Quadrature domains for harmonic functions

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ABSTRACT. This paper establishes a conjecture of Gustafsson, Sakai and Shapiro by showing that any quadrature domain (for harmonic functions) with respect to a signed measure is also a quadrature domain with respect to a positive measure.

1. Introduction

A bounded domain Ω in Euclidean space $\mathbb{R}^N$ $(N \geq 2)$ is called a *quadrature domain (for harmonic functions)* if there is a signed (Borel) measure μ, with compact support in Ω, such that

$$\int_{\Omega} h(x)dx = \int h\, d\mu \quad \text{for every integrable harmonic function } h \text{ on } \Omega. \tag{1.1}$$

We then say that Ω is a quadrature domain *with respect to* μ. A simple example of a quadrature domain is an open ball B; for (1.1) holds, with $\Omega = B$, if we choose μ to be the mass $\text{vol}(B)$ concentrated at the centre of B. However, there are many other possible choices of signed measure μ here, and many examples of quadrature domains other than balls. For example, (1.1) holds for an ellipse if we choose μ to be a suitably weighted form of one-dimensional measure on the line segment joining its foci (see, for example, [**7**]). Quadrature domains arise naturally, not only in potential theory, but also in several other areas of mathematical science; one such example is the Hele-Shaw moving boundary problem. A good indication of current research activity on the topic may be found in [**3**].

A significant question concerning the definition of quadrature domains is the following.

Problem If Ω is a quadrature domain with respect to a signed measure μ, is it also a quadrature domain with respect to some positive measure?

In the case where $N = 2$, Gustafsson, Sakai and Shapiro [**4**] showed that the answer is "yes"; their approach used conformal mappings together with deep results of Sakai [**5**],[**6**], concerning the geometrical classification of plane quadrature domains, which show that any such quadrature domain contains a finitely connected quadrature domain. This approach is clearly not available in higher dimensions. It was nevertheless conjectured in [**4**] that the answer is "yes" in any dimension.

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The purpose of this paper is to verify that conjecture. In fact, we will do this by establishing a stronger result, of independent interest, concerning the integrability of positive harmonic functions on quadrature domains. As is well known, positive harmonic functions need not be integrable on arbitrary bounded domains: for example, the harmonic function $h(x_1, x_2) = x_1 x_2 (x_1^2 + x_2^2)^{-2}$ is clearly not integrable on the square $(0, 1)^2$. By contrast, we will show that:

THEOREM 1. *Every positive harmonic function on a quadrature domain Ω is integrable on Ω.*

From this we will deduce, without difficulty (cf. [**4**], [**8**]), the desired result.

COROLLARY 1. *If Ω is a quadrature domain with respect to a signed measure μ with compact support in Ω, then it is also a quadrature domain with respect to some positive measure with compact support in Ω.*

Theorem 1 will be proved by a new approach which uses a key result concerning the regularity of free boundaries in conjunction with the potential theoretic notion of the Martin boundary. Even in the case $N = 2$, where the above results are known, our argument is more direct than that in [**4**].

2. Proof of Theorem 1

2.1. Let $B_R(z)$ denote the open ball in $\mathbb{R}^N$ of centre z and radius R, and let χ_A denote the characteristic function of a set $A \subseteq \mathbb{R}^N$. We will say that a subharmonic function v on $B_R(z)$ belongs to the class $P_R(z, M)$ if there is an open set $D \subset B_R(z)$ such that $z \in \partial D$ and

(i) $\Delta v = \chi_D$ in the sense of distributions (whence $v \in C^1(B_R(z))$),
(ii) $|v| \leq M$, and
(iii) $v = |\nabla v| = 0$ on $B_R(z) \backslash D$.

The following important estimate for functions in $P_1(z, M)$ is due to Caffarelli, Karp and Shahgholian (see Theorem I, or (3.11), in [**2**]).

Theorem A *There is a constant C_0, depending only on N, such that, if $u \in P_1(z, M)$, then $|u(x)| \leq C_0 M |x - z|^2$ on $B_1(z)$.*

We denote by $U\mu$ the Newtonian (if $N \geq 3$) or logarithmic (if $N = 2$) potential of a signed measure μ with compact support, normalized so that $-\Delta U\mu = \mu$ in the sense of distributions. In the case where $d\mu$ can be written as $f dx$ for some measurable function f we will also use the notation Uf.

