QUADRATURE IDENTITIES AND DEFORMATION OF QUADRATURE DOMAINS

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Abstract. We study the possibility of deforming quadrature domains into each other, and also discuss the possibility of changing the distribution in a quadrature identity from complex to real and from real to positive. The last question is in a sense also studied without the assumption that we have a quadrature domain.

1. Introduction

In this paper we will be interested in two things. We start by studying a question considered in the paper [4]. In that paper the class of domains which are quadrature domains for analytic functions with respect to one and the same real measure were under consideration, and for this case it was possible to prove that two domains in such a class can always be deformed into each other within the class. Here we will study the same question but for complex distributions with compact support in $\mathbb{R}^2$. This will be done in Section 3 by a direct argument similar to the one used in [4].

Another approach to this would be to show that a complex distribution always may be replaced by a real measure in a quadrature identity for analytic functions. That this is possible for a single quadrature domain turns out to be rather easy to prove. However, it is not obvious that for two domains which are quadrature domains with respect to the same complex distribution it is possible to change to the same real measure for both domains. We do not have a complete answer to this, but at-least obtain some necessary and sufficient conditions on when this is possible.

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Since we proved that a complex distribution may always be replaced by a real a similar question of whether a real measure may always be replaced by a positive one is natural to pose. This question is just as natural to study in any number of dimensions, and it has already been studied in [6]. There the authors were able to prove that it is in-fact always possible in two dimensions, and gave some sufficient conditions to guarantee it in the higher-dimensional case. They only treated the case for harmonic quadrature domains. Here we aim at relaxing the conditions in the higher-dimensional case, and also briefly discuss the case of analytic quadrature domains in two dimensions.

During the work with the latter questions above it became clear that it was more natural to study these without the assumption that we have a quadrature domain. What we mean here is simply that if we are given any open set Ω in $\mathbb{R}^N$ and a set of functions $V \subset C(\Omega)$ (real or complex-valued) and two classes A, B of Radon measures with compact support in Ω, then we would like to get necessary and sufficient conditions on the measures μ in A that guarantees the existence of a measure η in B such that

$$\int f d\mu = \int f d\eta \qquad \forall f \in V.$$

The two cases we are interested in are first in dimension two when V is a class of analytic functions, A is the class of complex Radon measures and B the class of real (signed) Radon measures. This is treated in Section 4. The second is in any number of dimensions and V is a class of harmonic functions, A the class of real (signed) Radon measures and B positive Radon measures, which is treated in Section 5.

I would also like to take the opportunity to thank first of all my supervisor Björn Gustafsson for his support, and also professor Makoto Sakai for several constructive comments which led to improvements of the paper. In particular professor Sakai pointed out to me the possibility of using a theorem of Bers (Theorem 2.2 part 1) in Lemma 4.4, which in turn made it possible to get rid of an unnecessary assumption in Theorem 4.6.

2. Basic definitions and notation

Let $\Omega \subset \mathbb{R}^N$ be open. We now define the following classes in Ω:

$H(\Omega) := \{\text{harmonic functions in } \Omega\}$,
$L^1(\Omega) := \{\text{Lebesgue-Integrable functions in } \Omega\}$, $m =$ Lebesgue-measure,
$P(\Omega) := \{\text{positive functions in } \Omega\}$,
If $N = 2$ we let $A(\Omega) := \{\text{holomorphic functions in } \Omega\}$,
$\mathcal{D}'(\Omega, \mathbb{R}) := \{\text{real-valued distributions in } \Omega\}$,
$\mathcal{D}'(\Omega, \mathbb{C}) := \{\text{complex-valued distributions in } \Omega\}$,
$\mathcal{E}'(\Omega, \mathbb{R}) := \{\text{real-valued distributions with compact support in } \Omega\}$,
$\mathcal{E}'(\Omega, \mathbb{C}) := \{\text{complex-valued distributions with compact support in } \Omega\}$.

There are natural identifications of $\mathcal{D}'(\Omega, \mathbb{R})$ and $\mathcal{E}'(\Omega, \mathbb{R})$ with subspaces of $\mathcal{D}'(\Omega, \mathbb{C})$ resp. $\mathcal{E}'(\Omega, \mathbb{C})$ which we will use.

We also order $\mathcal{D}'(\Omega, \mathbb{R})$ by saying that $u \leq v$ if $\langle u, \phi \rangle \leq \langle v, \phi \rangle$ for every $\phi \geq 0$ in $C_c^\infty(\Omega, \mathbb{R})$.

$$h_\varepsilon(x) := \begin{cases} \frac{1}{\varepsilon^N} C \exp(\frac{\varepsilon^2}{|x|^2 - \varepsilon^2}) & |x| < \varepsilon \\ 0 & |x| \geq \varepsilon \end{cases},$$

where C is s.t. $\int h_\varepsilon dm = 1$, h_ε are the standard smooth radial mollifiers.

$$\Phi(x) = \{\text{Newtonian kernel}\} := \begin{cases} C_2 \log|x| \text{ if } N = 2 \\ C_N |x|^{2-N} \text{ if } N \geq 3 \end{cases}$$

where C_N is s.t. $-\Delta\Phi = \delta$, the Dirac measure at the origin.

We now define $U^\mu = \{\text{Newtonian potential of } \mu\} = \Phi * \mu$.

An open set $\Omega \subset \mathbb{R}^N$ is called Greenian if it has a Green's function, or what amounts to the same thing, the function $\Phi(\bullet - y)$ has a subharmonic minorant for each $y \in \Omega$. Then

$$G(x, y) := \Phi(x - y) - h(x, y)$$

where $h(\bullet, y)$ is the largest subharmonic minorant to $\Phi(\bullet - y)$, and G is called the Green's function for Ω. Also if G is the Green's function for some domain we denote the Green-potential of μ by G^μ (for $\mu \in \mathcal{E}'(\mathbb{R}^N, \mathbb{R})$).

$$A(z) = \{\text{Cauchy kernel}\} := \frac{1}{\pi z}, \text{ s.t. } \frac{\partial}{\partial \bar{z}} A = \delta.$$

The Cauchy transform of μ will be denoted by $A^\mu := A * \mu$ (for $\mu \in \mathcal{E}'(\mathbb{R}^2, \mathbb{C})$).

