MOTHER BODIES OF ALGEBRAIC DOMAINS IN THE COMPLEX PLANE

TOMAS SJÖDIN

Abstract. We give a definition of a mother body of a domain in the complex plane, and prove some continuity properties of its potential in terms of the Schwarz function (which is explicitly assumed to exist). We end the article by studying the case of the ellipse, and use the previous results to prove existence and uniqueness of a mother body in this case, as well as a related existence result about graviequivalent measures for the ellipse.

1. Introduction

In the paper [2] Björn Gustafsson introduced the notion of a mother body for a domain $\Omega \subset \mathbb{R}^N$ (this was probably the first rigorous definition, although the notion of a mother body, a kind of potential theoretic skeleton, goes back at least to Zidarov [6]). The idea of a mother body is also implicit in the theory of quadrature domains. We will give exact definitions in the next section, but for now we may think of a mother body for Ω as a positive measure μ such that its logarithmic potential agrees with that of Ω (considered as a body of density one) in the sense that

$$U^{\mu} = U^{\chi_{\Omega}} \text{ outside of } \Omega,$$

and such that its support has zero Lebesgue-measure, and does not disconnect any part of Ω from the complement of Ω.

In the paper [2] existence and uniqueness of a mother body in case of convex polyhedra in arbitrary dimension was proved. In the present article we will focus on smooth domains in two dimensions. The smoothness will not be explicitly assumed, but we will assume the existence of a Schwarz function (to be defined), which implies a high degree of smoothness.

As in [2] we are interested in existence and uniqueness of a mother body for a given domain. As in the case of convex polyhedra, the natural way to attack the problem of uniqueness of a mother body makes it necessary to have some continuity results for the logarithmic potential U^{μ}. So this will be treated in section 3.

In section 4 we turn to the case of the ellipse to see an example of how a proof of uniqueness may look. (We start by proving existence, but this proof is standard and taken more or less directly from [4]).

The mother body for the ellipse is a measure with support on the line between the two focal points, and one may ask whether every positive measure graviequivalent to the ellipse as above must have a support encapsulating this line segment, where we by encapsulating here mean that the intersection between the unbounded component of the plane minus the support of the measure with this line segment is empty.

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This question was posed in [1] in connection with one-sided L^1-approximation by harmonic functions. Björn Gustafsson was later able to answer the question in the negative. His idea of proof was based on the Cauchy-Kowalevski theorem and a deformation argument (not published). Here we will prove the same result relying on other methods from the article [5]. This is done in section 5.

2. Basic notation and definitions

We will always be working in the complex plane $\mathbb{C}$, where we denote points by $z = x + iy$ (with subscripts when necessary). For $\Omega \subset \mathbb{C}$ open we define the following function classes:

$$\begin{aligned} A(\Omega) &:= \{\text{analytic functions in } \Omega\} \\ H(\Omega) &:= \{\text{harmonic functions in } \Omega\} \\ U(\Omega) &:= \{\text{superharmonic functions in } \Omega\} \\ S(\Omega) &:= \{\text{subharmonic functions in } \Omega\}. \end{aligned}$$

For $A \subset \mathbb{C}$ Lebesgue measurable we let

$$L^P(A) := \{p\text{-th power Lebesgue integrable functions in } A\},$$

and m will denote Lebesgue measure. For $A \subset \mathbb{C}$ general we put

$$\begin{aligned} C(A) &:= \{\text{continuous functions in } A\} \\ P(A) &:= \{\text{non-negative functions in } A\}. \end{aligned}$$

We also use superpositioning in the sense that for instance $UP(\Omega) := U(\Omega) \cap P(\Omega)$. We recall Green's formula in the plane in complex notation for functions and boundaries smooth enough:

$$\int_{\partial\Omega} f dz + g d\bar{z} = 2i \int_{\Omega} \left(\frac{\partial f}{\partial \bar{z}} - \frac{\partial g}{\partial z} \right) dm.$$

