ON THE STRUCTURE OF PARTIAL BALAYAGE

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ABSTRACT. We define partial balayage, construct it with classical potential-theoretic methods and establish the so called structure formula in full generality. Finally we give a stochastic construction of partial balayage.

1. Introduction

The aim of the paper is to study partial balayage of a (positive) Radon measure μ to density one. Loosely speaking, partial balayage means to find a measure in some sense maximal with the constraint that it is $\leq$ Lebesgue measure and is gravi-equivalent to μ outside the open set on which Lebesgue measure is attained.

In section 2 we define partial balayage in the same way as in [7], and give a proof of existence and uniqueness. We only use some basic distribution and potential theory to give direct arguments of existence and basic properties of partial balayage instead of relying on Sobolev-space methods and the solution of an obstacle problem as is done in [7]. Here we only consider positive measures, and for these we are able to prove the so called structure formula of partial balayage without any extra regularity assumptions on the measure. Another recent article, where partial balayage in the presence of an external field in the plane is studied, is [5].

In section 4 we turn to the probabilistic way of getting balayage. Here we assume that μ has a L^∞ density with respect to Lebesgue measure. In a way this problem has already been studied in [11] for the case when μ is a constant (> 1) times the characteristic function of some open bounded set. However, there seems to be a more direct approach which does not depend on the particular form of μ more than that it is in L^∞ and has compact support.

2. Partial Balayage

We start by introducing some notation.

All measures will be positive Radon measures with compact support unless otherwise stated.

$\mathcal{D}'(\mathbb{R}^n)$ is the space of real-valued distributions with its natural order:

$$u \leq v \text{ iff } \langle u, \phi\rangle \leq \langle v, \phi\rangle \quad \forall \phi \in C_c^\infty(\mathbb{R}^n), \phi \geq 0.$$

Also, U^η will denote the Newtonian potential of η, normalized so that $-\Delta U^\eta = \eta$ ($\eta \in \mathcal{E}'(\mathbb{R}^n)$ = distributions with compact support).

Finally m is Lebesgue-measure on $\mathbb{R}^n$, and

$$l_\varepsilon(x) := \frac{1}{m(B_\varepsilon)} \chi_{B_\varepsilon}(x),$$

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where $B_\varepsilon = B_\varepsilon(0) =$ the open ball with center 0 and radius ε.

One preliminary fact, perhaps easiest to see from standard elliptic regularity and Sobolev space theory, is that if we are given two functions $f, g \in L^\infty(m)$ with compact support, and we have $U^f = U^g$ on some set D (which need not be open), then $f = g$ a.e. on D.
To see this, suppose $f \in L^\infty(m)$ has compact support in a ball B in $\mathbb{R}^N$. Then U^f is the unique solution to

$$\begin{cases} -\Delta u = f & \text{in } B \\ u = U^f & \text{in } B \end{cases}.$$

So $u \in W^{2,2}(B')$ for every ball $B' \subset\subset B$ (see [6] theorem 8.8 p.183 for instance), and if $D \subset B'$ with $u = 0$ on D and $u \in W^{2,2}(B')$, then

$$\frac{\partial^2 u}{\partial x_j^2} = 0 \text{ a.e. on } D,$$

([6] lemma 7.7 p.152 applied twice).

We now fix a Radon measure $\mu \geq 0$ with compact support in $\mathbb{R}^n$ $(n \geq 2)$. In the following identities between functions and measures will be in the sense of distributions. Let

$$\Phi := \{u \in \mathcal{D}'(\mathbb{R}^n) : -\Delta u \leq 1, \quad u \leq U^\mu\}.$$

Lemma 2.1. *There is a unique maximal element V in Φ which can be taken to be of the form U^f where $f \in L^\infty(m)$ with $0 \leq f \leq 1$ and compact support. The set $\Omega := \{U^\mu > U^f\}$ is open and bounded. Furthermore, if we write $\mu = \mu_a + \mu_s$ where $\mu_a \ll m$, $\mu_s \perp m$, then $\mu_s(\Omega^c) = 0$ and $\mu_a \leq 1$ on Ω^c. In addition $f = 1$ on Ω, $f = \mu$ on $\overline{\Omega}^c$.*

We denote the function f from the lemma above by $\mathrm{Bal}(\mu, 1)$.