$\mathrm{Bal}(\mu, \partial\Omega)$ = Balayage of μ onto $\partial\Omega$ (where μ is a real signed Radon measure with compact support in Ω). We will only use balayage when $\partial\Omega$ is fairly smooth, and then it may be defined as the unique Radon measure η with support on $\partial\Omega$ such that

$$\int h d\mu = \int h d\eta \qquad \forall h \in H(\Omega) \cap C(\overline{\Omega}).$$

Furthermore we use the super-positioning principle such that for instance

$$HL^1(\Omega) = H(\Omega) \cap L^1(\Omega).$$

We also set the following conventions regarding measures. By a real measure we mean a real (possibly) signed measure always indicating when we mean positive. If we write something like $\mu = \alpha + i\beta$ is a complex measure it is to be understood that α and β are real measures. Also, by support we always mean closed support.

Definition 2.1. If $\mu \in \mathcal{E}'(\mathbb{R}^N, \mathbb{R})$, then we define the class $Q(\mu, HL^1)$ to consist of those open subsets $\Omega \subset\subset \mathbb{R}^N$ (i.e. compactly contained in) such that $\mu \in \mathcal{E}'(\Omega, \mathbb{R})$ and $\langle \mu, f \rangle = \int_\Omega f dm \ \forall f \in HL^1(\Omega)$.

If $N = 2$ we make a similar definition of $Q(\mu, AL^1)$ for $\mu \in \mathcal{E}'(\mathbb{R}^2, \mathbb{C})$ to consist of those open subsets $\Omega \subset\subset \mathbb{R}^2$ such that $\mu \in \mathcal{E}'(\Omega, \mathbb{C})$ and $\langle \mu, f \rangle = \int_\Omega f dm$ $\forall f \in AL^1(\Omega)$.

Elements in $Q(\mu, HL^1)$ are called quadrature domains for harmonic functions with respect to μ, and elements in $Q(\mu, AL^1)$ quadrature domains for analytic functions with respect to μ.

Remark: Note that we do not require quadrature domains to be connected, but since μ has compact support in Ω it follows that a finite number of components covers μ and there can't be any component that doesn't contain some part of μ, because the characteristic function of a component lies in the test class. Hence a quadrature domain has a finite number of components. The definition above is more restrictive than the one used in [4] where the measure is not required to have compact support in the domain.

Example: The most basic example of a quadrature domain is a ball which is a quadrature domain with respect to a point-mass at its center. In two dimensions we may use conformal mappings to get other examples. So let us look at $B_1(0)$ in $\mathbb{R}^2$, which belongs to $Q(\pi\delta, AL^1)$. Now let $f(z) = z + iz^2/2$ which is a conformal map of $B_1(0)$ onto some domain which we denote Ω. We have $f(0) = 0$ and $f'(z) = 1 + iz$ so $f'(0) = 1$. If $g \in AL^1(\Omega)$ then $g \circ f \in AL^1(B_1(0))$ and

$$\int_\Omega g dm = \int_{B_1(0)} (g \circ f)(z)|f'(z)|^2 dm(z).$$

We have $|f'(z)|^2 = (1-y)^2 + x^2$ so for $n \geq 0$:

$$\int_{B_1(0)} z^n |f'(z)|^2 dm = \int_{\theta=0}^{2\pi} \int_{r=0}^{1} r^n e^{in\theta}(1 - 2r\sin\theta + r^2) r dr d\theta =$$

$$\begin{cases} 3\pi/2 & \text{if } n = 0 \\ -i\pi/2 & \text{if } n = 1 \\ 0 & \text{if } n \geq 2. \end{cases}$$

Hence we get for $g \in AL^1(\Omega)$

$$\int_\Omega g dm = \int_{B_1(0)} (g \circ f)(z)|f'(z)|^2 dm(z) =$$

$$\frac{3\pi}{2}(g \circ f)(0) - \frac{i\pi}{2}(g \circ f)'(0) = \frac{3\pi}{2} g(0) - \frac{i\pi}{2} g'(0).$$

Therefore $\Omega \in Q(3\pi\delta/2 - i\pi\delta'/2, AL^1)$.

(This is a standard example, and many others may be produced by analytic maps. See [8] for instance).

The following theorem, where the first part is due to L.Bers [2] and the second to M.Sakai [8], will be of fundamental importance for us.

Theorem 2.2. .

(1) *If $\Omega \subset\subset \mathbb{R}^2$, then the linear span of the Cauchy kernels with poles in Ω^c is dense in $AL^1(\Omega)$.*
(2) *If $\Omega \subset\subset \mathbb{R}^N$ then the linear span of the Newtonian kernels and its first order partial derivatives with poles in Ω^c are dense in $HL^1(\Omega)$.*

Corollary 2.3. .

(1) $\Omega \in Q(\mu, HL^1)$ *if and only if* $U^\Omega = U^\mu$ *and* $\nabla U^\Omega = \nabla U^\mu$ *on* Ω^c.
(2) $\Omega \in Q(\mu, AL^1)$ *if and only if* $A^\Omega = A^\mu$ *on* Ω^c.

For the proof see [4]. (Notice that if μ is real then $A^\mu = A^\Omega$ is equivalent to $\nabla U^\mu = \nabla U^\Omega$.)

3. Deformation of quadrature domains

Here we will study quadrature domains in $\mathbb{R}^2$ with respect to the class AL^1. We start by introducing the following partial order on the subsets of $\mathbb{R}^N$: $\Omega_1 \prec \Omega_2$ means that $U^{\Omega_1} \geq U^{\Omega_2}$ in $\mathbb{R}^N$. Before we state and prove the main theorem of this section we need a few preliminaries.

Lemma 3.1. *For* $\varepsilon > 0$, *let* $B_\varepsilon = B_\varepsilon(0)$ *and let* $l_\varepsilon(x) := \frac{1}{m(B_\varepsilon)}\chi_{B_\varepsilon}(x)$.

(1) *A l.s.c. function* u *on* $\mathbb{R}^N$ *is superharmonic if and only if* $u * l_\varepsilon \leq u$ $\forall \varepsilon > 0$. *Furthermore* $u * h_\varepsilon \leq u$ *for any superharmonic function* u *on* $\mathbb{R}^N$.
(2) *If* $u \in \mathcal{D}'(\mathbb{R}^N, \mathbb{R})$ *and* $u \leq w$ *where* w *is superharmonic on* $\mathbb{R}^N$, *then* $u * l_\varepsilon \leq w$ *and* $u * h_\varepsilon \leq w$.