In particular if f is replaced by $f\bar{z}$ where f is analytic in Ω (and sufficiently smooth up to the boundary), and g by zero in the formula we get:

$$\int_{\partial\Omega} f(z)\bar{z} dz = 2i \int_{\Omega} f dm.$$

The language and basic results from distribution theory will be used extensively, and the following notation will be employed. In the definition below $\mathcal{F}$ stands for either $\mathbb{R}$ or $\mathbb{C}$.

$$\begin{aligned} C_0^\infty(\Omega, \mathcal{F}) &:= \{\phi : \Omega \to \mathcal{F} : \text{supp}(\phi) \subset \Omega, \phi \text{ is infinitely differentiable}\} \\ \mathcal{D}'(\Omega, \mathcal{F}) &:= \{\text{distributions "from" } C_0^\infty(\Omega, \mathcal{F}) \text{ to } \mathcal{F}\} \\ \mathcal{E}'(\Omega, \mathcal{F}) &:= \{\text{those elements in } \mathcal{D}'(\Omega, \mathcal{F}) \text{ which have compact support in } \Omega\}. \end{aligned}$$

We have natural injections $\mathcal{D}'(\Omega, \mathbb{R}) \subset \mathcal{D}'(\Omega, \mathbb{C})$ and $\mathcal{E}'(\Omega, \mathbb{R}) \subset \mathcal{E}'(\Omega, \mathbb{C})$ which will also be used whenever needed. We define the logarithmic kernel by

$$U(z) := \log |z|.$$

(Note that we omit the constant that ought to be in front of this for it to be a fundamental solution of the Laplace operator). For $\mu \in \mathcal{E}'(\mathbb{R}^2, \mathbb{C})$ we now define

$$U^\mu := U * \mu,$$

called the logarithmic potential of μ.

Now we give the definition of a mother body we will work with in this paper, and then make some comments on it.

Definition 2.1. Let $\Omega \subset \mathbb{C}$ be a bounded domain. By a mother body for Ω we mean a positive element $\mu \in \mathcal{E}'(\Omega, \mathbb{R})$ such that

(1) $U^{\mu} = U^{\chi_\Omega}$ in $\overline{\Omega}^c$.

(2) $m(\text{supp}(\mu)) = 0$

(3) each component of $\mathbb{C} \setminus \text{supp}(\mu)$ intersects $\overline{\Omega}^c$.

We remark that every element in $\mathcal{E}'(\Omega, \mathbb{C})$ may be seen in an obvious way as an element in $\mathcal{E}'(\mathbb{C}, \mathbb{C})$, and that U^{μ} is a smooth function outside the support of μ (or to be precise, has such a representation). By μ being positive we mean that

$$\langle \mu, \phi \rangle \geq 0 \quad \forall \phi \geq 0,$$

and any such μ is well known to have a representation as a positive measure (by measure we will always mean Radon measure).

In [2] the support was allowed to reach the boundary, and that is necessary if we have a boundary which is not smooth as in the case of polyhedra, but since we aim at assuming the existence of a Schwarz function this sort of problem will not occur (the boundary may nevertheless have certain type of singularities though, but will essentially be real analytic). The existence of a Schwarz function extendable across the entire boundary actually implies that the support cannot reach the boundary, but more about this later. Another difference in the definition is that it was also required that we should have (in our notation)

$$U^{\mu} \leq U^{\chi_\Omega}$$

everywhere. We will not assume this simply because we will not have any use of it, but for certain results, in particular with regards to inverse balayage, this is a natural part of the definition. We will not motivate the definition further, since this is already done well in [2].

3. Some continuity results

In this section we will throughout assume that $\Omega \subset\subset \mathbb{C}$ is a domain with piecewise C^1 boundary (in particular finitely connected, and with $\Omega = \text{int}(\overline{\Omega})$). We also fix a compact set

$$K \subset \Omega$$

with

$$m(K) = 0$$

and such that

$$\text{each component of } \mathbb{C} \setminus K \text{ intersects } \overline{\Omega}^c.$$

Furthermore we assume that there is a function

$$S \in C(\overline{\Omega} \setminus K) \cap A(\Omega \setminus K)$$

with

$$S(z) = \bar{z} \quad \forall z \in \partial\Omega,$$

and such that

$$S \in L^1(\Omega)$$

(since S is defined a.e. on Ω this makes sense). The function S will be referred to as the Schwarz function of Ω, and we refer the reader to the book [4] for more information about it.

