Duality Theory and Optimality Conditions for LPs

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The Primal, Dual Pair LPs

Primal (**P**)	Dual (**D**)
$$\begin{aligned} \text{maximize} \quad z(x) &= cx \\ \text{s. to} \quad Ax &\le b \\ \mathbb{R}^n \ni x &\ge 0 \end{aligned}$$	$$\begin{aligned} \text{minimize} \quad \nu(\pi) &= \pi b \\ \text{s. to} \quad \pi A &\ge c \\ \mathbb{R}^m \ni \pi &\ge 0 \end{aligned}$$
Data: $A \in \mathbb{R}^{m \times n}$; $c \in \mathbb{R}^n$; $b \in \mathbb{R}^m$	**Data:** $A^T \in \mathbb{R}^{n \times m}$; $b \in \mathbb{R}^m$; $c \in \mathbb{R}^n$

- Given any LP, there is another related LP, called the **dual**.
- Involve a different set of variables, but share the same data.
- The original LP is called the **primal**. Together, the two are referred to as a primal, dual pair of LPs.

The Dual Problem

Theorem

Gordons Theorem: *The system* $Ax < 0$ *has a solution iff the alternate system* $yA = 0,\ y \geq 0$ *has no nonzero solution.*

Theorem

Farkas Lemma: *The system* $Ax = b, x \geq 0$ *has a feasible solution iff the alternate system* $yA \leq 0; yb > 0$ *has no feasible solution.*

Example with Rational Economic Arguments

- The fertilizer manufacturer (**Primal**)

$$
\begin{array}{llcrl}
\text{Maximize } z(x) = & 15x_1 & + & 10x_2 & & \text{Item} \\
\text{S. to} & 2x_1 & + & x_2 \leq & 1500 & \text{RM 1} \\
 & x_1 & + & x_2 \leq & 1200 & \text{RM 2} \\
 & x_1 & & \leq & 500 & \text{RM 3} \\
 & x_1 \geq 0, & & x_2 \geq 0, & &
\end{array}
$$

  where the decision variables are
  - $x_1 =$ the tons of Hi-ph made per day.
  - $x_2 =$ the tons of Lo-ph made per day.
- Detergent company (**Dual**)
  - A detergent company in the area needs supplies of RM 1, 2, and 3.
  - The detergent manufacturer wants to persuade the fertilizer manufacturer to give up fertilizer making, and instead sell RM 1, 2, 3 to them at $\pi_i = \textbf{price/ton for RMi}, \ i = 1, 2, 3$.

Example with Rational Economic Arguments

- How can we make the deal acceptable to both? **Solve the dual LP:**

$$\begin{array}{rrrrrr}
\text{Minimize } v(\pi) = & 1500\pi_1 + & 1200\pi_2 + & 500\pi_3 & & \\
\text{s. to} & 2\pi_1 + & \pi_2 + & \pi_3 & \geq & 15 \\
 & \pi_1 + & \pi_2 & & \geq & 10 \\
 & \pi_1, & \pi_2, & \pi_3 & \geq & 0 \\
 & & \text{all } \pi_i \geq 0. & & &
\end{array}$$

- the fertilizer manufacturer will not find the price vector (π_1, π_2, π_3) acceptable unless $2\pi_1 + \pi_2 + \pi_3 \geq 15$ (for x_1).
- the detergent company needs to minimize the total cost $1500\pi_1 + 1200\pi_2 + 500\pi_3$ of acquiring the RMi's.
- **Dual Variables** are **Marginal Values**. Detergent manufacturer wants to make it the smallest value that will be acceptable to the fertilizer manufacturer.

Rules for Writing the Dual of a General LP

- The **right** and **wrong** types of inequalities for an LP!
  - **Right:** $(\text{maximize}; \quad \leq)$; $(\text{minimize}; \quad \geq)$.

Primal (**P**)	Dual (**D**)
Objective function (**maximize**) Objective function (**minimize**)	Objective function (**minimize**) Objective function (**maximize**)
Constraint $(=)$ Constraint (**Right**) Constraint (**Wrong**)	Variable (**unrestricted**) Variable $(\geq)$ Variable $(\leq)$
Variable (**unrestricted**) Variable $(\geq)$ Variable $(\leq)$	Constraint $(=)$ Constraint (**Right**) Constraint (**Wrong**)