Proof. 1. Since $u * l_\varepsilon(x) = \int u(y) l_\varepsilon(x-y)dm(y) = \frac{1}{m(B_\varepsilon)}\int_{B_\varepsilon(x)} u(y)dm(y)$ this is simply a reformulation of the super-mean-value inequality. The second statement is a direct consequence of the inequality $u(x) \leq \frac{1}{S(\partial B_\varepsilon)}\int_{\partial B_\varepsilon(x)} u dS$ where S is surface-area, and the argument may be found in most books on potential theory. It is actually true that u l.s.c. and $u * h_\varepsilon \leq u$ implies that u is superharmonic, but we will not need it.

2. Since $u \leq w \Rightarrow u * l_\varepsilon \leq w * l_\varepsilon$ this follows from (1). The proof for h_ε is identical. □

The following is our main result in this section.

Theorem 3.2. *If* $\mu \in \mathcal{E}'(\mathbb{R}^2, \mathbb{C})$ *and* $\Omega_0, \Omega_1 \in Q(\mu, AL^1)$, *then there is a one-parameter family* $\Omega(t) \in Q(\mu, AL^1)$ *s.t.* $\Omega(0) = \Omega_0$, $\Omega(1) = \Omega_1$ *and* $U^{\Omega(t)}$ *depends continuously on* t. *Furthermore there is a least upper bound* $\Omega_0 \vee \Omega_1$ *in* $Q(\mu, AL^1)$ *of* Ω_0 *and* Ω_1 *w.r.t.* $\prec$. *In other words we have* $\Omega_j \prec \Omega_0 \vee \Omega_1$ $(j = 0, 1)$ *and if* $D \in Q(\mu, AL^1)$ *with* $\Omega_j \prec D$ $(j = 0, 1)$ *then* $\Omega_0 \vee \Omega_1 \prec D$.

Proof. For $(t_0, t_1) \in \mathbb{R}^2$ define

$$F(t_0, t_1) := \{u \in \mathcal{D}'(\mathbb{R}^2, \mathbb{R}) : u \leq \min\{U^{\Omega_0} + t_0, U^{\Omega_1} + t_1\}, -\Delta u \leq 1\}.$$

To keep the notation less cumbersome we omit (t_0, t_1) in the first part of the proof since it is considered as fixed for the moment. The function

$$w := \min\{U^{\Omega_0} + t_0, U^{\Omega_1} + t_1\}$$

is superharmonic and continuous. If $t_0 \geq t_1$, then $w = U^{\Omega_1} + t_1$ far away since $U^{\Omega_0} = U^{\Omega_1}$ there, and the other way around if $t_0 \leq t_1$. In either case

$$w = U^{\nu} + \min\{t_0, t_1\}$$

where $\nu = -\Delta w$ is a positive Radon measure with compact support.

First of all F is nonempty, because if we mollify w with h_ε, then $w_\varepsilon = w * h_\varepsilon \leq w$ (and actually $w_\varepsilon = w$ far away) and $-\Delta w_\varepsilon \leq 1$ if ε is large enough. So $w_\varepsilon \in F$ for large ε.

We now prove that F has a largest element by a Perron-type of argument (see [1] [3] or [7] for instance for the facts needed). By adding $|x|^2/4$ to each element of F they become subharmonic, so $F \subset L^1_{loc}(\mathbb{R}^2)$ and each member has a unique representation as an upper semi-continuous (u.s.c.) function. Also, if $u_1, u_2 \in F$, then so does $\sup\{u_1, u_2\}$, therefore we may put:

$$v := \sup\{\phi : \phi \in F\}$$

and

$$V := \inf\{f : f \text{ is u.s.c. and } f \geq v\}.$$

Then $V \in F$, $V = v$ a.e. and hence is the largest element of F.

We now start by proving that V is in-fact superharmonic. By lemma 3.1 we have that $V_\varepsilon := V * l_\varepsilon \in F$ for each $\varepsilon > 0$, because $V \leq w$ implies $V_\varepsilon \leq w * l_\varepsilon \leq w$ and $-\Delta V_\varepsilon = (-\Delta V) * l_\varepsilon \leq 1$. (Note that $V_\varepsilon \in C(\mathbb{R}^2)$). Since $V \in L^1_{loc}(\mathbb{R}^2)$, $V_\varepsilon \to V$ in L^1_{loc} and also point-wise a.e. by Lebesgue's differentiation theorem. So if we look at $u := \sup_{\varepsilon>0}\{V_\varepsilon\}$ point-wise, then u is l.s.c. by construction ($\{x \in \mathbb{R}^2 : u(x) > a\} = \cup_{\varepsilon>0}\{x \in \mathbb{R}^2 : V_\varepsilon(x) > a\}$ is open $\forall a \in \mathbb{R}$).

We also have $u = V$ a.e. on $\mathbb{R}^2$, hence $u * l_\varepsilon = V * l_\varepsilon = V_\varepsilon \leq u$ for every $\varepsilon > 0$. But this implies that u is superharmonic and belongs to F. That is $0 \leq -\Delta u \leq 1$, and the first inequality implies that $-\Delta u$ may be represented as a positive Radon measure, whereas the second implies that this measure is absolutely continuous with respect to m. Also note that $-\Delta u = 0$ far away. So if we put $f = -\Delta u$ as a function in $L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ we may choose a representative that is identically zero outside some compact set and always fulfills $0 \leq f \leq 1$. All this implies that u is in-fact continuous and therefore we have $u = V$ everywhere by the choice of V (because $u + |x|^2/4 = V + |x|^2/4$ a.e and both are subharmonic). Now $V = U^f + \min\{t_0, t_1\}$ on $\mathbb{R}^2$, so $V \in W^{2,p}_{loc}(\mathbb{R}^2)$ for $1 \leq p \leq \infty$.