Before stating and proving our results for this section let us motivate the above assumptions, and explain what they have to do with mother bodies. So assume that Ω has a mother body μ. If we put $K := \operatorname{supp}(\mu)$, then by definition the assumptions about K above will hold. Now let us define the functions

$$u := -\frac{1}{2\pi}\left(U^{\mu} - U^{\chi_\Omega}\right)$$

and

$$S(z) := \bar{z} - 4\frac{\partial u}{\partial z}.$$

By construction we have

$$-\Delta u = -4\frac{\partial^2 u}{\partial z \partial \bar{z}} = \mu - 1 \text{ in } \Omega, \text{ and } S(z) = \bar{z} \text{ on } \partial\Omega.$$

Also

$$\frac{\partial}{\partial \bar{z}}\left(\bar{z} - 4\frac{\partial u}{\partial z}\right) = 0 \text{ in } \Omega \setminus K,$$

so we see that

$$S \in C(\overline{\Omega} \setminus K) \cap A(\Omega \setminus K)$$

as above. Finally we have by the local integrability of $1/|z|$ and Fubini's theorem that

$$\begin{aligned}
\int_\Omega |S|\, dm &= \int_\Omega \left|\bar{z} - 4\frac{\partial u}{\partial z}(z)\right| dm(z) \\
&= \int_\Omega \left|\bar{z} - \frac{1}{\pi}\left(\int_\Omega \frac{1}{z - z_0} dm(z_0) - \int \frac{1}{z - z_0} d\mu(z)\right)\right| dm(z) \\
&\leq \int_\Omega |\bar{z}|\, dm(z) + \frac{1}{\pi}\int_\Omega\int_\Omega \frac{1}{|z - z_0|} dm(z_0) dm(z) + \\
&+ \frac{1}{\pi}\int\int_\Omega \frac{1}{|z - z_0|} dm(z) d\mu(z_0) < \infty.
\end{aligned}$$

This ends the motivation, since it implies that $S \in L^1(\Omega)$.

We will use the following lemma valid in any number of dimensions.

Lemma 3.1. *Suppose $B \subset \mathbb{R}^N$ is open, and $u \in L^1_{loc}(B)$. If the distributional gradient $\nabla u \in (\mathcal{D}'(B, \mathbb{R}))^N$ has a representative in $(L^\infty(B))^N$, then u has a representative which is Lipschitz-continuous on B.*

Proof. Let

$$\eta_\varepsilon := \begin{cases} \frac{C}{\varepsilon^N} \exp\left(\frac{\varepsilon}{|x|^2 - \varepsilon}\right) & |x| < \varepsilon \\ 0 & |x| \geq \varepsilon, \end{cases}$$

where C is chosen so that

$$\int_{\mathbb{R}^N} \eta_\varepsilon dm = 1 \quad \varepsilon > 0.$$

Let $K \subset B$ be compact, for small $\varepsilon > 0$ we may look at the function

$$u_\varepsilon(y) := (u * \eta_\varepsilon)(y) = \int u(x)\eta_\varepsilon(x - y) dm(x) \quad y \in K.$$

Then we get that

$$|\nabla u_\varepsilon(y)| = |(\nabla u * \eta_\varepsilon)(y)| = \left| \int \nabla u(x) \eta_\varepsilon(x-y) dm(x) \right| \leq$$
$$\leq \int |\nabla u(x)| \, \eta_\varepsilon(x-y) dm(x) \leq ||\nabla u||_{L^\infty(B)},$$

so

$$||\nabla u_\varepsilon||_{L^\infty(K)} \leq ||\nabla u||_{L^\infty(B)}.$$

Since u_ε is smooth and defined in a neighbourhood of K for small ε we may now apply the mean value theorem from advanced calculus to get

$$|u_\varepsilon(x) - u_\varepsilon(y)| \leq ||\nabla u||_{L^\infty(B)} |x-y| \quad \forall x, y \in K.$$