Proof. First note that $\Phi \neq \emptyset$ because $U^{\mu_\varepsilon} \in \Phi$ for large ε $(\mu_\varepsilon := \mu * l_\varepsilon)$. If we add $|\cdot|^2/(2n)$ to the elements of Φ we get a saturated family of subharmonic distributions, which all have representatives as upper semicontinuous functions. From this it directly follows that Φ has a largest element (because we may look at "$\Phi + |\cdot|^2/(2n)$" which we see by standard potential theory (see [1], [4] or [9] for instance) that this class has a largest element, and we only need to remove $|\cdot|^2/(2n)$ from it to get our V).

Note that V as constructed above is an upper semicontinuous function with $V \leq U^\mu$ everywhere. Now

$$V_\varepsilon = V * l_\varepsilon \in \Phi \quad \forall \varepsilon > 0,$$

since

$$V \leq U^\mu \Rightarrow V_\varepsilon \leq U^{\mu_\varepsilon} \leq U^\mu,$$

and

$$-\Delta V_\varepsilon = (-\Delta V) * l_\varepsilon \leq 1 * l_\varepsilon = 1.$$

Note that $V_\varepsilon \to V$ in L^1_{loc} and m-a.e. by the Lebesgue differentiation theorem, so if we put

$$u := \sup\{V_\varepsilon : \varepsilon > 0\}$$

pointwise we have that u is lower semicontinuous and $u = V$ m-a.e., because $V_\varepsilon \le V$ and there is a set $A \subset \mathbb{R}^n$, $m(A^c) = 0$ and $V_\varepsilon(x) \to V(x)$ $\forall x \in A$, so for these we have:

$$V(x) = \lim_{\varepsilon\to 0} V_\varepsilon(x) \le \sup_{\varepsilon>0} V_\varepsilon(x) \le V(x) \quad \forall x \in A.$$

Hence

$$u * l_\varepsilon = V * l_\varepsilon = V_\varepsilon \le u,$$

so u is superharmonic, and belongs to Φ (because as distributions $u = V$). So $0 \le -\Delta u \le 1$. The first inequality says that $-\Delta u$ has a representation as a Radon measure, and the second that this measure must be absolutely continuous with respect to Lebesgue measure m. So, since $-\Delta u \equiv 0$ far away (since $U^{\mu_\varepsilon} \in \Phi$ for large ε and $U^{\mu_\varepsilon} = U^\mu$ a distance ε from $\mathrm{supp}(\mu)$) we get that we can take $-\Delta u$ to be a Borel measurable function f with

$$0 \le f \le 1$$

everywhere, and f has compact support. That is $u = U^f$, which is continuous.

Therefore we may conclude that $u \equiv V \equiv U^f$, since

$$\begin{aligned} V(x) &= \lim_{r\searrow 0} \frac{1}{m(B_r(x))} \int_{B_r(x)} V(y)dm(y) = \\ &= \lim_{r\searrow 0} \frac{1}{m(B_r(x))} \int_{B_r(x)} U^f(y)dm(y) = U^f(x) \quad \forall x \in \mathbb{R}^n. \end{aligned}$$

(Above we used that V is the sum of a subharmonic and a continuous function).

