, depending only on n and γ, such that $\lambda_x^{H_+}(W) \leq c_3$ when $x \in H_+ \cap \partial U$. Let $V = B(0,1) \cap H_+$. Since

$$h(x) = \lambda_x^V(H_+ \cap \partial B(0,1)) + \int_W h\, d\lambda_x^V \qquad (x \in V),$$

$$\int_W h\, d\lambda_x^V \leq \int_W h\, d\lambda_x^{H_+} \leq c_3 m \qquad (x \in H_+ \cap \partial U),$$

and

$$\lambda_x^V(H_+ \cap \partial B(0,1)) \leq C(n,\gamma)x_n \qquad (x \in H_+ \cap B(0, 1-\gamma/2)),$$

we see that

$$h \leq C(n,\gamma)\rho + c_3 m \quad \text{on} \quad H_+ \cap U,$$

whence

$$m = \sup_W h \leq C(n,\gamma)(1-c_3)^{-1}\rho.$$

Finally, we use the fact that $1 - v_F \leq h$ on $B(0,1)\backslash F$ to see (as in (11)) that

$$\begin{aligned}
\mathcal{C}(D(1)) - \mathcal{C}(F) &= \int_{D(1)} (1 - v_F)\, d\mu_{D(1)} \\
&\leq \int_W h\, d\mu_{D(1)} \\
&\leq C(n,\gamma)\rho\mu_{D(1)}(W) \\
&\leq C(n,\gamma)\left[l_{n-1}(W)\right]^{1+1/(n-1)}.
\end{aligned}$$

in view of (10) and our choice of ρ. □

Corollary 11 provides a sufficient condition for $\mathcal{P}_E$ to be two-dimensional. A necessary condition is that

$$\int_{\{|x'| \leq 1\}} |x'|^{-n} \operatorname{dist}((x', 0), E)dx' < \infty. \tag{21}$$

To see this, we note from Theorem 6 that, if $\mathcal{P}_E$ is two-dimensional, then there is a function u in $\mathcal{P}_E$ satisfying $u(x) \geq x_n |x|^{-n}$ on Ω, whence

$$u(x',0) \geq C(n)u(x',\mathrm{dist}((x',0),E)) \geq C(n)\mathrm{dist}((x',0),E)\,|x'|^{-n}$$

by Harnack's inequalities, and (21) now follows from the local integrability of u on L.

Combining Corollary 11 with the observed necessity of (21), we see that, for

$$E = L\backslash\left(\bigcup_k B(x^{(k)},r_k)\right), \quad \text{where} \quad x^{(k)} \in L\cap\partial B(0,2^{-k}) \quad \text{and} \quad r_k < 2^{-k},$$

the cone $\mathcal{P}_E$ is two-dimensional if and only if $\sum 2^{nk}r_k^n < \infty$.

Further illustrations of condition (b) in Theorem 8 may be found in [12]. In particular, it is shown there that, for

$$E = L\backslash\left\{x \in (0,1)\times\mathbb{R}^{n-2}\times\{0\} : \sqrt{x_2^2+...+x_{n-1}^2} < g(x_1)\right\},$$

where $n \geq 3$ and $g:(0,1)\to(0,\infty)$ is increasing, $\mathcal{P}_E$ is two-dimensional if and only if

$$\int_0^1 t^{-n}\,[g(t)]^{n-1}\,dt < \infty.$$

3.4. Boundary behaviour of the Green function

Up to now we have been discussing the relationship between the Euclidean and Martin boundaries of Denjoy domains. Next we consider the boundary behaviour of their Green functions. Let $x_0 \in \Omega$. We will say that $G_\Omega(x_0,\cdot)$ is *Lipschitz continuous at* 0 if there is a constant $C > 0$ such that $G_\Omega(x_0,x) \leq C|x|$ on some neighbourhood of 0, where $G_\Omega(x_0,\cdot)$ is interpreted as 0 on E. This definition is independent of the choice of x_0, in view of Harnack's inequalities. Since $G_{H_+}(x_0,x)/x_n$ has a finite (positive) limit at 0, it is clear that $G_{H_+}(x_0,\cdot)$ is Lipschitz continuous at 0. The next result shows that this remains true of $G_\Omega(x_0,\cdot)$ when Ω is sufficiently like H_+ near 0.

