Convexity and the Exterior Inverse Problem of Potential Theory

Stephen J. Gardiner and Tomas Sjödin

Abstract

Let Ω_1 and Ω_2 be bounded solid domains such that their associated volume potentials agree outside $\Omega_1 \cup \Omega_2$. Under the assumption that one of the domains is convex, it is deduced that $\Omega_1 = \Omega_2$.

1 Introduction

For any (positive Radon) measure μ with compact support in Euclidean space $\mathbb{R}^N$ $(N \geq 2)$, we define the usual potential

$$U^\mu(x) = \int U_y(x) d\mu(y) \qquad (x \in \mathbb{R}^N),$$

where $U_y(x) = |x-y|^{2-N}$ if $N \geq 3$, and $U_y(x) = \log(1/|x-y|)$ if $N = 2$. In the case where μ is the restriction of volume measure λ to a bounded Borel set A, we will write U^A in place of $U^{\lambda|_A}$. A domain Ω in Euclidean space $\mathbb{R}^N$ is called *solid* if it is bounded, $(\overline{\Omega})^\circ = \Omega$ and the complement, $\overline{\Omega}^c$, of $\overline{\Omega}$ is connected.

A long-standing open question, known as the *exterior inverse problem of potential theory*, asks: if $U^{\Omega_1} = U^{\Omega_2}$ on $(\Omega_1 \cup \Omega_2)^c$, where Ω_1 and Ω_2 are solid domains, does it follow that $\Omega_1 = \Omega_2$? (The answer is "no" if we omit the word "solid", as is obvious from the example of a ball and a suitably chosen concentric annular domain of equal measure.) An early result on this problem, due to Novikov [6], says that the answer is "yes" if we require both Ω_1 and Ω_2 to be convex (or, more generally, starlike with respect to a common point). More recently, Shahgholian [7] proved that it is enough here for $\Omega_1 \cap \Omega_2$ to be convex. Kondraškov [5] has shown that the answer to the question is also "yes" if one of the domains is a ball or an ellipsoid (cf.

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[1]; an elegant elementary proof for the case of a ball may be found in [9].) In this paper we show that convexity of one of the domains is sufficient to arrive at a positive answer.

Theorem 1 *Let* Ω_1 *be a solid domain and* Ω_2 *be a convex domain, and let* ν *be a measure such that* $\nu \geq \lambda|_{\Omega_2}$ *and* $\nu(\Omega_2^c) = 0$*. If* $U^{\Omega_1} = U^{\nu}$ *on* $(\Omega_1 \cup \Omega_2)^c$*, then* $\Omega_2 \subseteq \Omega_1$*.*

Corollary 2 *Let* Ω_1 *be a solid domain and* Ω_2 *be a convex domain. If* $U^{\Omega_1} = U^{\Omega_2}$ *on* $(\Omega_1 \cup \Omega_2)^c$*, then* $\Omega_1 = \Omega_2$*.*

There is no implication in either direction between the corollary and the result of Shahgholian mentioned above. However, it is worth noting that Theorem 1 only imposes an additional hypothesis on one of the domains in question. Zalcman [9] has conjectured a stronger version of Corollary 2 in which U^{Ω_1} and U^{Ω_2} are only assumed to agree near infinity. The proof of Theorem 1 will be given in Section 3, following some preliminary material in Section 2 concerning the notion of partial balayage, on which it is based.

2 Partial Balayage

Let $a_N = \sigma_N \max\{1, N-2\}$, where σ_N denotes the surface area of the unit sphere in $\mathbb{R}^N$, and let $q(x) = a_N |x|^2 / (2N)$. Thus $U^{\Omega} + q$ is harmonic on Ω, for any bounded open set Ω. If μ is a measure with compact support, it is easy to see that there is a greatest subharmonic minorant s_μ, say, of $U^\mu + q$ on $\mathbb{R}^N$ (using Theorem 3.7.5 of [2], for example). We need the following facts (see [3], [8]).

Theorem A *Let* μ *and* s_μ *be as above. Then:*
(i) the function $s_\mu - q$ *can be expressed as* $U^{f\lambda}$*, where* $f : R^N \to [0,1]$ *is a Borel function with compact support;*
(ii) the open set $\omega(\mu) = \left\{U^\mu > U^{f\lambda}\right\}$ *is bounded, and* $f\lambda = \lambda|_{\omega(\mu)} + \mu|_{\omega(\mu)^c}$*.*

The measure $f\lambda$ arising in Theorem A is called the *partial balayage of* μ *onto* λ, and will be denoted by $\widetilde{\mu}$. Obviously, $U^{\widetilde{\mu}} \leq U^\mu$. The decomposition formula in (ii) arises from the fact that s_μ must be harmonic on $\omega(\mu)$, by standard balayage arguments. It is clear from the lemma that, if we define $\Omega(\mu)$ to be the largest open set Ω for which $(\lambda - \widetilde{\mu})(\Omega) = 0$, then $\Omega(\mu)$ is bounded and contains $\omega(\mu)$, and

$$\widetilde{\mu} = \lambda|_{\Omega(\mu)} + \mu|_{\Omega(\mu)^c}. \tag{1}$$

The next result is a generalization of a fact due to Gustafsson (see pp.205-6 of [3]).