Hence, if we fix a small ε_0 we see that the family

$$\{u_\varepsilon : 0 < \varepsilon < \varepsilon_0\},$$

is uniformly bounded and uniformly equicontinuous on K. Since $u_\varepsilon \to u$ a.e. and since K was arbitrary it is therefore easy to see from the Ascoli-Arzéla theorem that we may take a continuous representative for u. But we also get on K

$$|u(x) - u(y)| \leq |u(x) - u(y) - u_\varepsilon(x) + u_\varepsilon(y)| + |u_\varepsilon(x) - u_\varepsilon(y)|$$
$$\leq |u(x) - u(y) - u_\varepsilon(x) + u_\varepsilon(y)| + ||\nabla u||_{L^\infty(B)} |x-y|.$$

So if we let $\varepsilon \to 0$ and since K is arbitrary we conclude that

$$|u(x) - u(y)| \leq ||\nabla u||_{L^\infty(B)} |x-y| \quad \forall x, y \in B,$$

so the proof is done. □

In the following theorem the notation introduced in the beginning of this chapter is used.

Theorem 3.2. *Suppose there is a positive measure η with $\operatorname{supp}(\eta) \subset K$ and such that the function*

$$u := \frac{-1}{2\pi} \left(U^\eta - U^{\chi_\Omega} \right)$$

is equal to 0 on Ω^c. Then we have

$$\frac{\partial u}{\partial z} = \frac{1}{4} \left(\overline{z} - S(z) \right) \ \text{ in } \Omega \setminus K.$$

If furthermore $B \subset \Omega$ is open and S is bounded on $B \setminus K$, then u is Lipschitz-continuous on B.

Remark 3.3. We note that, from the definition of u, $u \in C^1(\Omega \setminus K)$ and this implies that $\partial u / \partial z$ is defined pointwise on $\Omega \setminus K$, and hence $m-$a.e. on Ω. But it is also clear from the definition of u that this derivative defined pointwise almost everywhere on Ω has

$$\frac{\partial u}{\partial z} \in L^1(\Omega),$$

and that this L^1 function equals the distributional derivative of u (w.r.t. z) in $\mathcal{D}'(\Omega, \mathbb{R})$.

Proof. We start by proving that

$$\frac{\partial u}{\partial z} = \frac{1}{4}\left(\overline{z} - S(z)\right) \text{ in } \Omega \setminus K \text{ (and as elements in } \mathcal{D}'(\Omega, \mathbb{R})).$$

To see this we note that the function

$$4\frac{\partial u}{\partial z} + S(z) - \overline{z}$$

is zero on $\partial\Omega$, and analytic in $\Omega \setminus K$, and from this it follows that the identity holds pointwise on $\Omega \setminus K$. The equality in $\mathcal{D}'(\Omega, \mathbb{R})$ follows from the remark preceding the theorem.

Since u is real we also have

$$\frac{1}{2}\left|\nabla u(z)\right| = \left|\frac{\partial u(z)}{\partial z}\right| = \frac{1}{4}\left|S(z) - \overline{z}\right| \text{ on } B \setminus K,$$

so $\nabla u \in \left(L^\infty(B)\right)^2$, and hence u has a representative v that is a Lipschitz-continuous function. But by standard potential theory it is easy to see that we also have on B

$$u(x) = \lim_{r\to 0} \frac{1}{m(B_r(x))}\int_{B_r(x)} udm = \lim_{r\to 0} \frac{1}{m(B_r(x))}\int_{B_r(x)} vdm = v(x),$$

so $u \equiv v$ on B, and the proof is done. □

4. Existence and uniqueness of a mother body for the ellipse

We now turn to the ellipse, and we need to introduce some more notation. We let $a > b > 0$, $c := \sqrt{a^2 - b^2}$ and

$$\Omega := \left\{(x, y) : \frac{x^2}{a^2} + \frac{y^2}{b^2} < 1\right\}. \tag{1}$$

It is often more convenient to work with $z, \overline{z}$ instead of x, y, and it is easy to compute that

$$P(z, \overline{z}) := \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{1}{4a^2b^2}\left(2(a^2 + b^2)z\overline{z} - (a^2 - b^2)(z^2 + \overline{z}^2)\right).$$