Now set

$$\Omega = \{V < U^\mu\}.$$

Then Ω is open by the lower semicontinuity of $U^\mu - V$.
First note that

$$-\Delta V = 1 \text{ on } \Omega,$$

because if $x \in B_\varepsilon(x) \subset\subset \Omega$, and $-\Delta V = f < 1$ on some subset of $B_\varepsilon(x)$ with positive Lebesgue-measure, then with

$$\widetilde{V} := \begin{cases} V & \text{on } B_\varepsilon(x)^c \\ P - \frac{|\cdot|^2}{2n} & \text{on } B_\varepsilon(x) \end{cases} \tag{1}$$

where P is the Poisson integral of $V + |\cdot|^2/(2n)$ on $B_\varepsilon(x)$, then $\widetilde{V} > V$ on $B_\varepsilon(x)$ and belongs to Φ, which gives a contradiction. To see the above we note that

$$U := \begin{cases} V + \frac{|\cdot|^2}{2n} & \text{on } B_\varepsilon(x)^c \\ P = \text{Poisson-integral of } (V + \frac{|\cdot|^2}{2n}) & \text{on } B_\varepsilon(x) \end{cases}$$

is continuous and subharmonic. Also by the maximum principle we have $U > V + |\cdot|^2/(2n)$ on $B_\varepsilon(x)$, so we get that $\widetilde{V} = U - |\cdot|^2/(2n) > V$ on $B_\varepsilon(x)$.

Since $U^f = U^\mu$ on $\overline{\Omega}^c$ it follows that $f = \mu$ on $\overline{\Omega}^c$. Also let

$$\eta = \mu\lfloor\partial\Omega$$

and assume that η is not the zero measure. Now let

$$g = \mathrm{Bal}(\eta, 1)$$

as constructed above. Note that

$$v := U^f - U^\mu + U^\eta$$

satisfies $v \le U^\eta$ (since $U^f \le U^\mu$), and $-\Delta v = f - (\mu - \eta) \le 1$. Therefore since $v = U^\eta$ on Ω^c it follows that

$$D := \{U^\eta > U^g\} \subset \Omega.$$

But this implies that $D = \emptyset$ by the domination principle, since $U^\eta \le U^g$ on $\partial\Omega$, and hence this must hold everywhere. To be exact we know that $U^\eta = U^g$ on B^c for some large ball $B \supset\supset \Omega \supset D$, so

$$(*)\, G_B^\eta = U^\eta - h \le U^g - h = G_B^g \text{ on } \partial\Omega \supset \mathrm{supp}(\eta).$$

Since G_B^η is real-valued the domination principle says that the inequality $(*)$ holds on all of B. This clearly implies that $U^\eta \le U^g$ on all of $\mathbb{R}^n$. But by definition we also have $U^g \le U^\eta$, so $U^\eta \equiv U^g$, and hence $\eta = g$. This (together with the previous step) implies that $\mu_s(\Omega^c) = 0$, so the proof is done. □ □

Theorem 2.2. *The following can be said about the operation* $\mu \mapsto \mathrm{Bal}(\mu, 1)$*:*

(1) $\mu \le \eta \Rightarrow \mathrm{Bal}(\mu, 1) \le \mathrm{Bal}(\eta, 1)$
(2) $\mu_n \nearrow \mu \Rightarrow \mathrm{Bal}(\mu_n, 1) \nearrow \mathrm{Bal}(\mu, 1)$
(3) *With the notation from lemma 2.1 we have the structure formula* $\mathrm{Bal}(\mu, 1) = \chi_\Omega + \mu\lfloor\Omega^c$

(By $\mu_n \nearrow \mu$ *we mean increasing convergence in the weak***-topology.)*

Proof. (1) Let $f_\mu := \mathrm{Bal}(\mu, 1)$, $\Omega_\mu = \{U^\mu > U^{f_\mu}\}$ (we will use this notation for any measure μ). First of all, since $v := U^{f_\eta} - U^\eta + U^\mu$ fullfills $v \le U^\mu$ and $-\Delta v = f_\eta - (\eta - \mu) \le 1$ we get that $v \le U^{f_\mu}$, hence

$$U^\mu - U^{f_\mu} \le U^\eta - U^{f_\eta},$$

and so

$$\Omega_\mu \subset \Omega_\eta.$$

If we use the fact that

$$U^{f_\eta} - U^\eta + U^\mu \le U^\mu$$

which holds by the above, together with

$$-\Delta(U^{f_\eta} - U^\eta + U^\mu) \le 1,$$

we immediately get

$$U^{f_\eta} - U^\eta + U^\mu \le U^{f_\mu}.$$

Since $f_\eta = 1$ on Ω_η it follows trivially that $f_\mu \le f_\eta$ on Ω_η. But we also have