We now introduce the sets

$$\begin{aligned}
&\omega = \omega(t_0, t_1) := \{x \in \mathbb{R}^2 : V(x) < w(x)\},\\
&\Omega = \Omega(t_0, t_1) := \mathbb{R}^2 \setminus \operatorname{supp}(1 + \Delta V),\\
&D_0 := \{x \in \mathbb{R}^2 : U^{\Omega_0} + t_0 < U^{\Omega_1} + t_1\},\\
&D_1 := \{x \in \mathbb{R}^2 : U^{\Omega_0} + t_0 > U^{\Omega_1} + t_1\},\\
&S := \{x \in \mathbb{R}^2 : U^{\Omega_0} + t_0 = U^{\Omega_1} + t_1\}.
\end{aligned}$$

Note that Ω, ω, D_0 and D_1 are open and S is closed since all functions involved are continuous. Also note that $\omega \subset \Omega$, because if $\omega \setminus \Omega$ contained an open ball B, then on this ball neither the bound $V = w$ nor the bound $-\Delta V = 1$ are attained, which obviously contradicts the maximality of V. We next aim at proving that

$$V = U^{\Omega} + \min\{t_0, t_1\}.$$

First of all,

$$-\Delta V = 1 \text{ in } \Omega$$

by definition. We also have

$$V = U^{\Omega_j} + t_j \text{ on } \Omega^c \cap (D_j \cup S),$$

and therefore, since $V, U^{\Omega_j} \in W^{2,p}_{loc}$, we get

$$\nabla V = \nabla U^{\Omega_j} \text{ a.e. on } \Omega^c \cap (D_j \cup S)$$

$$-\Delta V = -\Delta U^{\Omega_j} = \chi_{\Omega_j} \text{ a.e. on } \Omega^c \cap (D_j \cup S).$$

Also $\Omega_j \cap D_j \subset \Omega$ $(j = 0, 1)$ because $-\Delta w = 1$ here.

This implies that $\Omega^c \cap D_j \cap \Omega_j = \emptyset$. On $\Omega_0 \cap \Omega_1$ we have $-\Delta w \geq 1$, because on $\Omega_0 \cap \Omega_1$,

$$\min\{U^{\Omega_0} + t_0, U^{\Omega_1} + t_1\} = U^{\Omega_0 \cap \Omega_1} + \min\{U^{\Omega_0 \setminus \Omega_1} + t_0, U^{\Omega_1 \setminus \Omega_0} + t_1\}.$$

Therefore $\Omega_0 \cap \Omega_1 \subset \Omega$, hence $\Omega^c \cap S \cap \Omega_0 \cap \Omega_1 = \emptyset$.

Summing up

$$\Omega^c = (\Omega^c \cap D_0) \cup (\Omega^c \cap D_1) \cup (\Omega^c \cap S) =$$

$$= (\Omega^c \cap D_0 \cap \Omega_0^c) \cup (\Omega^c \cap D_1 \cap \Omega_1^c) \cup (\Omega^c \cap S \cap (\Omega_0^c \cup \Omega_1^c)).$$

Therefore

$$-\Delta V = 0 \text{ a.e. on } \Omega^c.$$

All together this implies that $-\Delta V = \chi_{\Omega}$, and hence $V = U^{\Omega} + \min\{t_0, t_1\}$ because this holds on the unbounded component.

Also, since

$$A^{\Omega_j} = A^{\mu} \text{ on } \Omega_j^c$$

we have

$$A^{\Omega} = A^{\mu} \text{ on } \Omega^c.$$

So by Corollary 2.3

$$\Omega \in Q(\mu, AL^1).$$

The least upper bound is easily seen to be $\Omega(0, 0)$ in the above construction. As for the one-parameter family we note that two variables were introduced mainly for symmetry reasons, since obviously

$$\Omega(t_0 + \alpha, t_1 + \alpha) = \Omega(t_0, t_1) \forall \alpha \in \mathbb{R}$$

one dimension collapses automatically. If we now look at $\Omega(0, t)$, then for large t $\Omega(0, t) = \Omega_0$ and for small t $\Omega(0, t) = \Omega_1$ of-course. By translating and rescaling the parameter t this gives us our one-parameter family.

The only thing that remains is the continuity of $U^{\Omega(0,t)}$ with respect to t. We start by noting that

$$\bigcup_{(t_0,t_1)\in\mathbb{R}^2} \Omega(t_0,t_1) \subset\subset \mathbb{R}^2.$$

This is proved by choosing a ball B so large that

$$\Omega_0 \cup \Omega_1 \subset\subset B$$

and

$$\gamma := \mathrm{Bal}(\mu, \partial B) = \mathrm{Bal}(\chi_{\Omega_j}, \partial B)$$

may be mollified to a measure γ_ε s.t. $\gamma_\varepsilon \leq 1$ and $\mathrm{supp}(\gamma_\varepsilon) \cap \overline{(\Omega_0 \cup \Omega_1)} = \emptyset$. This implies that $U^{\gamma_\varepsilon} + \min\{t_0, t_1\} \in F(t_0, t_1)$, and therefore that

$$\Omega(t_0, t_1) \subset B \cup \mathrm{supp}(\gamma_\varepsilon),$$

independent of (t_0, t_1). We only prove that $U^{\Omega(0,t+\epsilon)} \searrow U^{\Omega(0,t)}$ as $\epsilon \searrow 0$ since the other cases are similar to handle. So given $\epsilon > 0$ then obviously

$$\min\{U^{\Omega_0}, U^{\Omega_1} + t + \epsilon\} \geq \min\{U^{\Omega_0}, U^{\Omega_1} + t\}.$$

Hence if we let V_ϵ denote the maximum of $F(0, t+\epsilon)$ (not to be confused with the mollification V_ε of V) and $V = V_0$, then $V_\epsilon > V$ and V_ϵ is decreasing with ϵ. So we may define $V' := \lim_{\epsilon\to 0} V_\epsilon$. Now since

$$V_\epsilon \leq \min\{U^{\Omega_0}, U^{\Omega_1} + t + \epsilon\} \text{ for every } \epsilon$$

we have

$$V' \leq \min\{U^{\Omega_0}, U^{\Omega_1} + t\},$$

and also $-\Delta V' \leq 1$, so $V' \in F(0,t)$ and therefore we have $V' \leq V \leq V'$, i.e. $V' = V$. But this immediately implies that $U^{\Omega(0,t+\epsilon)} \to U^{\Omega(0,t)}$ as $\epsilon \to 0$, which finishes the proof.