Lemma 3 *Let Ω_1 and Ω_2 be bounded open sets, where $\lambda(\partial\Omega_2) = 0$, and let ν be a measure such that $\nu \geq \lambda|_{\Omega_2}$, $\nu(\Omega_2^c) = 0$, and U^ν is continuous on Ω_2. If $U^\nu = U^{\Omega_1}$ on $(\Omega_1 \cup \Omega_2)^c$, and we denote by η the measure satisfying $U^\eta = \min\{U^{\Omega_1}, U^\nu\}$, then $\widetilde{\eta} = \lambda|_{\Omega(\eta)}$.*

Proof. Let $\Omega = \Omega(\eta)$ and

$$D_1 = \{U^{\Omega_1} < U^\nu\}, \quad D_2 = \{U^{\Omega_1} > U^\nu\}, \quad S = \{U^{\Omega_1} = U^\nu\},$$

$$A = (\Omega_1 \cap D_1) \cup (\Omega_2 \cap D_2) \cup (\Omega_1 \cap \Omega_2).$$

Then A is an open set. Since $U^\eta = U^{\Omega_1}$ on D_1, $U^\eta = U^\nu$ on D_2, and

$$U^\eta = U^{\Omega_1 \cap \Omega_2} + \min\{U^{\Omega_1 \backslash \Omega_2}, U^{\nu - \lambda|_{\Omega_1 \cap \Omega_2}}\}, \tag{2}$$

we see that $\eta|_A \geq \lambda|_A$ and so $\widetilde{\eta}|_A = \lambda|_A$. It follows that $A \subseteq \Omega$. Since $(\Omega_1 \cup \Omega_2)^c \subseteq S$, we see that

$$\Omega^c \subseteq A^c \subseteq A_1 \cup A_2 \cup \partial\Omega_2, \tag{3}$$

where

$$A_1 = \Omega_1^c \cap (D_1 \cup S) \quad \text{and} \quad A_2 = \overline{\Omega}_2^c \cap (D_2 \cup S).$$

On $A_1 \cap \Omega^c$ we have $U^{\widetilde{\eta}} = U^\eta = U^{\Omega_1}$, and since $U^{\widetilde{\eta}}$ and U^{Ω_1} both belong to the Sobolev space $W^{2,2}_{\text{loc}}(\mathbb{R}^N)$, we have $\widetilde{\eta}(A_1 \cap \Omega^c) = \lambda|_{\Omega_1}(A_1 \cap \Omega^c) = 0$. Since $U^{\widetilde{\eta}} = U^\eta = U^\nu$ on $A_2 \cap \Omega^c$ and U^ν is harmonic on $\overline{\Omega}_2^c$, we similarly have $\widetilde{\eta}(A_2 \cap \Omega^c) = 0$. Hence $\widetilde{\eta}(\Omega^c) = 0$, in view of (3) and the fact that $\lambda(\partial\Omega_2) = 0$. The result now follows on applying (1) to the measure η. ■

We denote a typical point x of $\mathbb{R}^N$ by (x', x_N), where $x' \in \mathbb{R}^{N-1}$ and $x_N \in \mathbb{R}$, and define

$$W_+ = \{x : x_N > 0\}, \quad W_- = \{x : x_N < 0\} \quad \text{and} \quad H = \{x : x_N = 0\}.$$

The following result is due to Gustafsson and Sakai [4]. We give a short proof here for the sake of completeness.

Lemma 4 *Let μ be a measure with compact support contained in $W_- \cup H$ and let $A = \{x' : (x', 0) \in \Omega(\mu) \cap H\}$. Then there is a continuous function $g : A \to (0, \infty)$ such that*

$$\Omega(\mu) \cap W_+ = \{(x', x_N) : x' \in A \text{ and } 0 < x_N < g(x')\}. \tag{4}$$