$$U^{f_\eta} = U^{f_\mu} + U^{f_{\eta-\mu}} \text{ on } \Omega_\eta^c,$$

(because $U^{f_\eta} = U^\eta$, $U^{f_\mu} = U^\mu$, and $U^{f_{\eta-\mu}} = U^{\eta-\mu}$ there, since $\Omega_\mu \cup \Omega_{\eta-\mu} \subset \Omega_\eta$), and since $f_\eta, f_\mu, f_{\eta-\mu} \in L^\infty(m)$ with compact support it follows that we have

$$f_\eta = f_\mu + f_{\eta-\mu} \text{ a.e. on } \Omega_\eta^c.$$

So since $f_{\eta-\mu}$ is positive we get $f_\eta \ge f_\mu$ a.e. on $\mathbb{R}^N$, and this finishes the proof of (1).

(2) If we set

$$f_n := \mathrm{Bal}(\mu_n, 1) \text{ and } f := \mathrm{Bal}(\mu, 1),$$

then we know by (1) that $f_n \leq f$ and f_n is increasing. Also

$$\int (f - f_n)dm = \int d(\mu - \mu_n) \searrow 0 \text{ as } n \to \infty.$$

Hence

$$\lim_{n\to\infty} (f - f_n) = 0 \quad m - a.e.$$

This proves (2).

(3) Let $\mu = \mu_a + \mu_s$ as before, and let $\Omega_1 \subset\subset \Omega_2 \ldots \subset\subset \Omega$ and $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$. Then put

$$\mu_n = \mu\lfloor\Omega_n + \mu_a\lfloor\Omega^c.$$

Note that since $\mu_s(\Omega^c) = 0$ we have that $\mu_n \nearrow \mu$.

Now let $f_n := \mathrm{Bal}(\mu_n, 1)$, $f = \mathrm{Bal}(\mu, 1)$, then by (2) we know that $f_n \nearrow f$ pointwise (a.e.). Also note that since the singular part of μ_n has compact support in Ω and $\mu_n \leq 1$ in a neighborhood of $\partial\Omega$ for each n it follows that $f_n = \mu_n = \mu_a$ on Ω^c for each n. But this means that $f = \mu_a$ on Ω^c as-well, and this concludes the proof. □ □

3. Remarks

• In this article I have decided to consider only partial balayage to density 1, but as in [8] if we instead take $\rho \geq 0$ in $L^\infty(\mathbb{R}^n)$ with $\rho > const. > 0$ in some neighborhood of ∞, then we can find a subharmonic continuous function s in $\mathbb{R}^n$ with $\Delta s = \rho$. Using this we can define the operation $\mathrm{Bal}(\mu, \rho)$ in an equivalent way as for the case of $\rho = 1$, and the proof of the corresponding results as above follows by essentially only replacing 1 by ρ and $|\cdot|^2/2n$ by s at appropriate places.

• In [7] μ was allowed to be a signed measure, and the theorem more or less still holds, but one difference beeing that V will of course not be superharmonic in general, and even worse not continuous, so the statement must necessarily be a bit more complicated. If we assume that the negative part of μ is in $L^\infty(m)$ then the theorem as it stands holds ($\mathrm{Bal}(\mu, 1) \geq 0$ will of course not hold in general). One difference here is that we have the result that $\mathrm{Bal}(\mu, 1) = \chi_\Omega + \mu\lfloor\Omega^c$ without any extra regularity assumptions on μ.

