Linear Optimization

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Overview

1. Egdes, One-Dimensional Faces, Adjacency of Extreme Points, Extreme Directions
2. Adjacency in a Primal Simplex Pivot Step
3. Optimality Criterion in the Primal Simplex Algorithm
4. Boundedness of Convex Polyhedra

Egdes, One-Dimensional Faces, Adjacency of Extreme Points, Extreme Directions

- Every One dimensional face or edge of a convex polyhedron either has 2, or 1 or no extreme points. We can classify into 3 cases
    (1) The edge is a line segment joining two extreme points which are said to be adjacency pair of extreme points. Are called bounded edges.
    (2) The edge is a half-line called extreme half-line or unbounded edge beginning at the extreme point on it. Its direction is said to be an extreme direction for the convex polyhedron.
    (3) The edge is an entire straight line.
- In LP we mostly meet the first two cases

Egdes, One-Dimensional Faces, Adjacency of Extreme Points, Extreme Directions Cont..

Definition

Geometric Definition of Adjacency: Two Extreme points x^1,x^2 of a convex polyhedron $K \subset R^n$ are adjacent if every point $\overline{x}$ on the line segment joining them satisfies If any pair of points x^3,$x^4 \in K$ satisfy $\overline{x} = \alpha x^3 + (1 - \alpha)x^4$ for some $0 < \alpha < 1$, then x^3 and x^4 both lies also on the line joining x^1 and x^2

Egdes, One-Dimensional Faces, Adjacency of Extreme Points, Extreme Directions Cont..

Definition

Algebraic Definition of Adjacency: Let K be the set of feasible solutions of

$$\begin{aligned} Dx &= d \\ Fx &\geq g \end{aligned} \tag{1}$$

where the inequality constraints include all the bound restrictions on individual variables, if any.

- Let x^1,x^2 be the two BFS of 1, and let $\overline{x}$ be an interior point with property $\alpha x^1 + (1-\alpha)x^2$ for some $0 < \alpha < 1$, eg $\overline{x} = (x^1 + x^2)/2$
- Let (S) denotes the active system at $\overline{x}$ that the system of all active constraints in 1 at $\overline{x}$ treated as a system of equations.
- x^1,x^2 are adjacent extreme points of K iff the set of all solutions of (S) is one dimensional, that is a straight line joining x^1, x^2.

Egdes, One-Dimensional Faces, Adjacency of Extreme Points, Extreme Directions Cont..

Definition

Algebraic Definition of Adjacency for System in Standard Form: Let K be the set of feasible solutions of the system in standard form

$$\begin{aligned} Ax &= b \\ x &\geq 0 \end{aligned} \tag{2}$$

Let x^1,x^2 be two BFS of 2. Let $\overline{x}$ be some points in the interior of the line segment joining x^1, x^2 (i.e a point of the fomr $\alpha x^1 + (1-\alpha)x^2$ for some $0 < \alpha < 1$, eg $\overline{x} = (x^1 + x^2)/2$). Then x^1,x^2 are adjacent on K iff the set of column vectors $\{A_{.j} : J \text{ such that } \overline{x}_j > 0\}$ is one less than its cardinality

Example 1

How to check if a given Feasible solution is on an edge:
For the following system, is the feasible solution $\bar{x} = (5, 10, 3, 0, 0)$ on an edge?

x_1	x_2	x_3	x_4	x_5	b
1	-1	-1	1	-2	-8
0	1	2	3	8	16
-1	1	1	8	-9	8

Table: $x_j \geq 0$ for all j

The index set of positive components in $\bar{x}$ is $J = \{1, 2, 3\}$
We need to find the rank of $\{A_{.1}, A_{.2}, A_{.3}\}$

Example 1 Cont..