Proof. Let $u = U^{\mu} - U^{\widetilde{\mu}}$. Thus $u \geq 0$. We may assume, by means of a limiting argument, that supp$\mu \subset W_-$, and so u is continuously differentiable on an open set containing $W_+ \cup H$. Let $\overline{u}(x) = u(x', -x_N)$. We note that $U^{\mu} - \overline{u} + q$ is subharmonic on W_+, and $U^{\widetilde{\mu}} + q$ is subharmonic on all of $\mathbb{R}^N$. Since

$$U^{\mu} - \overline{u} + q = U^{\mu} - u + q = U^{\widetilde{\mu}} + q \quad \text{on } H,$$

the function

$$v = \begin{cases} \max\{U^{\mu} - \overline{u} + q, U^{\widetilde{\mu}} + q\} & \text{on } W_+ \\ U^{\widetilde{\mu}} + q & \text{on } W_- \cup H \end{cases}$$

is a subharmonic minorant of $U^{\mu} + q$. Thus $v \leq U^{\widetilde{\mu}} + q$ by the definition of $U^{\widetilde{\mu}}$, whence $U^{\mu} - \overline{u} \leq U^{\widetilde{\mu}}$ on W_+ and so $u \leq \overline{u}$ there. It follows that $\partial u / \partial x_N \leq 0$ on H.

Let $\Omega_+ = \Omega(\mu) \cap W_+$. Since $u = 0$ on $\omega(\mu)^c$, and so on $\Omega(\mu)^c$, and since every point of $\partial \Omega(\mu)$ is the limit of some sequence of points of Lebesgue density of $\Omega(\mu)^c$, we see that $|\nabla u| = 0$ on $\partial \Omega_+ \cap W_+$. We note from (1) that Δu is constant in Ω_+, so the function $\partial u / \partial x_N$ is harmonic there, and hence $\partial u / \partial x_N \leq 0$ on Ω_+, by the maximum principle. Further, since u is nonconstant in each component of Ω_+, and $u = 0$ on $W_+ \backslash \Omega_+$, we actually have $\partial u / \partial x_N < 0$ on Ω_+. We now define

$$g(x') = \sup\{t > 0 : (x', t) \in \Omega_+\} \qquad (x' \in A).$$

Clearly $\Omega(\mu) \cap W_+$ lies under the graph of g. Conversely, if (x', x_N) lies under the graph of g and $x_N > 0$, then $u(x', x_N) > 0$ and so $(x', x_N) \in \omega(\mu) \subseteq \Omega(\mu)$. Thus (4) holds.

It remains to check that g is continuous. In fact, since $\Omega(\mu)$ is open and

$$\{x' : g(x') > c\} = \big\{x' : (x', c) \in \Omega(\mu)\big\} \qquad (c > 0),$$

it is clear that g is lower semicontinuous. On the other hand, if we apply the result of the previous paragraph with hyperplanes of varying orientation, we see that each point of $\partial \Omega_+ \cap W_+$ is the vertex of a vertical cone lying in $\Omega(\mu)^c$, and so g is also upper semicontinuous. ■

3 Proof of Theorem 1

Let Ω_1, Ω_2 and ν be as in the statement of Theorem 1. We begin by observing that we may assume U^{ν} to be continuous on Ω_2. To see this, let (ω_n) be an increasing sequence of regular open sets with union Ω_2 such that

$\overline{\omega}_n \subset \omega_{n+1}$ for each n, and let $\nu_n = (\nu - \lambda)|_{\omega_n \backslash \omega_{n-1}}$, where $\omega_0 = \emptyset$. If we define $u_n = U^{\nu_n}$ on ω_{n+1}^c, and extend u_n to $\mathbb{R}^N$ by solving the Dirichlet problem on ω_{n+1}, then the function $U^{\Omega_2} + \sum u_n$ is continuous on Ω_2, equals U^ν on Ω_2^c and can be expressed as $U^{\nu'}$ with $\nu' \geq \lambda|_{\Omega_2}$ and $\nu'(\Omega_2^c) = 0$.

Now let η be as in Lemma 3, and let $\Omega = \Omega(\eta)$. As we noted earlier, it follows from (2) that $\Omega_1 \cap \Omega_2 \subseteq \Omega$. We will suppose that

$$\lambda(\Omega_1 \backslash \Omega) > 0 \tag{5}$$

with a view to reaching a contradiction.

Let $D = \Omega \cup \Omega_2$. Our first step is to show that

$$U^{\widetilde{\eta}} = U^\Omega = U^\nu \quad \text{on} \quad D^c. \tag{6}$$

To see this, we note that $U^{\widetilde{\eta}} = U^\eta$ on Ω^c, since $\omega(\eta) \subseteq \Omega$, and $U^{\widetilde{\eta}} = U^\Omega$, by Lemma 3. On $D^c \backslash \Omega_1$, which coincides with $(\Omega_1 \cup \Omega_2)^c \cap \Omega^c$, we thus have $U^\nu = U^{\Omega_1} = U^\eta = U^{\widetilde{\eta}}$. The nonnegative function $U^{\Omega_1} - U^\Omega$ is superharmonic on Ω_1. It cannot be constant on Ω_1, in view of (5), so it is strictly positive there. Hence $U^{\Omega_1} > U^\Omega = U^{\widetilde{\eta}} = U^\eta = U^\nu$ on $D^c \cap \Omega_1$. We have now proved (6).