Theorem 13. *For a Denjoy domain* $\Omega = \mathbb{R}^n\backslash E$ *with* $0 \in E$*, and for any point* $x_0 \in \Omega$*, the following statements are equivalent:*
(a) $G_\Omega(x_0,\cdot)$ *is Lipschitz continuous at* 0*;*
(b) $\begin{cases} \sum 2^{k(n-2)}\left[\mathcal{C}(D_k)-\mathcal{C}(E_k)\right] < \infty & (n\geq 3) \\ \sum 2^{k}\left[\mathcal{C}(D_k)-\mathcal{C}(E_k)\right] < \infty & (n=2) \end{cases}$.

This result was first established for the case $n=2$ by Carleson and Totik [14], without reference to the work of Benedicks [10]. Below we will present a proof of this result in all dimensions, taken from [12], which explains why the same condition appears in both Theorems 8 and 13.

A few preliminary comments may serve to illuminate this phenomenon. Thinness (in the ordinary sense) of a set A at 0 may be characterized by the existence of a superharmonic function v on a neighbourhood of 0 such that

$$\liminf_{x\to 0, x\in A} v(x) > v(0) = \liminf_{x\to 0} v(x). \tag{22}$$

However, it may equivalently be characterized by the existence of a superharmonic function v on a neighbourhood of 0 such that

$$\liminf_{x\to 0, x\in A} \frac{v(x)}{|x|^{2-n}} > \liminf_{x\to 0} \frac{v(x)}{|x|^{2-n}}. \tag{23}$$

(When $n = 2$, we replace $|x|^{2-n}$ by $\log(1/|x|)$ in (23).) A similar duality occurs in the case of minimal thinness. By analogy with (23), a set $A \subset \Omega$ is minimally thin at $y \in \partial_1^M \Omega$ if and only if there is a positive superharmonic function v on Ω such that

$$\liminf_{x\to y, x\in A} \frac{v(x)}{M_\Omega(x,y)} > \liminf_{x\to y} \frac{v(x)}{M_\Omega(x,y)},$$

but an equivalent characterization is that there is a positive superharmonic function v on Ω such that

$$\liminf_{x\to y, x\in A} \frac{v(x)}{G_\Omega(x_0,x)} > \liminf_{x\to y} \frac{v(x)}{G_\Omega(x_0,x)}$$

(cf. (22)). It is this possibility of characterizing minimal thinness in terms either of minimal harmonic functions or of the Green function that allows us to connect the dimensionality of $\mathcal{P}_E$ with the boundary behaviour of G_Ω. We will now make this connection precise.

Proof. Suppose firstly that condition (b) of Theorem 13 holds, and let $x_0 = (0', 1)$. By Theorems 8 and 6 there is a point $y \in \mathcal{M}_E \cap \partial_1^M \Omega$ such that $\Omega \cap L$ is minimally thin at y. The associated minimal harmonic function u, say, must differ from its Poisson integral in one of the halfspaces, H_+, say. Hence $\Omega \backslash H_+$ is minimally thin at y, and it follows from Theorem 9.5.2 of [8] that

$$\limsup_{x\to y, x\in H_+} \frac{G_{H_+}(x_0,x)}{G_\Omega(x_0,x)} > 0.$$

In view of Remark 5, we know that any sequence of points in H_+ that converges to y must converge to 0 in the Euclidean topology, so