Memory matrix

$A_{.1}$	$A_{.2}$	$A_{.3}$				
			PC			
1	0	0	$\boxed{1}$	0	−1	PR
0	1	0	−1	1	1	
0	0	1	−1	2	1	
				PC		
1	0	0	1	0	−1	
1	1	0	0	$\boxed{1}$	0	PR
1	0	1	0	2	0	
1	0	0	1	0	−1	
1	1	0	0	1	0	
−1	−2	1	0	0	0	

$$-A_{.1} - 2A_{.2} + A_{.3} = 0$$
$$5A_{.1} + 10A_{.2} + 3A_{.3} = b$$

Example 1 Cont..

- $$\begin{aligned} -A_{.1} - 2A_{.2} + A_{.3} &= 0 \\ 5A_{.1} + 10A_{.2} + 3A_{.3} &= b \\ (5-\lambda)A_{.1} + (10-2\lambda)A_{.2} + (3+\lambda)A_{.3} &= b \end{aligned}$$
- Therefore $x(\lambda) = (5-\lambda, 10-2\lambda, 3+\lambda, 0, 0)^T$ satisfies equality constraints in the system $x(\lambda) \geq 0$ and is feasible to the system for all $\theta_1 = -3 \leq \lambda \leq 5 = \theta_2$
- $\overline{x}$ is obtained by fixing $\lambda = 0$
- The points $x(-3) = (8, 16, 0, 0, 0)^T$ and $x(5) = (0, 0, 8, 0, 0)^T$ are two extreme points on this edge, so the edge is bounded, which is the line joining points $x(-3)$ and $x(5)$

Adjacency in a Primal Simplex Pivot Step

- Consider a primal method of LP in standard form

$$\begin{aligned} \text{Minimize } &= cx \\ \text{subject to } \; Ax &= b \\ x &\geq 0 \end{aligned}$$

where A is a matrix of orders mxn and rank m, and $b \neq 0$.

- In each pivot step, the primal algorithm enters a single nonbasic variables into the present feasible basic vector.
- We start by assuming $x_B = (x_1, ..., x_m)$ and the entering variable is x_{m+1}, so the values $x_{m+2}, ..., x_n$ will remain 0

Adjacency in a Primal Simplex Pivot Step..

Canonical tableau wrt $x_B = (x_1, \ldots, x_m)$

BV	Basic x_1	$\ldots$	x_m	PC x_{m+1}	Other nonbasics	$-z$	Updated RHS
x_1	1	$\ldots$	0	$\bar{a}_{1,m+1}$	$\ldots$	0	$\bar{b}_1$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
x_m	0	$\ldots$	1	$\bar{a}_{m,m+1}$	$\ldots$	0	$\bar{b}_m$
$-z$	0	$\ldots$	0	$\bar{c}_{m+1}$	$\ldots$	1	$-\bar{z}$

$x_j \geq 0$ for all j, min z

"BV" is the abbreviation for "basic variable in this row." So, the present BFS is $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_m, \bar{x}_{m+1}, \ldots, \bar{x}_n)^T = (\bar{b}_1, \ldots, \bar{b}_m, 0, \ldots, 0)^T$ with objective value $= \bar{z}$.

We discuss some different cases

CASE 1:$(\overline{a}_{1,m+1}, ..., \overline{a}_{m,m+1}) \leq 0$. so there are no ratios to compute in the minimum ratio sub step in this primal simplex pivot step.

- All entries in the PC are ≤ 0, $x(\lambda)$ remains ≥ 0 for all $\lambda > 0$: so $x(\lambda)$ is feasible to this problem for all $\lambda \geq 0$ as $\lambda \to \infty$, $z(\lambda) \to -\infty$, unboundness termination criterion has been satisfied.
- The direction of unboundness is $y = (-\overline{a}_{1,m+1} ..., -\overline{a}_{1,m+1}, 1, 0, ..., 0)^T$
- It satisfy $Ay = 0$, $y \geq 0$, and $cy = \overline{c}_{m+1} < 0$.