Let

$$E = \Omega_2 \backslash \Omega \quad \text{and} \quad \mu = \nu + \lambda\,|_E\,,$$

whence $D = \Omega \cup E$. Clearly $U^\Omega = U^{\widetilde{\eta}} \leq U^\eta \leq U^\nu$, so

$$U^D = U^\Omega + U^E \leq U^\nu + U^E = U^\mu, \tag{7}$$

and from (6) we see that

$$U^D = U^\Omega + U^E = U^\nu + U^E = U^\mu \text{ on } D^c. \tag{8}$$

We note from (7) that $U^D + q$ is a subharmonic minorant of $U^\mu + q$, so $U^D \leq U^{\widetilde{\mu}} \leq U^\mu$. The nonnegative function $U^{\widetilde{\mu}} - U^D$ vanishes on D^c, by (8), and hence on $\mathbb{R}^N$, since it is subharmonic on D. Thus

$$\widetilde{\mu} = \lambda\,|_D \quad \text{and} \quad D \subseteq \Omega(\mu). \tag{9}$$

Further,

$$0 = (\lambda - \widetilde{\mu})(\Omega(\mu)) = \lambda(\Omega(\mu) \backslash D) \geq \lambda((\Omega(\mu) \backslash \overline{\Omega}_2) \backslash \Omega) = (\lambda - \widetilde{\eta})(\Omega(\mu) \backslash \overline{\Omega}_2),$$

so $\Omega(\mu) \backslash \overline{\Omega}_2 \subseteq \Omega \subseteq D$. In view of (9) we thus see that

$$D \backslash \overline{\Omega}_2 = \Omega(\mu) \backslash \overline{\Omega}_2. \tag{10}$$

We now claim that $\partial\Omega_1 \subset \overline{D}$. For, if this were not the case, we could choose an open ball $B \subset \overline{D}^c$ that intersects $\partial\Omega_1$. Since $U^{\Omega_1} \geq U^{\eta} \geq U^{\widetilde{\eta}} = U^{\nu}$ on D^c, by (6), the function $U^{\Omega_1} - U^{\nu}$, which is nonnegative and superharmonic on B and attains the value 0 on $B \backslash \Omega_1$, must vanish identically on B. This leads to a contradiction, as $B \cap \Omega_1 \neq \emptyset$.

Next we claim that $\Omega_1 \backslash \overline{\Omega}_2 \subset D$. For, if there were a point $y \in \Omega_1 \backslash (D \cup \overline{\Omega}_2)$, then we could assume (by choosing a suitable coordinate system) that the closest point of $\overline{\Omega}_2$ to y is 0, and $y = (0', |y|)$. Let $y_0 = (0', t_0)$, where $t_0 = \sup\{t : (0', t) \in \Omega_1\}$. Then $t_0 > |y|$ and $y_0 \in \partial\Omega_1 \subset \overline{D} \subseteq \overline{\Omega(\mu)}$, by the preceding paragraph and (9). Also, $y \in \Omega(\mu)^c$, by (10). Since $\text{supp}\mu = \overline{\Omega}_2 \subset W_- \cup H$, Lemma 4 now yields the desired contradiction.

In view of the previous paragraph we see that

$$\lambda(\Omega_1 \backslash \Omega) = \lambda((\Omega_1 \cup \Omega_2) \backslash D) \leq \lambda(\partial\Omega_2) = 0,$$

which contradicts (5). Thus $\lambda(\Omega_1 \backslash \Omega) = 0$ and so $\Omega_1 \subseteq \Omega$, by the definition of Ω. Since $\lambda(\Omega_1) = \lambda(\Omega)$, and Ω_1 is solid, it follows that $\Omega = \Omega_1$. Hence $U^{\nu} - U^{\Omega}$ is a nonnegative superharmonic function on $\overline{\Omega}^c$ which attains the value 0, so $U^{\nu} = U^{\Omega} = U^{\Omega_1}$ there, and therefore $\lambda(\Omega_2 \backslash \overline{\Omega}_1) = 0$. It follows that $\Omega_2 \subset \overline{\Omega}_1$, and so $\Omega_2 \subseteq \left(\overline{\Omega}_1\right)^{\circ} = \Omega_1$, as required.

The corollary is immediate, since $\lambda(\Omega_1) = \lambda(\Omega_2)$ in this case.

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School of Mathematical Sciences
University College Dublin
Belfield, Dublin 4, Ireland.
e-mail: stephen.gardiner@ucd.ie and tomas.sjodin@ucd.ie