$$\limsup_{x\to 0, x\in H_+} \frac{G_{H_+}(x_0,x)}{G_\Omega(x_0,x)} > 0.$$

Hence, by Theorem 9.3.3(ii) of [8], $G_\Omega(x_0,\cdot)/G_{H_+}(x_0,\cdot)$ has a finite minimal fine limit at 0 relative to H_+, and so a finite nontangential limit there, by Theorem 9.7.4 of [8]. Since $G_\Omega(x_0,x) \le C(n) G_\Omega(x_0,(0',|x|))$, by Theorem 1, we now see that $G_\Omega(x_0,x)/|x|$ is bounded above on $B(0,1/2) \cap \Omega$, and so $G_\Omega(x_0,\cdot)$ is Lipschitz continuous at 0.

Conversely, suppose that condition (a) holds, and let y be the minimal Martin boundary point of Corollary 4 (with $D = \Omega$). Since

$$\limsup_{x\to y, x\in H_+} \frac{G_{H_+}(x_0,x)}{G_\Omega(x_0,x)} \geq C(n) \limsup_{t\to 0+} \frac{t}{G_\Omega(x_0,(0',t))} > 0,$$

by hypothesis, we can apply Theorem 9.5.2 of [8] again to see that $\Omega \backslash H_+$, and hence $L \cap \Omega$, is minimally thin at y. Theorem 6 now shows that $\mathcal{P}_E$ is two-dimensional, and condition (b) follows by Theorem 8. □

4. Applications

4.1. Approximation of positive harmonic functions

Let $D \subset \mathbb{R}^n$ be a Greenian domain. In this section, which is based on [26], we shall study the following approximation problem: *when is $\mathcal{HPL}^p(D)$ dense (with respect to uniform convergence on compact subsets in D) in $\mathcal{HP}(D)$?* We will deal with the cases $0 < p \leq 1$ and $p = \infty$ (so the case $p = \infty$ refers to approximation by bounded positive harmonic functions).

In all cases there are some trivial counterexamples. In the case $p = \infty$ the punctured ball $D = B(0,1) \setminus \{0\}$ is a counterexample. Indeed it is easy to see that $\overline{\mathcal{HPL}^\infty(D)} \neq \mathcal{HP}(D)$ if $\partial D \setminus \operatorname{supp}(\lambda_x^D)$ is non-empty, because the latter set is removable for bounded harmonic functions. Of course, a point such as the origin in the case of the punctured ball corresponds to exactly one Martin boundary point, which is minimal, and the associated minimal harmonic function has the same singularity as the fundamental solution of the Laplacian, and so is integrable. However, we easily obtain counterexamples in the case where $0 < p \leq 1$ if we allow unbounded domains D, for then the halfspace $D = H_+$ is a domain for which $\overline{\mathcal{HPL}^p(D)} \neq \mathcal{HP}(D)$ (cf. Section 1).

We now give a general theorem which relates the above approximation problem to "topological" properties of the Martin boundary. Later we will see that Denjoy domains play a role in constructing non-trivial counterexamples to the above approximation properties.

Theorem 14. *Let $D \subset \mathbb{R}^n$ be a Greenian domain.*

(1) *Let $0 < p \leq 1$, and $A_p = \{y \in \partial_1^M D : M_D(\cdot, y) \in \mathcal{L}^p(D)\}$. Then $\overline{\mathcal{HPL}^p(D)} = \mathcal{HP}(D)$ if and only if $\partial_1^M D \subset \overline{A_p}^M$.*

(2) *$\overline{\mathcal{HPL}^\infty(D)} = \mathcal{HP}(D)$ if and only if $\partial_1^M D \subset \operatorname{supp}(\nu_1)$.*

Proof. (1) Firstly we note that, if D_k is an exhaustion of D, then

$$A_p = \partial_1^M D \cap \{y \in \partial^M D : \lim_{k\to\infty} \int_{D_k} (M_D(x,y))^p dx < \infty\},$$

so A_p is a Borel set.