• It is of interest to give some results assuring that $\mathrm{supp}(\mu) \subset \overline{\Omega}$ or $\mathrm{supp}(\mu) \subset \Omega$. To say something we start by noting that if μ is singular with respect to Lebesgue measure, or $\mu \geq C\chi_D$, $\mathrm{supp}(\mu) \subset \overline{D}$ where D is an open set and $C > 1$ is a constant, then trivially $\mathrm{supp}(\mu) \subset \overline{\Omega}$. This is so because $\mu\lfloor\Omega^c \leq 1$.

In the second case, $\mu \geq C\chi_D$ with $C > 1$ and $\mathrm{supp}(\mu) \subset \overline{D}$, and also ∂D satisfies an inner ball condition so that for every $x \in \partial D$ there is an open ball $B \subset D$ with $x \in \overline{B}$, then

$$\mathrm{Bal}(\mu, 1) \geq \mathrm{Bal}(C\chi_D, 1) \geq \mathrm{Bal}(C\chi_B, 1) = \chi_{\widetilde{B}},$$

where $\widetilde{B}$ is the ball with the same center as B and such that $m(\widetilde{B}) = Cm(B)$, which follows from the mean value property of harmonic functions. Since $x \in \widetilde{B}$ it follows that $\overline{D} \subset \Omega$, and therefore by assumption $\mathrm{supp}(\mu) \subset \Omega$.

• It is known that if $\mathrm{supp}(\mu) \subset \Omega$ then $m(\partial\Omega) = 0$. This need not be the case in general however. To see this, let D denote an open set in $\mathbb{R}^n$ with $m(\partial D) > 0$. Let

$$\Omega_t := \left\{ U^{t\chi_D} > U^{\mathrm{Bal}(t\chi_D, 1)} \right\},$$

where $t > 1$. Then

$$\mathrm{Bal}(t\chi_D, 1) = \chi_{\Omega_t},$$

and hence

$$m(\Omega_t) = tm(D).$$

So for small $t > 1$ we have

$$tm(D) < m(D) + m(\partial D),$$

and for these t it follows that

$$m(\partial\Omega_t) > 0,$$

because

$$\partial\Omega_t \supset \partial D \setminus \Omega_t.$$

It might be worth noting that we however always have

$$\nu_x^D(\Omega_t^c) = 0 \quad \forall t > 1,$$

where ν_x^D is the harmonic measure on ∂D with respect to $x \in D$. To prove this we let

$$\mu := \mathrm{Bal}(\chi_D, \partial D)$$

be the classical balayage of χ_D onto ∂D. Then (we assume that D is connected, the general case needs some trivial modifications) ν_x^D and μ are mutually absolutely continuous, and it is known that ν_x^D is singular with respect to Lebesgue measure (see [3]). Hence μ is singular with respect to Lebesgue measure. Let

$$\eta_\varepsilon := (1-\varepsilon)\chi_D + \varepsilon\mu.$$

Then we have $U^{\eta_\varepsilon} \leq U^{\chi_D}$ everywhere for every $\varepsilon > 0$. Also define

$$\Omega_{t,\varepsilon} = \left(U^{t\eta_\varepsilon} > U^{\mathrm{Bal}(t\eta_\varepsilon,1)}\right),$$

then we must have $\mathrm{Bal}(t\eta_\varepsilon) = \chi_{\Omega_{t,\varepsilon}}$ if $t(1-\varepsilon) > 1$, because then $D \subset \Omega_{t,\varepsilon}$, and since $t\varepsilon\mu \perp m$ we also have $t\varepsilon\mu(\Omega_{t,\varepsilon}^c) = 0$.

Since $m(\Omega_{t,\varepsilon}) = tm(D)$, and

$$U^{\chi_{\Omega_{t,\varepsilon}}} = tU^{\chi_D} \text{ on } \Omega_{t,\varepsilon}^c,$$

$$U^{\chi_{\Omega_{t,\varepsilon}}} \leq tU^{\chi_D} \text{ everywhere},$$

it actually follows that

$$\Omega_{t,\varepsilon} = \Omega_{t,0} = \Omega_t$$

for t such that $t(1-\varepsilon) > 1$. Hence we have proved that $\mu(\Omega_t^c) = 0$ for every $t > 1$, and this implies that $\nu_x^D(\Omega_t^c) = 0$ for every $t > 1$.