2.2. We will now use Theorem A to establish the following property of quadrature domains.

LEMMA 1. *Let Ω be a quadrature domain with respect to a signed measure μ, let $u = U\mu - U\chi_\Omega$ and $d_\Omega(x) = \mathrm{dist}(x, \mathbb{R}^N \backslash \Omega)$. Then there is a constant $C_1 \in (0,1)$, depending only on Ω and μ, such that*

$$B_{C_1 d_\Omega(x)}(x) \cap \{u > 0\} \neq \emptyset \qquad (x \in \Omega).$$

PROOF. Let

$$R_0 = 2^{-1}\mathrm{dist}\{\mathrm{supp}\mu, \partial\Omega\} \quad \text{and} \quad M_0 = \sup\{|u(x)| : d_\Omega(x) \leq R_0\}.$$

Clearly $u \in C^1(\mathbb{R}^N \backslash \mathrm{supp}\mu)$. Also, by considering (1.1) for Newtonian (or logarithmic) kernels with poles outside Ω, and for their first order partial derivatives, we see that $u = |\nabla u| = 0$ on $\mathbb{R}^N \backslash \Omega$. Hence $u \in P_{R_0}(y, M_0)$ for every $y \in \partial\Omega$. For every $y \in \partial\Omega$, the function $x \mapsto R_0^{-2} u(y + R_0 x)$ belongs to $P_1(0, M_0 R_0^{-2})$, so it follows from Theorem A that

$$|u(x)| \leq C_0 M_0 R_0^{-2} \{d_\Omega(x)\}^2 \qquad (d_\Omega(x) \leq R_0). \tag{2.1}$$

Suppose now, for the sake of contradiction, that there is no constant C_1 as in the statement of the lemma. Then there exist sequences (x_n) in Ω and (c_n) in $(0,1)$ such that $c_n \to 1$ and

$$B_{c_n d_\Omega(x_n)}(x_n) \subset \{u \leq 0\} \qquad (n \in \mathbb{N}).$$

By redefining (x_n), if necessary, we can easily arrange that $d_\Omega(x_n) < R_0/2$ for all n. For each n we can choose $y_n \in \partial B_{d_\Omega(x_n)}(x_n) \cap \partial\Omega$.

Let $z_n = (x_n - y_n)/|x_n - y_n|$ and

$$v_n(x) = \frac{u(d_\Omega(x_n)x + y_n)}{\{d_\Omega(x_n)\}^2} \qquad (x \in \mathbb{R}^N).$$

Then $\Delta v_n = 1$ and $|v_n(x)| \leq C_0 M_0 R_0^{-2} |x|^2$ on $B_1(z_n)$, by (2.1). By passing to a suitable subsequence, we can arrange that (z_n) converges to some point $z_0 \in \partial B_1(0)$ and (v_n) converges locally uniformly on $B_1(z_0)$ to a function v such that

$$\Delta v = 1 \quad \text{and} \quad |v(x)| \leq C_0 M_0 R_0^{-2} |x|^2 \quad \text{on} \quad B_1(z_0). \tag{2.2}$$

Since $u \leq 0$ on $B_{c_n d_\Omega(x_n)}(x_n)$ for each n, we see that $v_n \leq 0$ on $B_{c_n}(z_n)$ for each n, and hence $v \leq 0$ on $B_1(z_0)$. It follows from the maximum principle that

$$v(x) - \frac{|x - z_0|^2 - 1}{2N} \leq 0 \qquad (x \in B_1(z_0)),$$

whence

$$v(tz_0) \leq \frac{-t}{2N} \qquad (0 < t < 1).$$

This yields the desired contradiction, in view of (2.2). □

2.3. In order to complete the proof of Theorem 1, we will use the notion of the Martin boundary Δ of Ω, an account of which may be found in Chapter 8 of [**1**]. We denote by M the Martin kernel defined by

$$M(x, y) = \lim_{z \to y} \frac{G_\Omega(x, z)}{G_\Omega(x_0, z)} \qquad (x \in \Omega, y \in \Delta),$$

where G_Ω is the Green kernel for Ω and x_0 is some fixed reference point in Ω.