□

Remark: In the construction above we used μ very little, and in-fact if there are two different distributions μ_1, μ_2 such that $\Omega_0, \Omega_1 \in Q(\mu_i, AL^1)$ for $i = 1, 2$ then the family $\Omega(t) \in Q(\mu_i, AL^1)$ for $i = 1, 2$.

Also, the first part of the proof is not dimension-dependent, and it gives an alternative to the obstacle-problem approach used in [4]. Except for this difference the proof above is essentially based on the one for real measures in [4].

Example: There are simple examples of quadrature domains

$$\Omega_0, \Omega_1 \in Q(\mu, AL^1)$$

where one is connected and the other one is not.
We may for instance take $\mu = \pi\delta_0 + 2S\lfloor_{\partial B_2(0)}$ in $\mathbb{R}^2$, where S is arc-length. Now we may take

$$\Omega_0 = B_3(0)$$

and

$$\Omega_1 = B_1(0) \cup (B_{\sqrt{41}/2}(0) \setminus \overline{B_{3/2}(0)}).$$

This leads to a natural question that we do not have an answer for:

If Ω_0 and Ω_1 are connected must $\Omega(t_0, t_1)$ constructed as above consist of only connected domains?
If for instance $\Omega_0 \cap \Omega_1$ is connected this is trivially so, but in general we do not know. It is mainly for this reason that we do not require quadrature domains to be connected.

4. Complex to real identities

In this section it will be natural to use complex notation, and we will write $z = x + iy$ for a point in the complex plane $\mathbb{C}$. The following elementary lemma which is an immediate consequence of the Cauchy-Riemann equations will also be used.

Lemma 4.1. *Let $u + iv$ be analytic in Ω, where Ω is a domain (i.e. open and connected) in $\mathbb{C}$. Let $a \in \Omega$ be fixed. In that case we have for any $z \in \Omega$*

$$(*)\ v(z) = v(a) + \int_\gamma (-\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy)$$

for any piecewise C^1 Jordan arc γ in Ω between a and z.

Theorem 4.2. *Let $\Omega \subset \mathbb{C}$ be open. Suppose $\mu = \alpha + i\beta$ (α,β real Radon measures) has compact support in Ω.*
Then there is a real Radon measure η with compact support in Ω such that

$$\int f d\mu = \int f d\eta \qquad \forall f \in A(\Omega)$$

if and only if $\beta(U) = 0$ for each component U of Ω.

Proof. We start by assuming that Ω is connected. Fix $a \in \Omega$ and for each $z \in \operatorname{supp}(\mu)$ a piecewise C^1 Jordan arc γ_z between a and z in such a way that all γ_z are contained in a fixed compact $K \subset \Omega$ and the lengths of all γ_z are bounded by a constant $l < \infty$. (This is possible since Ω is connected and $\operatorname{supp}(\mu)$ is compact.)
Now define $T : C_c^\infty(\Omega, \mathbb{R}) \to \mathbb{R}$ by

$$\langle T, \phi \rangle := \int \phi d\alpha - \int \int_{\gamma_z} (-\frac{\partial \phi}{\partial y}dx + \frac{\partial \phi}{\partial x}dy)d\beta.$$

Then

$$|\langle T, \phi \rangle| \leq \sup_K |\phi| ||\alpha|| + (\sup_K |\frac{\partial \phi}{\partial y}| + \sup_K |\frac{\partial \phi}{\partial x}|) l ||\beta||.$$

T is obviously linear, so by the above $T \in \mathcal{E}'(\Omega, \mathbb{R})$, and it may now be mollified to a measure η with compact support in Ω. Let $f = u + iv \in A(\Omega)$. Then

$$\begin{aligned}
\int f d\eta &= \int u d\eta + i \int v d\eta = \langle T, u \rangle + i \langle T, v \rangle \\
&= \int u d\alpha - \int\int_{\gamma_z} (-\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy)d\beta + i \int v d\alpha - i \int\int_{\gamma_z} (-\frac{\partial v}{\partial y}dx + \frac{\partial v}{\partial x}dy)d\beta \\
&= \int u d\alpha - \int (v(z) - v(a))d\beta(z) + i \int v d\alpha - i \int (u(a) - u(z))d\beta(z) \\
&= \int u d\alpha + i \int u d\beta + i \int v d\alpha - \int v d\beta = \int f d\mu.
\end{aligned}$$

Here we used that $\int v(a)d\beta(z) = v(a) \int d\beta = 0$ and similarly for u.

If Ω is not connected we first note that by compactness the support of μ must be contained in a finite number of components of Ω. Let these be denoted $U_1, ..., U_n$.

By the above there are for each $j = 1, ..., n$ a real measure η_j with compact support in U_j such that
$$\int f d\mu\lfloor U_j = \int f d\eta_j \qquad \forall f \in A(U_j).$$
If we put $\eta = \sum_{j=1}^{n} \eta_j$ we get the required measure. In the other direction we get
$$\int \chi_U d\mu = \int \chi_U d\alpha + i \int \chi_U d\beta = \int \chi_U d\eta$$
which obviously implies $\int d\beta\lfloor U = 0$. □

Remark: What we actually proved above was the following. If $V \subset A(\Omega)$ and μ is as in the theorem, then the condition that $\beta(U) = 0$ for each component U of Ω is sufficient to guarantee the existence of a real measure η with compact support in Ω such that
$$\int f d\mu = \int f d\eta \qquad \forall f \in V$$
and if $\chi_U \in V$ for each component it is also necessary.

Corollary 4.3. *Suppose $\mu \in \mathcal{E}'(\mathbb{R}^2, \mathbb{C})$ and $\Omega \in Q(\mu, AL^1)$. Then there is a real Radon measure η with compact support in Ω such that $\Omega \in Q(\eta, AL^1)$.*

Proof. Since we may mollify μ to a measure we may without loss of generality assume that it already is a complex Radon measure $\mu = \alpha + i\beta$ with α,β real Radon measures with compact support in Ω. By definition we have
$$\int_\Omega \chi_U dm = \int \chi_U d\mu = \int d\alpha\lfloor U + i \int d\beta\lfloor U$$
for every component U of Ω. Hence $\int d\beta\lfloor U = 0$, and we may apply Theorem 4.2. □

In connection with the deformation of quadrature domains it would be good to have conditions on when it is possible to change a complex measure to the same real measure for two different domains simultaneously. We begin with an approximation lemma that is a simple consequence of Runge's theorem (Bers theorem for the integrable case). For simplicity we consider only bounded domains.