Table: The rules

Example

$$\text{minimize} \quad z(x) = -3x_1 - 4x_2 + 5x_3 - 6x_5 + 7x_6$$

s. to

$$x_1 + x_2 - x_3 - 2x_4 + 3x_6 = -17$$

$$-x_1 - x_2 + 2x_3 - 4x_4 + 6x_5 - 3x_6 \geq -18$$

$$2x_1 - 3x_2 + 3x_3 - x_4 + 4x_5 \leq 40$$

$$x_2 \geq 0,\ x_3 \leq 0,\ 2 \leq x_4 \leq 15,\ 0 \leq x_5 \leq 6, \quad x_6 \geq 3,$$

$$x_1 \text{ unrestricted}$$

x_1	x_2	x_3	x_4	x_5	x_6	RHS		Associated dual var.
1	1	−1	−2	0	3	=	−17	π_1
−1	−1	2	−4	6	−3	$\geq$	−18	π_2
2	−3	3	−1	4	0	$\leq$	40	π_3
0	0	0	1	0	0	$\geq$	2	π_4
0	0	0	1	0	0	$\leq$	15	π_5
0	0	0	0	1	0	$\leq$	6	π_6
0	0	0	0	0	1	$\geq$	3	π_7
−3	−4	5	0	−6	7	$= z$ minimize		

x_1, x_4, x_6 unrestricted; $x_2, x_5 \geq 0$, $x_3 \leq 0$

The dual problem in detached coefficient form is

π_1	π_2	π_3	π_4	π_5	π_6	π_7	RHS		Assoc. var.
1	−1	2	0	0	0	0	=	−3	x_1
1	−1	−3	0	0	0	0	$\leq$	−4	x_2
−1	2	3	0	0	0	0	$\geq$	5	x_3
−2	−4	−1	1	1	0	0	=	0	x_4
0	6	4	0	0	1	0	$\leq$	−6	x_5
3	−3	0	0	0	0	1	=	7	x_6
−17	−18	40	2	15	6	3	$= v$ maximize		

π_1 unrestricted, $\pi_2, \pi_4, \pi_7 \geq 0$, and $\pi_3, \pi_5, \pi_6 \leq 0$.

Complementary Pairs in a Primal, Dual Pair of LPs

- Each inequality constraint in the primal LP corresponds to its own slack variable; in fact, after rearranging the terms, if the inequality is put in the form

$$\text{(some expression in the variables)} \geq 0,$$

- the left-hand side of this inequality is the expression for the primal slack variable corresponding to this inequality. This primal inequality constraint corresponds to a sign-restricted dual variable in the dual problem.
- This **primal slack variable (PSV)** and the corresponding **sign-restricted dual variable (DV)**form a **complementary pair**. (Similarly for the dual problem).

What Is the Importance of Complementary Pairs?

- **Pairs**
  - (**PSV** ≥ 0, sign restricted **DV** ≥ 0).
  - (sign restricted **PV** ≥ 0, **DSV** ≥ 0).
- **Importance**
  - *complementary slackness* condition or property states that a pair of primal, dual feasible solutions is optimal to the respective problems iff at least one quantity in every complementary pair for these solutions is 0 (**Theorem 5.5**).
  - given an arbitrary pair of primal, dual feasible solutions, the duality gap in this pair, defined as a measure of how far these solutions are from being optimal to the respective problems, is shown to be equal to the sum of the products of various complementary pairs in this pair (**Theorem 5.6**).

Complementary Pairs for LPs in Standard Form

Primal (**P**)	Dual (**D**)
minimize $z(x) = cx$	maximize $\nu(\pi) = \pi b$
s. to $Ax = b$	s. to $\pi A \leq c$
$\mathbb{R}^n \ni x \geq 0$	$\mathbb{R}^m \ni \pi$ **unrestricted**
$A \in \mathbb{R}^{m \times n};\ c \in \mathbb{R}^n;\ b \in \mathbb{R}^m$	$A^T \in \mathbb{R}^{n \times m};\ b \in \mathbb{R}^m;\ c \in \mathbb{R}^n$

- The various complementary pairs in this primal, dual pair of LPs are $(x_j, \bar{c}_j = c_j - \pi A_{.j})$, $j = 1, \cdots, n$.

Example

Consider the LP in standard form in the following detached coefficient tableau

x_1	x_2	x_3	x_4	x_5	x_6	RHS	Associated dual var.
1	2	3	-2	1	16	17	π_1
0	1	-4	1	1	1	2	π_2
0	0	1	-2	1	0	1	π_3
3	11	-15	10	4	57	$= z$ minimize	

$x_j \geq 0$ for all j

Here is the dual problem in detached coefficient tableau form.

