Polyhedral Geometry (4.1-4.7)

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Convex Polyhedron

The intersection of a finite number of half-spaces is a geometric object known as a *convex polyhedron*.

- Every convex polyhedron is a set of feasible solutions of a system consisting of a finite number of linear inequalities.
- The set of feasible solutions of any linear programming problem is a convex polyhedron.

A bounded convex polyhedron is known as a convex polytope

An inequality view

Any convex polyhedron K, can be viewed as the set of feasible solution of some

$$\begin{aligned} Dx &= d \\ Fx &\geq g \end{aligned}$$

Note that it can be written as a general system of constraints:

$$\begin{aligned} Dx &\geq d \\ Dx &\leq d \iff -Dx \geq -d \\ Fx &\geq g \\ &\iff \\ Ax &\geq b \end{aligned}$$

A bit of perspective

An *affine space* of R^n is the set of feasible solutions of a general system of linear equations. Hence a special case of a convex polyhedron is an affine space.

Affine spaces are the focus of discussion in linear algebra textbooks... we will study convex polyhedra that are not affine spaces. - Murphy

Active/Inactive constraints at a feasible solution $\bar{x}$

Let K be the set of feasible solutions of the general system of constraints (or the solution space of a convex polyhedron)

$$A_{i\cdot}x \begin{cases} = b_i & \text{for } i = 1, \ldots, p \\ \geq b_i & \text{for } i = p+1, \ldots, m \end{cases} \tag{1}$$

consisting of inequalities and possibly equality constraints. For a feasible solution $\bar{x} \in K$, a constraint is considered

- *Active* (tight) if it in (1) is satisfied as an equation.
- *Inactive* (slack) if it in (1) is satisfied as a strict inequality.

Example

Example

Consider the following system of constraints in the point $\bar{x} = (x_1, x_2) = (3, 9)$:

$$\begin{aligned} 2x_1 + x_2 &\leq 15 \text{(Active)} \\ x_1 + x_2 &\leq 12 \text{(Active)} \\ x_1 &\leq 5 \text{(Non-active)} \\ x_1 &\geq 0 \text{(Non-active)} \\ x_2 &\geq 0 \text{(Non-active)} \end{aligned}$$

Minimal representation

General system of constraints K:

$$Ax \geq b \tag{2}$$

An inequality constraint is said to be a *redundant inequality constraint* if its removal from the system does not change its set of feasible solutions.

A system of constraints that does not contains any redundant constraints is said to be a *minimal representation* of that convex polyhedron K.

Example

Example

Consider the following system of constraints:

$$
\begin{aligned}
2x_1 + x_2 &\leq 15 \\
x_1 + x_2 &\leq 12 \\
x_1 &\geq 2 \\
x_1 &\geq 0 \text{(Redundant)} \\
x_2 &\geq 0
\end{aligned}
$$

Minimal representation:

$$
\begin{aligned}
2x_1 + x_2 &\leq 15 \\
x_1 + x_2 &\leq 12 \\
x_1 &\geq 2 \\
x_2 &\geq 0
\end{aligned}
$$

Boundary point/Interior point

Consider a general system of constraints K:

$$Ax \geq b \tag{3}$$

A points $\bar{x} \in K$ is said to be

- A boundary point of k if and only if there is at least one active constraint at $\bar{x}$
- An interior point of K if there are no active constraints at $\bar{x}$

(Analogous to a convex set)

Supporting hyper plane

A supporting hyperplane for a convex polyhedron K is a hyperplane H in R^n satisfying:

- K is completely contained on one of side of K
- $H \cap K \neq \emptyset$

Face of a polyhedron

A *face* of a convex polyhedron K is either

- The empty set $\emptyset$
- K itself
- Intersection $H \cap K$ of K with a supporting hyperplane H

Faces of K other than $\emptyset$ and K are called *proper faces*.
A single point that by itself is a face of a convex polyhedron is called an *extreme point*.

Theorem

Theorem

The set of optimum solutions of a linear program is a face of its set of feasible solutions.

The set of optimum solutions of an LP is usually referred to as its *optimum face*. (Note: Even if the LP has no solutions $\emptyset$ is still an face of K)

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Algebraic characterisation of extreme point

Let K be the convex polyhedron, which is the set of feasible solution of

$$Dx = d$$
$$Fx \geq g$$

Definition

Let $\bar{x} \in K$. Denote the active constraints at $\bar{x}$ as S. $\bar{x}$ is said to be an *basic feasible solution* (BFS, extreme point) if and only if it is the unique solution for S or equivalently the set of column vector of variables in S is linearly independent. S is called the active system in point $\bar{x}$.

If $\bar{x}$ is an BFS of K it is said to be non-degenerate if S is square and degenerate if the number of linear equalities is larger than the number of variables.

Example

Example

Consider the following system of constraints in the point $\bar{x} = (x_1, x_2) = (3, 9)$:

$$
\begin{aligned}
2x_1 + x_2 &\leq 15 \\
x_1 + x_2 &\leq 12 \\
x_1 &\leq 5 \\
x_1, x_2 &\geq 0
\end{aligned}
$$

The point $\bar{x}$ is an non-degenerate BFS since the active system is a square matrix of full rank.

$$
\begin{aligned}
2x_1 + x_2 &\leq 15 \\
x_1 + x_2 &\leq 12
\end{aligned}
$$

System in standard form

Consider the system K in standard form

$$Ax = b$$
$$x \geq 0$$

where A is a $m \times n$-matrix. A *basic vector* for the standard form is a vector x_B of m variables whose associated column vectors are linearly independent.

Obtaining a basic solution

The *basic solution* of the standard form with respect to x_B is obtained by fixing each non-basic variable at its lower bound 0. Then solve for the values of the basic variables.

Non-basic vector: $x_D = 0$
Basic vector: $x_B = B^{-1}b$

Characterisation of solutions

In:
Non-basic vector: $x_D = 0$
Basic vector: $x_B = B^{-1}b$
x_B, B are said to be:

- *Feasible basic vector; feasible basis* if the solution is feasible, i.e., $B^{-1}b \geq 0$
- *Infeasible basic vector; infeasible basis* otherwise
- *Non-degenerate* if all the entries in $B^{-1}b$ are non-zero
- *Degenerate* if at least one entry in $B^{-1}b$ is zero

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The purification routine

Importance

- Interior point methods finds a feasible point of an LP, the purification routine makes it an BFS (or an optimal BFS)
- It can be shown that if the set of feasible solution of the system is bounded, then every feasible solution $\bar{x}$, is a convex combination of basic feasible solutions of the system

General idea of the purification routing

Consider the LP in standard form:

$$\begin{aligned} &\min c'x \\ &\text{s.t. } Ax = b \\ &\qquad\quad x \geq 0 \end{aligned}$$

1. Start from a feasible solution with r non-active constraints. Check if the active system is linear dependent, if not: stop (i.e., we have a BFS!)
2. Since the solution is not unique (the active system is not square). The non-active constraints can be written as a linear combination of the active constraints. Hence we have a space where the active constraints are always for-filled.
3. Move in this active system until a previous non-active constraint gets active. We have found an extreme point in this particular restricted space. Now we have $r - 1$ non-active constraints. Move to step 1.

Theorems of standard form

Consider the LP in standard form

$$\min c'x$$
$$Ax = b$$
$$x \geq 0$$

Theorem

If an LP in standard form has a feasible solution, then it has a BFS.

Theorem

If an LP in standard form has optimum feasible solution, then it has a BFS which is also optimal.

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