Lemma 4.4. *Suppose $\Omega_1, \Omega_2 \subset\subset \mathbb{C}$ are open and $f \in A(\Omega_1 \cap \Omega_2)$. For any compact set $K \subset \Omega_1 \cap \Omega_2$ and $\varepsilon > 0$ there are rational functions $r_i \in A(\Omega_i)$ $(i = 1, 2)$ such that $|f - (r_1 + r_2)| < \varepsilon$ on K. If furthermore $\Omega_i = \mathrm{int}(\overline{\Omega}_i)$ $(i = 1, 2)$ we may choose each $r_i \in C(\overline{\Omega}_i)$. Finally if $f \in AL^1(\Omega_1 \cap \Omega_2)$ then we may take $r_i \in AL^1(\Omega_i)$.*

Proof. Let $\{U_j\}_{j=1}^{\infty}$ denote the components of K^c. In each U_j we may assume that there is a point a_j in $(\Omega_1 \cap \Omega_2)^c$, because otherwise we may add U_j to K. This a_j then also belongs to some Ω_i^c trivially. By Runge's theorem we have
$$|f(z) - \sum_{k=1}^{m} \frac{p_k(z)}{(z - a_{j_k})^{n_k}}| < \varepsilon \qquad \text{on } K$$
for some choice of the polynomials p_k and constants m,j_k,n_k. Defining
$$A_1 := \{k \in \{1, ..., m\} : a_{j_k} \in \Omega_1^c\}$$
$$A_2 := \{1, ..., m\} \setminus A_1,$$
and
$$r_i(z) := \sum_{k \in A_i} \frac{p_k}{(z - a_{j_k})^{n_k}} \qquad (i = 1, 2)$$

then the first part easily follows. As for the second part we only need to note that with the notation as above, if $a_j \in \Omega_i^c$, then there is some $a'_j \in (\overline{\Omega}_i)^c \cap U_j$ by the topological assumption. So we may assume that in the r_i's $a_{j_k} \in (\overline{\Omega}_i)^c$ when $k \in A_i$ $(i = 1, 2)$. Finally, if $f \in AL^1(\Omega_1 \cap \Omega_2)$, if we refer to Bers theorem (Theorem 2.2 part 1) then we see that we may take p_k constant, and $n_k = 1$ for every k above, and this finishes the proof. □

Theorem 4.5. *Suppose $\Omega_1, \Omega_2 \subset\subset \mathbb{C}$ and $\mu = \alpha + i\beta$ is a complex Radon measure with compact support in $\Omega_1 \cap \Omega_2$. Then a necessary and sufficient condition for the existence of a real Radon measure η with compact support in $\Omega_1 \cap \Omega_2$ such that*

$$(1) \qquad \int f d\mu = \int f d\eta \qquad \forall f \in A(\Omega_1) \cup A(\Omega_2)$$

is that $\beta(U) = 0$ for each component U of $\Omega_1 \cap \Omega_2$. In that case we also have automatically

$$(2) \qquad \int f d\mu = \int f d\eta \qquad \forall f \in A(\Omega_1 \cap \Omega_2).$$

Proof. That $\beta(U) = 0$ for each component U of $\Omega_1 \cap \Omega_2$ is sufficient is obvious by Theorem 4.2. We start by proving that (1) ⇒ (2) (which holds regardless of whether $\beta(U) = 0$ for each component U of $\Omega_1 \cap \Omega_2$ or not). Let $f \in A(\Omega_1 \cap \Omega_2)$ and $\varepsilon > 0$. Then we may by Lemma 4.4 choose $r_i \in A(\Omega_i)$ $(i = 1, 2)$ with

$$|f - (r_1 + r_2)| < \varepsilon \qquad \text{on } \operatorname{supp}(\mu) \cup \operatorname{supp}(\eta),$$

so

$$|\int f d\mu - \int f d\eta| = |\int f d\mu - \int (r_1 + r_2) d\mu + \int (r_1 + r_2) d\eta - \int f d\eta|$$

$$\leq |\int (f - (r_1 + r_2)) d\mu| + |\int (f - (r_1 + r_2)) d\eta| \leq \varepsilon(||\mu|| + ||\eta||).$$

Hence

$$\int f d\mu = \int f d\eta.$$

Now if U is a component of $\Omega_1 \cap \Omega_2$, then $\chi_U \in A(\Omega_1 \cap \Omega_2)$, and the proof is done. □

Theorem 4.6. *If $\Omega_1, \Omega_2 \in Q(\mu, AL^1)$ where $\mu = \alpha + i\beta$ is a complex Radon measure with compact support, then a necessary and sufficient condition for the existence of some real Radon measure η with compact support such that $\Omega_1, \Omega_2 \in Q(\eta, AL^1)$ is that $\beta(U) = 0$ for each component U of $\Omega_1 \cap \Omega_2$.*

Proof. The proof is equivalent to the previous one of theorem 4.5. The only thing we need to note is that we may take $r_i \in AL^1(\Omega_i)$ by Lemma 4.4 in the proof above. □

We remark that if either $\Omega_1 \cap \Omega_2$ is connected or Ω_1 and Ω_2 differ only on a set of Lebesgue measure zero we may replace μ with the same real measure η. (In the first case this follows immediately from theorem 4.2 and the following remark and in the second by Corollary 4.3 because $\Omega_1 \cap \Omega_2 \in Q(\mu, AL^1)$).