π_1	π_2	π_3	RHS		Associated primal var.
1	0	0	$\leq$	3	x_1
2	1	0	$\leq$	11	x_2
3	-4	1	$\leq$	-15	x_3
-2	1	-2	$\leq$	10	x_4
1	1	1	$\leq$	4	x_5
16	1	0	$\leq$	57	x_6
17	2	1	$= v$ maximize		

- the complementary pairs in this primal, dual pair are

$$(x_1, \bar{c}_1 = 3 - \pi_1) \tag{1}$$

$$(x_2, \bar{c}_2 = 11 - 2\pi_1 - \pi_2) \tag{2}$$

$$(x_3, \bar{c}_3 = -15 - 3\pi_1 + 4\pi_2 - \pi_3) \tag{3}$$

$$(x_4, \bar{c}_4 = 10 + 2\pi_1 - \pi_2 + 2\pi_3) \tag{4}$$

$$(x_5, \bar{c}_5 = 4 - \pi_1 - \pi_2 - \pi_3) \tag{5}$$

$$(x_6, \bar{c}_6 = 57 - 16\pi_1 - \pi_2) \tag{6}$$

Complementary Pairs for LPs in Symmetric Form

Primal (**P**)	Dual (**D**)
minimize $z(x) = cx$	maximize $\nu(\pi) = \pi b$
s. to $Ax \geq b$	s. to $\pi A \leq c$
$\mathbb{R}^n \ni x \geq 0$	$\mathbb{R}^m \ni \pi \geq 0$
$A \in \mathbb{R}^{m \times n};\ c \in \mathbb{R}^n;\ b \in \mathbb{R}^m$	$A^T \in \mathbb{R}^{n \times m};\ b \in \mathbb{R}^m;\ c \in \mathbb{R}^n$

- The various complementary pairs in this primal, dual pair of LPs are
  - $(A_{i.}x - b_i, \pi_i)\,,\ i = 1, \cdots, m.$
  - $(x_j, c_j - \pi A_{.j})\,,\ j = 1, \cdots, n.$

Complementary Pairs for LPs in Bounded Variable Standard Form

Consider the primal LP

$$\textbf{Minimize } z = cx \tag{7}$$
$$\textbf{subject to } Ax = b \tag{8}$$
$$\ell \leq x \geq k \tag{9}$$

- where A is an $m \times n$ matrix and $\ell = \ell(\ell_j),\ k = (k_j)$ are finite vectors satisfying $\ell < k$.
- The problem in matrix notation in detached coefficient form is

x	RHS	Associated dual var.
A	$= b$	$\pi = (\pi_1, \cdots, \pi_m)$
I	$\geq \ell$	$\mu = (\mu_1, \cdots, \mu_n)$
I	$\leq k$	$\nu = (\nu_1, \cdots, \nu_n)$
c	$= z$ minimize.	

- where I is a unite matrix matrix of order n.
- Associating with the primal constraints, the dual variables as shown in the above tableau; the dual problem in matrix notation is

$$\textbf{Maximize } v = \pi b + \mu\ell + \nu k \tag{10}$$

$$\textbf{subject to } \pi A + \mu + \nu = c \tag{11}$$

$$\pi \text{ unrestricted}, \ \mu \geq 0, \ \nu \leq 0. \tag{12}$$

- Denoting as $-\delta = (-\delta_1, \cdots, -\delta_n)$, the dual problem is

$$\begin{aligned} \textbf{Maximize } v &= \pi b + \mu\ell - \delta k && (13)\\ \textbf{subject to } \pi A + \mu - \delta &= c && (14)\\ \pi \text{ unrestricted}, \ \mu, \delta &\geq 0. && (15) \end{aligned}$$

- There are n dual constraints, if you show them individually, the dual problem becomes

$$\begin{aligned} \textbf{Maximize } v &= \pi b + \mu\ell - \delta k && (16)\\ \textbf{subject to } \pi A_{.j} + \mu_j - \delta_j &= c && (17)\\ \pi \text{ unrestricted}, \ \mu, \delta &\geq 0. && (18) \end{aligned}$$

- The various complementary pairs in this primal, dual pair of LPs are $(x_j - \ell_j, \mu_j)$, $j = 1, \cdots, n$; $(k_j - x_j, \delta_j)$, $j = 1, \cdots, n$.

Theorem

Duals of equivalent LPs are equivalent

$$\begin{aligned} \text{Minimize} \quad & z(x) = cx \\ \text{s. to} \quad & Dx = d \\ & Fx \geq g, \end{aligned} \tag{5.9}$$

where D is a matrix of order $m \times n$ and F is a matrix of order $p \times n$. Its dual is

$$\begin{aligned} \text{Minimize} \quad v(\pi, \mu) \; &= \; \pi d + \mu g \\ \text{s. to} \quad \pi D + \mu F \; &= c \\ \mu &\geq 0, \end{aligned} \tag{5.10}$$