Then we may write

$$\Omega = \{z : P(z, \overline{z}) < 1\},\ \partial\Omega = \{z : P(z, \overline{z}) = 1\}.$$

In particular the Schwarz function satisfies

$$P(z, S(z)) = 1 \text{ on } \partial\Omega.$$

Solving this equation gives the Schwarz function for the ellipse, which is given by one branch of

$$S(z) := \frac{a^2 + b^2}{c^2}z - \frac{2ab}{c^2}(z^2 - c^2)^{1/2}.$$

If we replace P above by some other polynomial in $z, \overline{z}$ it is often possible to find the Schwarz function explicitly from this formula, and there will only be a finite number of points which are possible poles and/or branch points of the Schwarz function in this case. Hence, for algebraic domains (i.e. domains with algebraic boundary) the continuity of the potential produced by a mother body is not a problem as our results from the previous section shows. Here we shall only consider the ellipse. Another interesting article where mother bodies for algebraic domains are discussed is [3]. We refer to [4] for more information about the Schwarz function.

It is now time to show existence of a mother body for Ω. This construction is well known, and in this case taken from [4] more or less.

Suppose now that we remove the segment $[-c, c]$ from $\mathbb{C}$, and let $S(z)$ be the single-valued branch of the Schwarz-function in $\mathbb{C} \setminus [-c, c]$ which takes the value $\bar{z}$ on $\partial\Omega$. From Stokes theorem we immediately get for f analytic in a neighbourhood of $\overline{\Omega}$:

$$\int_\Omega f dm = \frac{1}{2i} \int_{\partial\omega} f(z) S(z) dz$$

for any open simply connected set ω which contains $-c$ and c (with reasonable boundary). Notice that it is only the term

$$-\frac{2ab}{c^2}(z^2 - c^2)^{1/2} = -\frac{2ab}{c^2}|z^2 - c^2|^{1/2} e^{i((\arg(z+c)+\arg(z-c))/2)}$$

of $S(z)$ that contributes to this integral, and if we extend this part of $S(z)$ to $[-c, c]$ from above we get

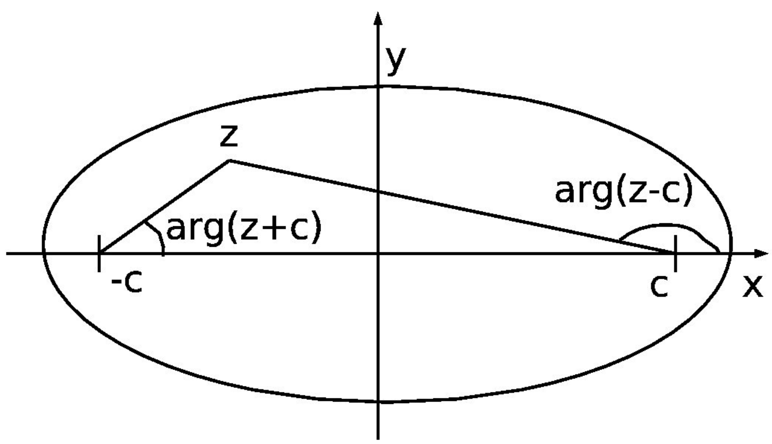
$$-i\frac{2ab}{c^2}\sqrt{c^2 - x^2},$$

and from below we get the same thing but with reversed sign. So if we take $\omega_n \searrow [-c, c]$ we see that

$$\int_\Omega f dm = \frac{2ab}{c^2} \int_{-c}^{c} f(x)\sqrt{c^2 - x^2} dx,$$

for any f analytic in a neighbourhood of $\overline{\Omega}$. Hence since this is a positive measure it is easy to verify that this gives us a mother body μ for Ω with

$$d\mu = \frac{2ab}{c^2}\sqrt{c^2 - x^2} dx \text{ on } [-c, c].$$



Notice also that the formula above implies that every pair of ellipses of constant density, having the same focal points and the same total mass, produce the same potential outside the largest one. So this is another way to see that we without loss of generality may assume that all our mother bodies have support in Ω.