Example 2

Consider the following canonical tableau for an LP wrt the basic vector x_1, x_2 in which nonbasic x_3 has been selected as the entering variable in the primal simplex algorithm. $m = 2$ and $x_{m+1} = x_3$

BV	x_1	x_2	x_3	Other nonbasics	$-z$	Updated RHS
			PC			
x_1	1	0	-2	...	0	0
x_2	0	1	-1	...	0	3
$-z$	0	0	-3	...	1	-10

$x_j \geq 0$ for all j, min z

Example 2 Cont..

- The present BFS is $\overline{x} = (0, 3, 0, ..., 0)^T$ with objective $\overline{z} = 10$. PC has no positive entry, so unboundness termination criterion
- Unboundness edge generated
  $\{x(\lambda) = (\overline{b}_1 - \overline{a}_{1,m+1}, ..., \overline{b}_m - \overline{a}_{m,m+1}, \lambda, \lambda, 0, ..., 0)^T\}$
  $\{x(\lambda) = (0 + 2\lambda, 3 + \lambda, \lambda, \lambda, .., 0)^T : \lambda \geq 0\}$
- its Direction of unboundness
  $y = (2, 1, 1, 0, ..., 0)^T$ satisfies $c\overline{y} = \overline{c}_3 = -3 < 0$
  The objective value is $z(\lambda) = 10 - 3\lambda \to \infty$ as λ increases along this unboundness edge

Case 2

The PC has at least one positive entry and it has positive entry in at least one row i in which $b_i = 0$

- For $t = 1$ to m, the value of the basis variable x_t in $x(\lambda)$ is $\overline{b} + \lambda \overline{a}_{t,m+1}$ and to keep it non-negative, we need
    (a) $\lambda \leq (\overline{b}_t / \overline{a}_{t,m+1})$ for all t such that $\overline{a}_{t,m+1} > 0$
    (b) The quantity $(\overline{b}_t / \overline{a}_{t,m+1})$ is called the ratio in this primal simplex pivot step in rwo t. is computed only if $\overline{a}_{t,m+1}$ is positive.
    (c) The maximum value of λ when PC has at least one positive is $\theta = \text{minimum}\{\overline{b}_t / \overline{a}_{t,m+1}$ over t such that $\overline{a}_{t,m+1} > 0\}$
- θ is called minimum ratio, so $x(\lambda)$ is feasible to the problem for all $0 \leq \lambda \leq \theta$

When we do not obtain a new feasible solution to the problem but remain the same with minimum ratio 0, is called DEGENERATE PIVOT STEP

BV	x_1	x_2	x_3	Other nonbasics	$-z$	Updated RHS	Ratio
			PC				
x_1	1	0	-2	...	0	1	
x_2	0	1	$\boxed{1}$	...	0	0	0 PR
$-z$	0	0	-3	...	1	-10	$\theta = 0$
x_1	1	2	0	...	0	1	
x_3	0	1	1	...	0	0	
$-z$	0	3	0	...	1	-10	

$x_j \geq 0$ for all j, min z

PC has positive entry in row 2 with RHS 0. The ratio is 0, the minimum ratio θ is 0, and is degenerate step. No change of solution.

Optimality Criterion in the Primal Simplex Algorithm

Consider the LP in standard form

$$\begin{aligned} \text{Minimize} z(x) &= cx \\ \text{subject to} \quad Ax &= b \\ x &\geq 0 \end{aligned} \tag{3}$$

where A is a matrix of order $m \times n$ and the rank m. $c \in R^n$ is the row vector of original cost coefficients of the variables $x = (x_1, ..., x_n)^T$

Optimality Criterion in the Primal Simplex Algorithm Cont..