4. Probabilistic part

We again start by introducing some notation. All the preliminaries needed may be found in [2] or [10] for instance. Now we will let $\mu \geq 0$ be a function in $L^\infty(m)$ with compact support in $\mathbb{R}^n$. (P^x, X_t) will denote a standard Brownian motion with respect to a filtration $\mathcal{F}_t$ satisfying the usual conditions (right continuity and such that $\mathcal{F}_t$ contains every set N such that for every x $P^x(N) = 0$) and $P^x\{X_0 = x\} = 1$.

Let θ_b denote the set of bounded stopping times with respect to $\mathcal{F}_t$. For every $\tau \in \theta_b$ we define

$$U_\tau(x) := E^x\left(-\tau + \int_0^\tau \mu(X_s)ds\right).$$

We also put

$$u(x) := \frac{1}{2} \sup_{\tau \in \theta_b} U_\tau(x).$$

$(U_0(x) \equiv 0 \Rightarrow u \geq 0.)$
Recall that Itô's formula easily leads to that for $v \in C^2(\mathbb{R}^n)$ and $\tau \in \theta_b$ we have

$$v(x) = E^x(v(X_\tau)) + \frac{1}{2} E^x \left(\int_0^\tau -\Delta v(X_s) ds \right).$$

So especially if $f \in C_c^1(\mathbb{R}^n)$ for instance, then we have

$$U^f(x) = E^x(U^f(X_\tau)) + \frac{1}{2} E^x \left(\int_0^\tau f(X_s) ds \right). \tag{2}$$

But it is easy to see by an approximation argument that this must then hold for all $f \in L^\infty(m)$ with compact support.

Theorem 4.1. *Let*

$$u(x) = \frac{1}{2} U_{\tau_\Omega}(x),$$

where $\Omega = \{u > 0\}$, *and* $\tau_\Omega = \inf \{t > 0 : X_t \notin \Omega\}$. *Then we have*

$$u = U^\mu - U^\eta,$$

where $\eta = \mathrm{Bal}(\mu, 1)$.

That is, the optimal stopping problem given by:
Find $\tau_0 \in \theta_b$ *such that*

$$\frac{1}{2} \sup_{\tau \in \theta_b} U_\tau(x) = \frac{1}{2} U_{\tau_0}(x),$$

has a solution $\tau_0 = \tau_\Omega$. *Furthermore we have*

$$\mathrm{Bal}(\mu, 1) = \Delta \frac{1}{2} U_{\tau_\Omega}(x) + \mu.$$

Proof. Let

$$\widetilde{u} := U^\mu - U^\eta, \quad \eta = \mathrm{Bal}(\mu, 1).$$

We know that

$$\eta = \chi_D + \mu \lfloor D^c,$$

where $D = \{\widetilde{u} > 0\}$. Formula (2) gives that

$$\widetilde{u}(x) = E^x(\widetilde{u}(X_\tau)) + \frac{1}{2} E^x \left(\int_0^\tau ((\mu - 1) \lfloor D)(X_s) ds \right) \quad \forall \tau \in \theta_b.$$

So

$$\widetilde{u}(x) \geq \frac{1}{2} E^x \left(\int_0^\tau ((\mu - 1) \lfloor D)(X_s) ds \right) \geq \{\mu \leq 1 \text{ on } D^c\} \geq$$
$$\frac{1}{2} E^x \left(\int_0^\tau (\mu - 1)(X_s) ds \right) = \frac{1}{2} E^x \left(-\tau + \int_0^\tau \mu(X_s) ds \right) =$$
$$\frac{1}{2} U_\tau(x) \quad \forall \tau \in \theta_b.$$

So $\widetilde{u} \geq u$, but also by the above, since $\widetilde{u} = \frac{1}{2} U_{\tau_D}(x)$ we have that $\widetilde{u} \equiv u$. Hence $D = \Omega$ and the theorem is proved. □ □

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