Theorem 4.7. *Suppose $\mu = \alpha + i\beta$ is a complex Radon measure with compact support in $\mathbb{C}$ and $\Omega_1, \Omega_2 \in Q(\mu, AL^1)$. Also assume that $\Omega_1 \cap \Omega_2$ consists of a finite number of components $U_1, ..., U_n$ where $\overline{U}_j \cap \overline{U}_k = \emptyset$ if $j \neq k$ and each U_j fulfills that $\partial D \cap \partial U_j \neq \emptyset$ where D is the unbounded component of $(\overline{\Omega_1 \cap \Omega_2})^c$. In that case we have $\beta(U_j) = 0$ for $j = 1, ..., n$.*

Proof. By the topological assumptions made we have that if $j \neq k$ then $\overline{U}_k$ lies in the unbounded component of $\mathbb{C} \setminus \overline{U}_j$. Let us now define V_j to be the union of U_j and all the bounded components of $\mathbb{C} \setminus U_j$. Then choose open $W_j's$ with $V_j \subset\subset W_j$ and $W_k \cap W_j = \emptyset$ if $j \neq k$. It is easy to see that it is enough to prove that $\beta(V_j) = 0$ for each j. Let

$$V = \begin{cases} U^{\Omega_1} & \text{on } D \setminus \Omega_1 \\ U^{\Omega_2} & \text{on } D \setminus \Omega_2 \\ 0 & \text{otherwise} \end{cases}$$

$$\nu := -\Delta V$$

so that

$$\nu \in \mathcal{E}'(\mathbb{C}, \mathbb{R})$$

with support on

$$\bigcup_{j=1}^{n} \partial V_j.$$

It follows that $A^\nu = A^\mu$ on D, and hence if we put

$$f_j := \begin{cases} 1 & \text{on } W_j \\ 0 & \text{otherwise} \end{cases}$$

we get:

$$\langle \alpha + i\beta, f_j\rangle = \alpha(V_j) + i\beta(V_j) = \langle \nu, f_j\rangle \in \mathbb{R}$$

so

$$\beta(V_j) = 0.$$

□

5. Real to positive identities

In this section we look at similar questions as the one in the previous section, but instead we are mainly interested in changing a real measure to a positive one, and for test-classes of harmonic functions instead of analytic ones. This question is natural to study in any number of dimensions, and that is what we intend to do. The question is also natural to consider for quadrature domains for analytic test functions, and we end by proving that we may find another real measure making these into quadrature domains for harmonic test functions, reducing the problem to the above one. In this section we will assume that Ω is connected because it is no essential loss, and makes some statements less cumbersome.

Theorem 5.1. *Suppose $\Omega \subset \mathbb{R}^N$ is a Greenian open domain and μ is a real (possibly signed) Radon measure with compact support in Ω, then the following are equivalent.*

1. *$G^\mu > 0$ in $\Omega \setminus K$ for some compact $K \subset \Omega$, where G is the Green's function of Ω.*
2. *There is a positive Radon measure $\eta \neq 0$ with compact support in Ω such that*
$$\int h d\mu = \int h d\eta \qquad \forall h \in H(\Omega).$$
3. *$\int d\mu > 0$ and there is a compact $K \subset \Omega$ such that*
$$\int h d\mu \leq (\int d\mu) \sup_K |h| \qquad \forall h \in H(\Omega).$$
4. $$\int h d\mu > 0 \qquad \forall h \in HP(\Omega) \setminus \{0\}$$

Proof. (3) $\Rightarrow$ (2) because if we consider $H(\Omega)$ as a subspace of $C(\Omega)$, then the linear functional $\Lambda(h) := \frac{1}{\int d\mu} \int h d\mu \leq \sup_K |h|$ may be extended to the whole $C(\Omega)$ s.t. $\Lambda(f) \leq \sup_K |f|$ $\forall f \in C(\Omega)$ by the Hahn-Banach theorem. But for $f \in C(\Omega)$ and $f \geq 0$ on K we then have

$$\sup_K f - \Lambda(f) = \Lambda(\sup_K f - f) \leq \sup_K(\sup_K f - f) \leq \sup_K f,$$

so $\Lambda(f) \geq 0$ which proves (3) $\Rightarrow$ (2) (by Riesz's theorem).

(2) $\Rightarrow$ (1) Suppose that (2) holds but not (1). Then there is a sequence

$$\{x_n\} \subset \{x \in \Omega : G^\mu(x) \leq 0\}$$

which has no limit point in Ω. We now let

$$M(x, y) := \frac{G(x, y)}{G(a, y)} \qquad (a \in \Omega \text{ fixed }, y \in \Omega).$$

Then

$$\int M(t,x_n)d\mu(t) = \frac{1}{G(a,x_n)} \int G(t,x_n)d\mu(t) \leq 0.$$

It is well known by Martin boundary theory (or seen directly from Harnack's convergence theorem and a diagonal argument) that some subsequence of $\{M(t,x_n)\}$ converges u.c. to a function $h \in HP(\Omega)$ with $h(a) = 1$. But then we have

$$0 \geq \int hd\mu = \int hd\eta > 0$$

which gives a contradiction.

(1) $\Rightarrow$ (4). We choose a compact K such that $G^\mu > 0$ on $\Omega \backslash K$ and $\text{supp}(\mu) \subset \text{int}(K)$ (this may even be needed to guarantee that G^μ is well defined on $\Omega \setminus K$). Now let $K \subset \Omega_1 \subset\subset \Omega$ be open, then it is well known that any $h \in HP(\Omega)$ may be represented in Ω_1 as a potential G^ν for some positive measure with support on $\partial\Omega_1$. For $h \not\equiv 0$ we get

$$\int hd\mu = \int G^\nu d\mu = \int G^\mu d\nu > 0.$$

That is $\int hd\mu > 0 \ \forall h \in HP(\Omega), h \not\equiv 0$.

(4) $\Rightarrow$ (3) Suppose that (4) holds but not (3). Then if we let

$$\Omega_n \nearrow \Omega \ (\Omega_1 \subset\subset \Omega_2 \subset\subset \cdots \text{ and } \Omega = \cup_{n=1}^{\infty} \Omega_n)$$

there is by assumption for each n a function $k_n \in H(\Omega)$ such that $\int k_n d\mu > (\int d\mu) \sup_{\Omega_n} |k_n|$. If we let

$$h_n := (\sup_{\Omega_n} |k_n| - k_n)/(\sup_{\Omega_n} |k_n| - k_n(a))$$

where a is a fixed point in Ω_1, then $h_n > 0$ on Ω_n, $h_n(a) = 1$ and $\int h_n d\mu < 0$ for every n. Now applying Harnack's convergence theorem together with Cantor's diagonal argument gives us a subsequence that converges uniformly on compact subsets of Ω to some $h \in HP(\Omega)$ with $h(a) = 1$. But then (if we still denote this subsequence by h_n) we get

$$\int h_n d\mu \to \int h d\mu \text{ so } \int h d\mu \leq 0$$

which gives a contradiction. □

Remark: In analogy with Theorem 4.2 we may note that we actually have proved the following: If $V \subset H(\Omega)$ and μ is as in the theorem, then that $G^\mu > 0$ on $\Omega \setminus K$ for some compact $K \subset \Omega$ is a sufficient condition for the existence of a positive η such that

$$\int h d\mu = \int h d\eta \qquad \forall h \in V.$$

If furthermore $HP(\Omega) \subset V$ it is also necessary.