Suppose now that $h \in \mathcal{HPL}^p(D)$, and assume without loss of generality that $h(x_0) = 1$, where x_0 is the reference point of $M_D(\cdot, \cdot)$. Then, by Tonelli's theorem and Jensen's inequality,

$$\begin{aligned}\int_{\partial_1^M D} \int_D (M_D(x,y))^p dx d\nu_h(y) &\leq \int_D \left(\int_{\partial_1^M D} M_D(x,y) d\nu_h(y) \right)^p dx \\ &= \int_D (h(x))^p \, dx < \infty,\end{aligned}$$

and so ν_h is carried by A_p.

Let us now introduce two convex subcones of $\mathcal{HP}(D)$ as follows: $\mathcal{K}_1$ denotes the cone of all (finite) positive linear combinations of functions of the form $M_D(\cdot, y)$ for $y \in A_p$, and $\mathcal{K}_2$ denotes the cone of all those positive harmonic functions h which have a representing measure (not necessarily the one carried by $\partial_1^M D$) supported by $\overline{A_p}^M$.

We note that

$$\mathcal{K}_1 \subset \mathcal{HPL}^p(D) \subset \mathcal{K}_2,$$

and $\mathcal{K}_2$ is closed by weak*-compactness. We will now prove that $\overline{\mathcal{K}_1} = \mathcal{K}_2$ by an application of the Hahn-Banach theorem (about separation of convex cones). Suppose η is a signed Radon measure with compact support in D such that $\int k d\eta \geq 0$ for all $k \in \mathcal{K}_1$. In particular, by assumption,

$$\int M_D(x,y) d\eta(x) \geq 0 \quad (y \in A_p),$$

and by continuity this also holds for all $y \in \overline{A_p}^M$. Hence, if $h \in \mathcal{K}_2$, so that h has a representing measure ν supported by $\overline{A_p}^M$, then

$$\int h d\eta = \int \int M_D(x,y) d\nu(y) d\eta(x) = \int \int M_D(x,y) d\eta(x) d\nu(y) \geq 0.$$

This proves that $\mathcal{K}_2 \subset \overline{\mathcal{K}_1}$, and hence the first part of the theorem is proved.

(2) This follows by analogous reasoning using the fact that $\mathcal{HPL}^\infty(D)$ is precisely the set of functions representable in the form $\int M_D(\cdot, y) f(y) d\nu_1(y)$, where f is a non-negative bounded Borel measurable function. □

Example 15. Theorem 14 shows that topological information about the minimal Martin boundary points is crucial for these approximation questions. We now claim that the Denjoy domain

$$\Omega = \mathbb{C} \setminus \left(\bigcup_{k \in \mathbb{Z}} \left[\frac{1}{k+3/4}, \frac{1}{k+1/4} \right] \cup \{0\} \right)$$

from Example 7 has the property that the two distinct minimal points y_+ and y_- corresponding to 0 are isolated in $\partial_1^M \Omega$. This is, of course, equivalent to saying that the two distinct minimal points w_+ and w_- corresponding to ∞ for the domain D

in the same example are isolated in $\partial_1^M D$. For reasons of notational convenience we will establish this latter assertion. We again choose 0 as our reference point.

To show this, we note from symmetry that every sequence $x^{(k)}$ in $\mathbb{R}\cap D$ converging to ∞ satisfies

$$M_D(\cdot,x^{(k)}) \to \frac{1}{2}\left(M_D(\cdot,w_+)+M_D(\cdot,w_-)\right).$$

It follows by applying Harnack's inequality to the circles $\partial B(k,1)$ that no sequence $\left(x^{(k)}\right)$ with $x^{(k)}\in\partial B(k,1)$ can converge to either w_+ or w_-. From this the claim follows easily using the maximum principle.

We note that Ancona's interest in this example stemmed from the fact that $\partial_1^M\Omega$ is not dense in $\partial^M\Omega$ in this case.