Let $\overline{x}$ be a BFS for this LP associated with the basic vector x_B, with objective value $z(\overline{x})$. For notational convenience, we assume that $x_B = (x_1, ..., x_m)$

Canonical tableau wrt $x_B = (x_1, \ldots, x_m)$

BV	PC							Updated RHS
	x_1	$\ldots$	x_m	x_{m+1}		x_n	$-z$	
x_1	1	$\ldots$	0	$\bar{a}_{1,m+1}$	$\ldots$	$\bar{a}_{1n}$	0	$\bar{b}_1$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$
x_m	0	$\ldots$	1	$\bar{a}_{m,m+1}$	$\ldots$	$\bar{a}_{mn}$	0	$\bar{b}_m$
$-z$	0	$\ldots$	0	$\bar{c}_{m+1}$	$\ldots$	$\bar{c}_n$	1	$\bar{z}$

$x_j \geq 0$ for all j, min z

BV is basic variables, BFS is
$\overline{x} = (\overline{x}_1, ..., \overline{x}_m, \overline{x}_{m+1}, ..., \overline{x}_n)^T = (\overline{b}_1, ..., \overline{b}_m, 0, ..., 0)^T$

Optimality Criterion in the Primal Simplex Algorithm Cont..

Theorem 1

In the canonical tableau wrt the feasible basic vector x_B, iff all the relative cost coefficients $\overline{c}_j \geq 0$ for all j, then the present BFS $\overline{x}$ is an optimum solution for this LP

Theorem 2

Starting with a feasible basic vector for an LP 3, the primal simplex algorithm (with techniques for solving cycling as necessary) terminates with either an optimum basic feasible solution or with an extreme half-line of the set of feasible solutions along which the objective function diverges to $-\infty$, after a finite number of pivot steps

Optimality Criterion in the Primal Simplex Algorithm Cont..

Theorem 3

When the primal simplex method is applied to solve an LP in which the objective function is to be minimized, after a finite number of pivot steps, it terminates with one of three outcomes: (1) conclusion that the LP is infeasible, (2) with an extreme half-line in the set of feasible solutions, along which the objective function diverges to $-\infty$, (3) with an optimum basic feasible solution of the problem.

Boundedness of Convex Polyhedra

Given

$$\begin{aligned} Dx &= d \\ Fx &\geq g \end{aligned} \tag{4}$$

- Let K denote the set of feasible solutions of the general system of linear constraints 4 in $x = (x_j : j = 1, .., n)$ where the inequalities include all the bounds constraints on individual variables if any.
- An important mathematical question is to determine whether K is bounded or not
- That is to develop conditions under which we can conclude that is is bounded or not.
- This leads to a system known as homogeneous system corresponding to 4

Boundedness of Convex Polyhedra Cont..

- The simple rules for obtaining the homogeneous system are
  (a) Change the variables x in 4 into y.
  (b) Change every RHS constants in every constraint (including any bounds on individual variables) into 0
  (c) The result is the homogeneous system, which is

$$\begin{aligned} Dy &= 0 \\ Fy &\geq 0 \end{aligned} \tag{5}$$

- The homogeneous system is always feasible because 0 is a feasible solution for it.
- The important thing is to determine whether it has a nonzero feasible solution

Boundedness of Convex Polyhedra Cont..

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f $\overline{x}$ is a feasible solution of the original system 4 and $\overline{y}$ is a nonzero feasible solution of the homogeneous system 5 corresponding to it, it can be verified directly that $\overline{x} + \lambda\overline{y}$ is also feasible to 4 for all $\lambda \geq 0$. Thus every nonzero feasible solution of the homogeneous system 5 is the direction for a feasible half-line for 4 at every feasible solution of 4.

FundamentalTheoremaboutUnboundednessofConvexPolyhedra:

When K is the set of feasible solutions of 4 is nonempty, it is bounded iff 0 is the only feasible solution of the homogeneous system 5 associated with it.

If the LP in standard form

$$\begin{aligned} \text{Minimize} \quad z(x) &= cx \\ Ax &= b \\ x &\geq 0 \end{aligned}$$

has a feasible solution, then the objective function z is unbounded below in it iff there exists an extreme homogeneous solution y satisfying $cy < 0$.

The End