Corollary 5.2. *If μ is a real (possibly signed) Radon measure with compact support in $\mathbb{R}^N$, and $\Omega \in Q(\mu, HL^1)$ (and Ω connected). Then a sufficient condition for the existence of a positive Radon measure $\eta \neq 0$ with compact support in $\mathbb{R}^N$ such that $\Omega \in Q(\eta, HL^1)$ is that $G^\mu > 0$ on $\Omega \setminus K$ for some compact $K \subset \Omega$. (Here G is the Green's function of Ω). If $\Omega = \mathrm{int}(\overline{\Omega})$ it is also necessary.*

Proof. The first part is obvious by Theorem 5.1. As for the second part let us assume that η exists. We may now choose a compact $K \subset \Omega$ such that $\mathrm{supp}(\mu) \cup \mathrm{supp}(\eta) \subset K$ and each component of K^c intersects Ω^c. If now $\Omega = \mathrm{int}(\overline{\Omega})$ it follows that if O is a component of K^c then there is some open ball B in $O \cap \Omega^c$. Since $U^\mu = U^\eta$ on B it follows by harmonic continuation that this holds on all of O. But this implies that $G^\mu = G^\eta > 0$ on $O \cap \Omega$, hence $G^\mu > 0$ on $\Omega \setminus K$. □

We may also note that since both $U^\Omega = U^\mu$ and $\nabla U^\Omega = \nabla U^\mu$ on $\partial\Omega$ for $\Omega \in Q(\mu, HL^1)$ it seems highly likely that this implies that $G^\mu \geq 0$ in a neighborhood of $\partial\Omega$. One difference between this condition and the ones given in [6] is that although this is a stronger result to guarantee the existence of a positive measure, [6] gives conditions to guarantee $HP(\Omega) \subset L^1(\Omega)$ for quadrature domains, and the above does not. In two dimensions it is proved in [6] that η always exists by the characterization of quadrature domains known only for $N = 2$ (see [8] and [5]).

A natural question to ask is now if the same is true for the class AL^1 in two dimensions ([6] deals with HL^1). We end this article with a proof of this fact using the powerful characterization of quadrature domains in two dimensions due to Sakai.

Theorem 5.3. *Let $\mu \in \mathcal{E}'(\mathbb{C}, \mathbb{C})$, $\Omega \in Q(\mu, AL^1)$. Then there is a $\mu' \in \mathcal{E}'(\mathbb{C}, \mathbb{R})$ such that $\Omega \in Q(\mu', HL^1)$.*

Proof. By Corollary 4.3 we may without loss of generality assume that $\mu \in \mathcal{E}'(\mathbb{C}, \mathbb{R})$. It is known (see [9] Theorem 1.7 and [6]) that Ω contains a finitely connected quadrature domain $\Omega' \in Q(\mu, AL^1)$. Call the components of $\mathbb{C} \setminus \Omega'$ $S_1, ..., S_n$. We start by proving that $U^\Omega - U^\mu$ is constant on each component S_j.

For this we must refer to [9] again for the result that $\partial\Omega'$ has finite length and is in-fact the subset of a real-analytic variety, which contains at most a finite set of singularities. The only result we need from this is that it implies that each pair of points $z_1, z_2 \in S_j$ may be connected by a piecewise C^1 Jordan arc $\gamma \subset S_j$. Since by assumption

$$\nabla U^{\Omega} = \nabla U^{\mu} \text{ on } S_j$$

and $U^{\Omega} - U^{\mu}$ is C^1 in some neighborhood of $\partial\Omega'$ it follows by a simple line integral that

$$U^{\Omega}(z_1) - U^{\mu}(z_1) = U^{\Omega}(z_2) - U^{\mu}(z_2)$$

which proves the statement that $U^{\Omega} - U^{\mu}$ is constant on S_j. We let c_j denote the constant value of $U^{\Omega} - U^{\mu}$ on S_j, then we choose a set $\Omega'' \subset\subset \Omega'$ such that $\mathbb{C} \setminus \Omega''$ consists of n components $W_1, ..., W_n$ such that each $S_j \subset W_j$. To finish the proof we define

$$T := \sum_{j=1}^{n} c_j \Delta \chi_{W_j}.$$

Then $T \in \mathcal{E}'(\Omega', \mathbb{R})$ and

$$U^T = \begin{cases} -c_j & \text{on } W_j \\ 0 & \text{in } \Omega'' \end{cases}$$

(We may note that $c_j = 0$ for the unbounded component S_j, but we do not need it here). Now we just need to put

$$\mu' = \mu - T.$$

Then it follows that

$$U^{\mu'} = U^{\Omega'} = U^{\Omega} \text{ on } \Omega'^c$$

and

$$\nabla U^{\mu'} = \nabla U^{\mu} = \nabla U^{\Omega'} = \nabla U^{\Omega} \text{ on } \Omega'^c$$

and the proof is done. □

Remark 1: In connection with the earlier results we may note that here we cannot expect to get μ' as in the theorem above for two domains $\Omega_1, \Omega_2 \in Q(\mu, AL^1)$ simultaneously such that $\Omega_1, \Omega_2 \in Q(\mu', HL^1)$. This is because this would imply that $U^{\Omega_1} = U^{\Omega_2}$ on $(\Omega_1 \cup \Omega_2)^c$. But this is not true even for two concentric annuli Ω_1, Ω_2 with $\Omega_1 \cap \Omega_2 \neq \emptyset$ and $m(\Omega_1) = m(\Omega_2)$ which are in $Q(\mu, AL^1)$ for some measure μ (μ can be chosen as some constant times arc-length on some common circle in $\Omega_1 \cap \Omega_2$). Then in the bounded component of $(\Omega_1 \cup \Omega_2)^c$ both U^{Ω_1} and U^{Ω_2} are constant but different unless $\Omega_1 = \Omega_2$.

Remark 2: By the results of [6] we may even take $\mu' \geq 0$ in the theorem above.

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