- where $\pi = (\pi_1, \cdots, \pi_m)$, $\mu = (\mu_1, \cdots, \mu_m)$ are row vectors of dual variables.
- To transform (5.9) into symmetric form, we need to replace the system of equations $Dx = d$ by the opposing pair of inequalities $Dx \geq d$, $Dx \leq d$. **Equivalent of (5.9) is:**

x^+	x^-	RHS	Associated dual vector
D	$-D$	$\geq d$	δ
$-D$	D	$\geq -d$	γ
F	$-F$	$\geq g$	ν
c	$-c$	$= z$ minimize	

$$x^+, x^- \geq 0$$

Its dual is:

$$\begin{aligned} \text{Maximize} \quad & (\delta - \gamma)d + \nu g \\ \text{s. to} \quad & (\delta - \gamma)D + \nu F \leq c \\ & -[(\delta - \gamma)D + \nu F] \leq -c \\ & \delta,\ \gamma,\ \nu \geq 0. \end{aligned} \tag{5.11}$$

The second constraint here is the same as $[(\delta - \gamma)D + \nu F] \geq c$, which together with the first is equivalent to $(\delta - \gamma)D + \nu F = c$. Also, when $\delta,\ \gamma \geq 0$, $\delta - \gamma = \pi$ is a vector of unrestricted variables. Then we see that (5.11) is equivalent to the dual (5.10) of the original LP (5.9).

Theorem

In a primal, dual pair of LPs, both problems may be infeasible.

$$\begin{aligned} \text{Minimize} \quad z &= 2x_1 - 4x_2 \\ \text{s. to} \quad x_1 - x_2 &= 1 \\ -x_1 + x_2 &= 2 \\ x_1, x_2 &\geq 0. \end{aligned}$$

Its dual is

$$\begin{aligned} \text{Maximize} \quad v &= \pi_1 + 2\pi_2 \\ \text{s. to} \quad \pi_1 - \pi_2 &\leq 2 \\ -\pi_1 + \pi_2 &\leq -4 \\ \pi_1, \pi_2 &\geq 0. \end{aligned}$$

Theorem

The Weak Duality Theorem: *Consider a primal, dual pair of LPs in which the primal is the minimization problem with objective function* $z(x)$ *and the dual is the maximization problem with objective function* $v(\pi)$*. If* $\bar{x}$, $\bar{\pi}$ *are, respectively, primal, dual feasible solutions, then* $z(\bar{x}) \geq \nu(\bar{\pi})$*. So, for any dual feasible solution* $\bar{\pi}, v(\bar{\pi})$ *is a lower bound for the minimum objective value in the primal. Likewise, for any primal feasible solution* $\bar{\pi}, v(\bar{\pi})$ *is an upper bound for the maximum objective value in the dual.*

Direct consequences of the weak duality theorem

- **Sufficient Optimality Criterion for LP:**
  - Let $\bar{x}$ be feasible solution LP. If $\exists\bar{\pi}$ for its dual such that $z(\bar{x}) = \nu(\bar{\pi})$, then $\bar{x}$ and $\bar{\pi}$ are optimum for primal and dual respectively.
- **Dual Infeasibility and Primal Unboundedness:**
  - If the primal (**minimization**) is feasible and $z(x) \to -\infty$ on its feasible solution set, then the dual must be infeasible.
- **Primal Infeasibility and Dual Unboundedness:**
  - If the dual (**maximization**) is feasible and $\nu(\pi) \to +\infty$ on its feasible solution set, then the primal must be infeasible.
- **Bounds on Objective Values in a primal, dual pair of LPs:**
  - (the minimum objective value in the minimization LP)$\geq$ (the maximum objective value in the maximization LP).

Theorem

The (Strong) Duality Theorem of LP: *In a primal, dual pair of LPs, (1) if one has an optimum solution, the other does also, and the two optimum objective values are equal, (2) if one of the problems is feasible and has the objective unbounded ("'below"' if the problem is a minimization problem, "'above"' if it is a maximization problem), then the other problem is infeasible and vice versa.*

Theorem

The Complementary Slackness Theorem: *In a primal, dual pair of LPs, let* $(\bar{x}, \bar{\pi})$ *be a primal, dual feasible solution pair. They are optimal to the respective problems if at least one quantity in every complementary pair is 0 in* $(\bar{x}, \bar{\pi})$*. These conditions are known as complementary slackness conditions for optimality.*

How Various Algorithms Solve LPs

- For a given feasible solution to an LP to be an optimum solution, it must satisfy three conditions.
  1. Primal feasibility.
  2. Dual feasibility.
  3. Complementary slackness property.
- The *primal simplex algorithm* starts with an initial BFS and searches only among BFSs for an optimum solution; so it maintains primal feasibility throughout. The *optimality termination condition* is *the dual feasibility* condition and *the unboundedness termination condition* establishes *dual infeasibility*.
- The *dual simplex algorithm*.
- The IPMs (interior point methods). The most popular: *primal-dual IPMs:* (See later chapters!) .

Thank you for your attention!