We now turn to the uniqueness. If we look at the above formula for $S(z)$ and use that if we are given a mother body η for Ω, then we know from the previous section that we must have

$$S(z) = \bar{z} - 4\frac{\partial u}{\partial z}(z),$$

where

$$u = -\frac{1}{2\pi}\left(U^{\eta} - U^{\chi_\Omega}\right).$$

This gives us that $4\overline{\frac{\partial u}{\partial z}} = z - \overline{S(z)}$, and if we fix $z_{\partial\Omega} \in \partial\Omega$ we get (in $\mathbb{C} \setminus \mathrm{supp}(\eta)$):

$$4u(z_0) = 4\left(\int_{z_{\partial\Omega}}^{z_0} \frac{\partial u}{\partial z} dz + \int_{z_{\partial\Omega}}^{z_0} \overline{\frac{\partial u}{\partial z}} d\bar{z}\right) =$$

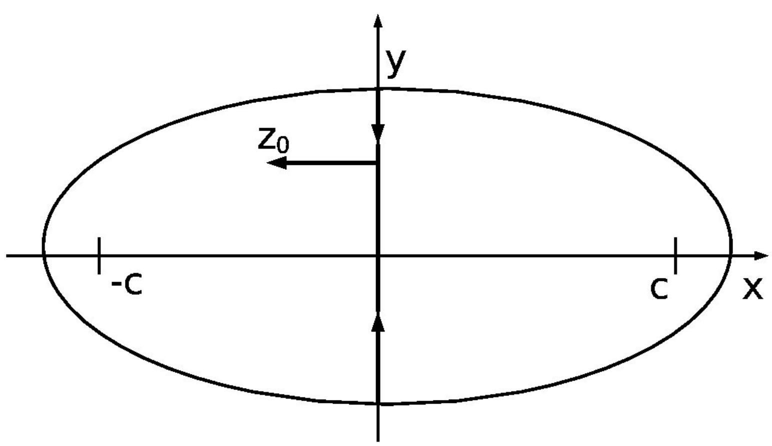
$$\int_{z_{\partial\Omega}}^{z_0} \bar{z} dz + \int_{z_{\partial\Omega}}^{z_0} z d\bar{z} - \int_{z_{\partial\Omega}}^{z_0} S(z) dz - \int_{z_{\partial\Omega}}^{z_0} \overline{S(z)} d\bar{z}$$

$$= |z_0|^2 - |z_{\partial\Omega}|^2 - 2Re\int_{z_{\partial\Omega}}^{z_0} S(z)dz.$$

(Notice that this is independent of the path of integration in $\mathbb{C} \setminus \mathrm{supp}(\eta)$ and also of $z_{\partial\Omega} \in \partial\Omega$.)

We now notice that at each point $z_0 \in \Omega$ there are at most two possible values of $u(z_0)$. If say $z_0 = x_0 + iy_0$ then we can get these two values performing the integration as above along the curves first starting from ib respectively $-ib$ and then going along the $y-$axis to iy_0, then in a straight line from iy_0 to $x_0 + iy_0$. We should of-course start at the branch of $S(z)$ taking the value $\overline{z}$ on $\partial\Omega$. There is however a set where there is only one possible value, for instance is $[-c, c]$ part of this set. We call this set A and we now determine what it looks like. The set A is hence determined by the points in which the two integrals along curves such as the ones described above coincide.

To determine this set we see that by symmetry it is sufficient to consider points in $\{(x, y) : x \leq 0, y \geq 0\}$ and given a point z_0 here we take the integral from $-ib$ to iy_0 and then along the straight line from iy_0 to z_0 parallel to the real axis, then we get the other value in the same way but starting from ib instead (see Figure 2).