Example 16. We now give non-trivial examples of domains D_p (that is, bounded domains for which $\operatorname{supp}(\lambda_x^{D_p})=\partial D_p$) where $\overline{\mathcal{HPL}^p(D_p)}\neq\mathcal{HP}(D_p)$.

We begin with the case $p=\infty$. The Denjoy domain Ω of Example 15 has the property that $\overline{\mathcal{HPL}^\infty(\Omega)}\neq\mathcal{HP}(\Omega)$, because the points y_+ and y_- are not in the support of ν_1 (see Theorem 14 (2)). If we define $D_\infty=B(0,1)\cap\Omega$, we get a bounded domain with $\operatorname{supp}(\lambda_x^{D_\infty})=\partial D_\infty$, yet $\overline{\mathcal{HPL}^\infty(D_\infty)}\neq\mathcal{HP}(D_\infty)$.

However, every positive harmonic function on D_∞ is integrable. To treat the case where $0<p\leq 1$ we fix such a p, choose $k\in\mathbb{N}$ such that $pk\geq 1$, and define $D_p=\{z\in\mathbb{C}: z^{2k}\in D_\infty\}$. Then

$$D_p=B(0,1)\setminus\bigcup_{j=0}^{4k-1}\left\{\sqrt[2k]{r}\exp\left(i\frac{\pi j}{2k}\right): r\in E\cap(0,\infty)\right\}.$$

(The boundary of D_p in $B(0,1)$ consists of subsets of $4k$ rays emanating from 0 with angular spacing $\pi/2k$.) From the construction of the Martin boundary it is easy to see that, analogously to the case of D_∞, the origin corresponds to $4k$ distinct points in $\partial_1^M D_p$, and these points are isolated in $\partial_1^M D_p$.

Further, the growth of the corresponding minimal functions is comparable to

$$\operatorname{Im}\left(\frac{1}{z^{2k}}\right)=\frac{1}{r^{2k}}\sin(2\theta k)\text{ on }0<\theta<\pi/2k,$$

where $z=re^{i\theta}$. Since $pk\geq 1$, we get

$$\int_0^1\int_0^{\pi/2k}\left(\frac{1}{r^{2k}}\sin(2\theta k)\right)^p r\,dr\,d\theta=\int_0^1\frac{1}{r^{2pk-1}}dr\int_0^{\pi/2k}\sin^p(2\theta k)d\theta=+\infty.$$

By applying Theorem 14 (1) we now see that $\overline{\mathcal{HPL}^p(D_p)}\neq\mathcal{HP}(D_p)$.

4.2. Minimal harmonic functions associated with an irregular boundary point

As is well known, the minimal harmonic functions on the unit ball B are simply multiples of the Poisson kernel with arbitrary boundary pole; that is, they are multiples of the functions

$$v_z: x\mapsto\frac{1-|x|^2}{|x-z|^n}\qquad(z\in\partial B).$$

We note that v_z continuously vanishes on $\partial B \backslash \{z\}$ but tends to ∞ along a tangential approach region to z. Now let $U = B \backslash \{x_0\}$, where $x_0 \in B$. The minimal harmonic functions on U comprise all the minimal harmonic functions on B together with multiples of

$$v_{x_0} : x \mapsto G_B(x, x_0) = u_{x_0}(x) - H^B_{u_{x_0}}(x),$$

where

$$u_y(x) = \begin{cases} |x-y|^{2-n} & (n \geq 3) \\ -\log|x-y| & (n = 2) \end{cases}$$

and H^V_f denotes the solution to the Dirichlet problem on V with boundary function f. Clearly v_{x_0} continuously vanishes on $\partial U \backslash \{x_0\}$ and tends to ∞ at x_0. This observation, concerning the irregular boundary point x_0 of U, illustrates the following general fact. Let U be a (Greenian) domain with an irregular boundary point x_0 (so $\mathbb{R}^n \backslash U$ is thin at x_0), and define

$$G_U(x, x_0) = u_{x_0}(x) - H^U_{u_{x_0}}(x) \qquad (x \in U).$$

If u is a positive multiple of $G_U(\cdot, x_0)$, then u is a minimal harmonic function on U and $u(x) \to \infty$ as $x \to x_0$ outside a set which is thin at x_0. Brelot [11] observed that the converse to this statement also holds when $n = 2$, but the corresponding question in higher dimensions has remained open until the following recent result [21].