Now let Ω be a quadrature domain with respect to a signed measure μ, and let $u = U\mu - U\chi_\Omega$. Since $u = 0$ on $\partial\Omega$ we can rewrite u as a difference of two Green potentials, namely $u = G_\Omega\mu - G_\Omega\chi_\Omega$. Let y be any point of Δ and choose a sequence (z_n) in Ω that converges to y. By Lemma 1 there is a positive constant C_1, and a sequence (w_n) of points such that $|z_n - w_n| < C_1 d_\Omega(z_n)$ and $u(w_n) > 0$. By passing to a subsequence, if necessary, we may assume that (w_n) also converges, to some point y' of Δ. It now follows from Harnack's inequalities, and then Fatou's lemma, that there are positive constants C_2 (depending only on N and C_1) and C_3 (depending only on suppμ, x_0 and Ω) such that

$$\begin{aligned}
\int_\Omega M(x,y)dx &= \int_\Omega \lim_{n\to\infty} \frac{G_\Omega(x,z_n)}{G_\Omega(x_0,z_n)}dx \\
&\leq C_2 \int_\Omega \lim_{n\to\infty} \frac{G_\Omega(x,w_n)}{G_\Omega(x_0,w_n)}dx \\
&\leq C_2 \liminf_{n\to\infty} \frac{G_\Omega\chi_\Omega(w_n)}{G_\Omega(x_0,w_n)} \\
&= C_2 \liminf_{n\to\infty} \frac{G_\Omega\mu(w_n) - u(w_n)}{G_\Omega(x_0,w_n)} \\
&\leq C_2 \liminf_{n\to\infty} \int_\Omega \frac{G_\Omega(x,w_n)}{G_\Omega(x_0,w_n)} d\mu(x) \\
&= C_2 \int M(x,y')d\mu(x) \\
&\leq C_2 C_3 \|\mu\|.
\end{aligned}$$

Since any positive harmonic function h on Ω can be expressed in the form

$$h(x) = \int_\Delta M(x,y)d\nu(y) \qquad (x \in \Omega),$$

where ν is a measure on Δ (see Corollary 8.1.12 of [**1**] for this elementary representation), we conclude from Tonelli's theorem that

$$\begin{aligned}
\int_\Omega h(x)dx &= \int_\Delta \int_\Omega M(x,y)dx\, d\nu(y) \\
&\leq C_2C_3 \|\mu\| \|\nu\| = C_2C_3 \|\mu\| h(x_0),
\end{aligned}$$

and Theorem 1 is now established.

2.4. It remains to deduce Corollary 1. Let Ω be a quadrature domain with respect to a signed measure μ, let (Ω_n) be an exhaustion of Ω by open sets such that supp$\mu \subset \Omega_1$ and $\overline{\Omega}_n \subset \Omega_{n+1}$ for each n, and let $x_0 \in \Omega_1$. We claim that, for some n_0,

$$\int h\, d\mu > 0 \quad \text{for every positive harmonic function } h \text{ on } \Omega_{n_0}. \tag{2.3}$$

To see this, suppose otherwise. Then, for each n, we can choose a positive harmonic function h_n on Ω_n such that $\int h_n d\mu \leq 0$ and, in addition, $h_n(x_0) = 1$. In view of Harnack's convergence theorem we could then find a subsequence of (h_n) that converges locally uniformly on Ω to a positive harmonic function h. By Theorem 1, the function h must be integrable on Ω, and so we arrive at the contradictory conclusion that $0 \geq \int h d\mu = \int h(x)dx > 0$.

We now define

$$L(f) = \int h_f d\mu \qquad (f \in C(\partial\Omega_{n_0})),$$

where h_f is the solution to the Dirichlet problem on Ω_{n_0} with boundary data f. From (2.3) we see that L is a positive linear functional on $C(\partial\Omega_{n_0})$. Thus, by the Riesz representation theorem, there is a (positive) measure λ on $\partial\Omega_{n_0}$ such that $\int h_f d\mu = \int f d\lambda$ for all $f \in C(\partial\Omega_{n_0})$. In particular, $\int h(x)dx = \int h d\mu = \int h d\lambda$ for every integrable harmonic function h on Ω, as required.

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