The difference of the two integrals is

$$2Re\int_{-b}^{y_0} S(iy)idy+2Re\int_{0}^{x_0} S(x+iy_0)dx-2Re\int_{b}^{y_0} S(iy)idy-2Re\int_{0}^{x_0} S(x+iy_0)dx$$

{notice that the second and fourth integral not are equal because we are on different branches of S}

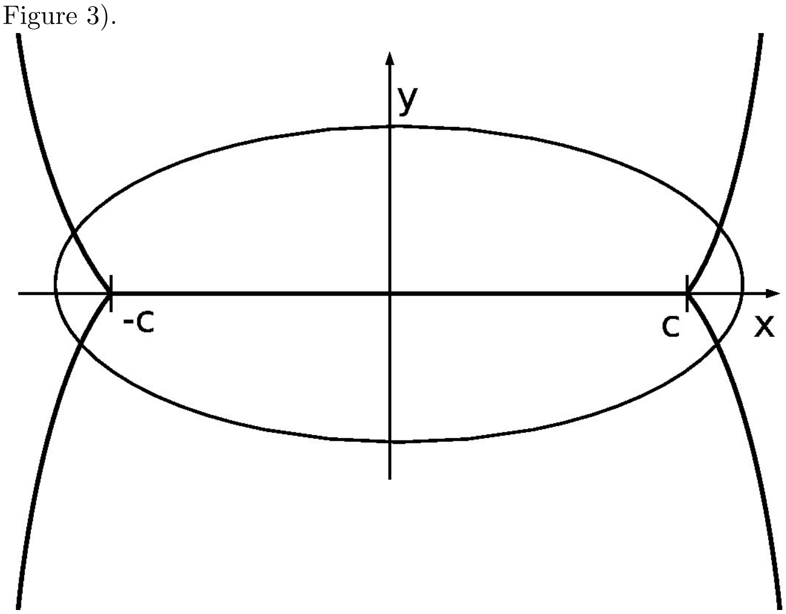
$$= 2Re\int_{-b}^{y_0}\frac{2ab}{c^2}\sqrt{y^2+c^2}e^{i3\pi/2}idy + 2Re\int_{y_0}^{b}\frac{2ab}{c^2}\sqrt{y^2+c^2}e^{i\pi/2}idy+$$

$$+4Re\int_{x_0}^{0}\frac{2ab}{c^2}|(x+iy_0)-c^2|^{1/2}e^{i(\arg(z+c)+\arg(z-c))/2}dx =$$

={here we have changed the branch of S in one integral along the x-direction and added the two together.}=

$$=\begin{cases} \frac{8ab}{c^2}\left(\int_0^{y_0}\sqrt{y^2+c^2}dy+\int_{x_0}^{0}|(x+iy_0)^2-c^2|^{1/2}\cos\left(\frac{\arg(z+c)+\arg(z-c)}{2}\right)dx\right) \\ \text{if } (y_0\le b). \\ \frac{8ab}{c^2}\left(\int_0^{b}\sqrt{y^2+c^2}dy+\int_{x_0}^{0}|(x+iy_0)^2-c^2|^{1/2}\cos\left(\frac{\arg(z+c)+\arg(z-c)}{2}\right)dx\right) \\ \text{if } (y_0> b). \end{cases}$$

This function is obviously zero on $-c\le x_0\le c, y_0=0$, furthermore if $x_0=0$ and $y_0>0$ it is strictly positive, and it decreases strictly towards minus infinity as we let $x_0\to-\infty$. Hence this gives us that the set A lies on $[-c,c]$ and four curves (one in each quadrant) starting at the branch point of S and going towards infinity (see Figure 3).



Suppose now that $z\in \text{supp}(\eta)\setminus A$, and we choose a small $\varepsilon>0$ such that $\overline{B}_\varepsilon(z)\cap A=\emptyset$. Let us in $B_\varepsilon(z)$ denote the two possible values of $u(z)$ by $u_1(z)$ respectively $u_2(z)$, where these functions are chosen to be continuous in $B_\varepsilon(z)$. Now we see that if $u(z)=u_1(z)$ then this must clearly be true everywhere in $B_\varepsilon(z)$, since otherwise u would not be continuous. But this would also imply that $-\Delta u=1$

in $B_\varepsilon(z)$, which contradicts that $z \in \text{supp}(\eta)$. We therefore may conclude that $\text{supp}(\eta) \subset A$, and this immediately gives that $\eta = \mu$, where μ was the mother body we constructed in the beginning of this section. We may now state this as a theorem.