Theorem 17. *Let $n \geq 3$. There is a domain U with irregular boundary point 0, and a minimal harmonic function u on U, such that $u(x) \to \infty$ as $x \to 0$ outside a set which is thin at 0, yet u is* **not** *a multiple of $G_U(\cdot, 0)$.*

We will shortly outline a construction motivated by the theory of Denjoy domains, which was used in the proof of Theorem 17. First we will indicate the significance of this result for fine potential theory.

Recall that, for nonnegative superharmonic functions on a Greenian domain U, the Riesz decomposition says that the following conditions are equivalent:

(i) the only nonnegative harmonic minorant of u is 0;
(ii) the only nonnegative subharmonic minorant of u is 0;
(iii) there is a Borel measure μ on U such that $u = \int G_U(\cdot, y) d\mu(y)$.

The *fine topology* on $\mathbb{R}^n$ is the coarsest topology which renders all superharmonic functions continuous. Fuglede and others have developed, since around 1970, an elegant and powerful theory of finely harmonic and finely superharmonic functions on fine domains (that is, finely connected, finely open sets), an account of which may be found in [17]. The fine topology counterparts of conditions (ii) and (iii) above were shown to be equivalent by Fuglede [18], and so either can be used as the definition of a fine potential. Also, it is obvious that (ii) implies (i). However, it has been a long-standing open question whether the fine topology counterparts of conditions (i) and (ii) are actually equivalent. This question was first raised by Fuglede in 1972 (see p105 of [17]), and further emphasized in [19]. We now explain, using an argument from [19], how Theorem 17 leads to a negative answer to this question.

Let U and u be as in Theorem 17. The set $U_0 = U \cup \{0\}$ is then a fine domain. Further, if we define $u(0) = +\infty$, then u certainly satisfies the fine topology analogue of the supermeanvalue property at 0, and so u is "finely superharmonic" on U_0. Any non-negative finely harmonic minorant v of u on U_0 is actually harmonic on the open set Ω (by Theorem 10.16 of [17]), so $v = cu$ for some $c \in [0,1]$ by the minimality of u on U. Since $v(x_0)$ is finite, we must have $c = 0$ and so $v = 0$. Thus the fine topology version of property (i) above holds. However, if u were the fine potential of a measure μ on U_0, then $\mu(U) = 0$ by the harmonicity of u on U, and we would be led to the contradictory conclusion that u is a multiple of $G_U(\cdot, 0)$. Thus the fine topology version of property (iii) (equivalently, (ii)) fails to hold.

We now outline one approach to proving Theorem 17. Let B' denote the unit ball in $\mathbb{R}^{n-1}$, and let $V = B' \times \mathbb{R}$. We are going to exploit the translational invariance of V and the thinness of $\overline{V}$ at infinity when $n \geq 4$. (A more intricate approach is required when $n = 3$: see [21].) Let α denote the square root of the first eigenvalue of $-\sum_{i=1}^{n-1} \partial^2/\partial x_i^2$ on B', and ϕ be the corresponding eigenfunction which satisfies $\phi(0) = 1$. The function

$$h : (x', x_n) \mapsto \phi(x') e^{\alpha x_n}$$

is then a minimal harmonic function on V that vanishes on ∂V. We extend h to be a subharmonic function on all of $\mathbb{R}^n$ by defining $h = 0$ on $\mathbb{R}^n \backslash V$. Let $\widetilde{V} = \mathbb{R}^n \backslash V$. It is enough to establish the following.