Theorem 4.1. *For the ellipse* $\Omega = \left\{(x,y) : \frac{x^2}{a^2} + \frac{y^2}{b^2} < 1\right\}$ *there is a unique mother body* μ. *It is given by*

$$d\mu = \frac{2ab}{c^2}\sqrt{c^2 - x^2}dx \text{ on } [-c, c].$$

5. Another existence result for the ellipse

We will make some further remarks about the ellipse in this section. A question related to the one above, and especially to the uniqueness of the mother body is the following. Since we have our mother body μ we can produce several positive measures supported by say a simple closed smooth arc, which give the same potential as χ_Ω outside Ω. This can be done by choosing a domain D such that $\text{supp}(\mu) \subset D \subset\subset \Omega$, where ∂D is a simple closed smooth curve, and then choosing the measure to be $\text{Bal}(\mu, \partial D)$. This construction relies on that $\text{supp}(\mu) \subset D$. The question is now whether there exists any set $D \subset\subset \Omega$, where ∂D is a simple closed smooth curve, and some positive measure η with support on ∂D which gives the same potential as χ_Ω outside Ω, but $\text{supp}(\mu) \not\subset \overline{D}$? The answer is yes, and we conclude this paper by proving this. This result should be compared with Example 1 in [1]. We still let Ω denote the fixed ellipse given by (1).

Theorem 5.1. *There is a domain* $D \subset\subset \Omega$, *where* ∂D *is a simple closed smooth curve such that*

(1) $[-c, c] \setminus \overline{D} \neq \emptyset$,
(2) *there is a positive measure* η *with* $\text{supp}(\eta) \subset \partial D$ *and*

$$U^\eta = U^{\chi_\Omega} \text{ on } \Omega^c.$$

Proof. Let

$$\omega := \Omega \setminus \{x + iy : y \leq -|x|\}.$$

So ω is simply Ω minus a wedge. The only important thing is that $0 \in \partial\omega$ and no other point of $[-c, c]$ belongs to $\partial\omega$, and also that ω is strongly starshaped with respect to some point $a \in \omega$. We already know that on any C^1 Jordan arc in ω between $-c$ and c there is a complex Radon measure γ s.t.

$$\int f d\mu = \int f d\gamma \tag{2}$$

for every f analytic in Ω. Now by theorem 4.2 in [5] and the fact that any harmonic function on Ω has a harmonic conjugate it follows that there is a real Radon measure ν with support in ω such that

$$\int h d\mu = \int h d\nu \tag{3}$$

for every h harmonic in Ω. And by harmonic continuation it is easy to see that (3) in-fact holds for any $h \in C(\overline{\omega}) \cap H(\omega)$. If we now let $\mu_\varepsilon := \mu\lfloor([-c, -\varepsilon] \cup [\varepsilon, c])$ for some $0 < \varepsilon < c$ we immediately get for any $h \in C(\overline{\omega}) \cap HP(\omega)$ which is strictly positive in ω that:

$$\int h d\nu \geq \int h d\mu_\varepsilon > 0. \tag{4}$$

But since ω is strongly starshaped it is trivial to see that this class is dense with respect to uniform convergence on compact subsets in ω in the class of all positive harmonic functions in ω. Hence (4) holds for any such $h \neq 0$. Now we may apply theorem 5.1 of [5] to get that there is some positive Radon measure η' with support in ω s.t.

$$\int h d\mu = \int h d\eta'$$

for every $h \in C(\overline{\omega}) \cap H(\omega)$. But this immediately implies that

$$U^{\eta'} = U^{\mu} = U^{\chi_\Omega}$$

on Ω^c. If we now take $D \subset\subset \omega$ with ∂D a smooth simple closed curve, and such that $\operatorname{supp}(\eta') \subset D$ we may choose η as

$$\eta := \operatorname{Bal}(\eta', \partial D),$$

and the proof is done. □

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Department of Mathematics
Royal Institute of Technology
S-100 44 Stockholm, Sweden.
E-mail address: tomas@math.kth.se