Proposition 18. *There is a minimal positive harmonic function u^* on a domain $U^* \supset \mathbb{R}^n \backslash \partial V$ such that u^* continuously vanishes on ∂U^* and satisfies $u^* \geq h$, $u^* = H_{u^*}^{\widetilde{V}}$ (where $u^*(\infty) = 0$) and*

$$|x'|^{n-2} u^*(x', 0) \to +\infty \quad \text{as} \quad |x'| \to \infty. \tag{24}$$

Theorem 17 follows from this proposition using the Kelvin transform and inversion in ∂B: the resulting minimal harmonic function u is defined on a domain U which has 0 as an irregular boundary point, so u has a fine limit at 0 which must be infinite, by (24). However, u cannot be a multiple of $G_U(\cdot, 0)$ because of its rapid growth along the x_n-axis on approach to 0.

Next, using the symmetries of V, it is not difficult to see that (24) is equivalent to

$$\int_{\partial V} u^* \, d\sigma = \infty, \tag{25}$$

where σ denotes surface area measure on ∂V. This guides our choice of U^*, as follows. We fix $\beta \in (0, 1/4)$ and define

$$A_{k,\beta} = \{(x', x_n) \in \partial V : \left| x_n - 2^k \right| < \beta e^{-\alpha 2^{k-1}}\} \quad (k \in \mathbb{N})$$

and $U^* = V \cup \widetilde{V} \cup (\cup_k A_{k,\beta})$. The point here is that, since $h(x', x_n) \approx (1 - |x'|) e^{\alpha x_n}$ on V, if there is a harmonic function u^* on U^* such that $u^* \geq h$, then Harnack's inequalities will show that

$$u^*(x', x_n) \geq C(n, \beta) e^{-\alpha 2^{k-1}} e^{\alpha 2^k} = C(n, \beta) e^{\alpha 2^{k-1}} \quad \text{on } A_{k,\beta/2},$$

and so (25) automatically holds.

It therefore remains to show that there is a minimal harmonic function u^* on U^* such that u^* vanishes on ∂U^*, $u^* \geq h$ and $u^* = H_{u^*}^{\widetilde{V}}$. This is analogous to Case (2) of Theorem 6, where there was a minimal harmonic function on Ω which vanishes on E, majorizes $x \mapsto x_n^+ |x|^{-n}$ and equals its Poisson integral in H_-.

The approach taken in [21] to showing this is as follows. We define the cylindrical annular sets

$$W_k = \{(x', x_n) : ||x'| - 1| < e^{-\alpha 2^{k-1}} \text{ and } \left|x_n - 2^k\right| < \beta e^{-\alpha 2^{k-1}}\}$$

and, for any nonnegative continuous function f on $\mathbb{R}^n$, the functions

$$H_j f = \begin{cases} H_f^{\cup_1^j W_k} & \text{on } \cup_1^j W_k \\ f & \text{elsewhere} \end{cases},$$

$$H_0 f = \begin{cases} h + H_f^V & \text{on } V \\ H_f^{\widetilde{V}} & \text{on } \widetilde{V} \\ f & \text{on } \partial V \end{cases}.$$

(We always interpret $f(\infty)$ as 0.) Next we inductively define a sequence (s_j) of continuous functions on $\mathbb{R}^n$ by

$$s_0 = h, \quad s_{2j-1} = H_j s_{sj-2}, \quad s_{2j} = H_0 s_{2j-1}.$$

It is not difficult to see, using the maximum principle, that each function s_j is subharmonic and the sequence is increasing. Less obvious is the fact that the sequence converges on $\mathbb{R}^n$: this has to be verified using appropriate estimates of harmonic measure. Once this is done it can be checked that the limit function u^* does indeed have all of the desired properties.

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