The Decomposition Principle (7.4-7.7)

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The Decomposition Principle

- Decomposing a problem into manageable subproblems by using special structures.
- In LP Benders, Danzig-Wolfe, Lagrangian relaxation are equivalent

The decomposition algorithm

$$\begin{aligned} \min \mathbf{cx} \\ \text{s.t. } \mathbf{Ax} \leq \mathbf{b} \\ \mathbf{x} \in X \end{aligned}$$

Decomposing with respect to extreme points (BFS):

$$\mathbf{x} = \sum_{j=1}^{t} \lambda_j \mathbf{x}_j$$

$$\sum_{j=1}^{t} \lambda_j = 1$$

$$\lambda_j \geq 0 \quad \forall j = 1, \ldots, t$$

Recall problems

Masterproblem

$$\min \sum_{j=1}^{t} (\mathbf{c}\mathbf{x}_j)\lambda_j$$
$$\sum_{j=1}^{t} (\mathbf{A}\mathbf{x}_j)\lambda_j \leq \mathbf{b}$$
$$\sum_{j=1}^{t} \lambda_j = 1$$
$$\lambda_j \geq 0 \quad \forall j = 1, \ldots, t$$

Subproblem

$$\max(\mathbf{w}\mathbf{A} - \mathbf{c})\mathbf{x} + \alpha$$
$$\text{s.t.}\, \mathbf{x} \in X$$

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The case of an unbounded region X

What to do when X is unbounded?
Decomposing with respect to extreme points (BFS) and extreme rays:

$$\mathbf{x} = \sum_{j=1}^{t} \lambda_j \mathbf{x}_j + \sum_{j=1}^{l} \mu_j \mathbf{d}_j$$

$$\sum_{j=1}^{t} \lambda_j = 1$$

$$\lambda_j \geq 0 \quad \forall j = 1, \ldots, t$$

$$\mu_j \geq 0 \quad \forall j = 1, \ldots, l$$

Master problem

$$\min \sum_{j=1}^{t} \lambda_j \mathbf{cx}_j + \sum_{j=1}^{l} \mu_j \mathbf{cd}_j$$

$$\text{s.t.} \sum_{j=1}^{t} \lambda_j \mathbf{cx}_j + \sum_{j=1}^{l} \mu_j \mathbf{cd}_j = \mathbf{b} \sum_{j=1}^{t} \lambda_j = 1$$

$$\lambda_j \geq 0 \quad \forall j = 1, \ldots, t$$

$$\mu_j \geq 0 \quad \forall j = 1, \ldots, l$$

Only change in sub problem is that if an unbounded solution is found. The extreme ray generated will be used in master problem.

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Block diagonal structure

$$\begin{aligned} \min \sum_{i=1}^{T}\sum_{j=1}^{t} \lambda_j \mathbf{cx}_j + \sum_{i=1}^{T}\sum_{j=1}^{l} \mu_j \mathbf{cd}_j \\ \text{s. t.} \sum_{i=1}^{T}\sum_{j=1}^{T_i} \lambda_j \mathbf{cx}_j + \sum_{i=1}^{T}\sum_{j=1}^{l} \mu_j \mathbf{cd}_j = \mathbf{b} \\ \sum_{j=1}^{T_i} \lambda_j = 1 \quad \forall j = 1, \dots, T \\ \lambda_j \geq 0 \quad \forall j = 1, \dots, t \forall j = 1, \dots, T \\ \mu_j \geq 0 \quad \forall j = 1, \dots, l \forall j = 1, \dots, T \end{aligned}$$

Different lower bound:

$$\mathbf{cx} \geq \mathbf{wb} + \alpha 1 - \sum_i (z_{ik} - \hat{c}_{ik})$$

Relations between other decomposition procedures

Benders

$$\begin{aligned} \min \mathbf{cx} \\ \text{s.t. } \mathbf{Ax} = \mathbf{b} \\ \mathbf{x} \in X = \{\mathbf{x}{:}\mathbf{Dx} \geq \mathbf{d}, \mathbf{x} \geq \mathbf{0}\} \end{aligned}$$

Dual

$$\begin{aligned} \max \mathbf{wb} + \mathbf{vd} \\ \text{s.t. } \mathbf{wA} + \mathbf{vD} \leq \mathbf{c} \\ \mathbf{w} \text{ unrestricted } , \mathbf{v} \geq \mathbf{0} \end{aligned}$$

Assume w is fixed

$$\max_{w}(\mathbf{wx} + \max \mathbf{vd}$$
$$\text{s. t.}\, \mathbf{wA} + \mathbf{vD} \leq \mathbf{c}$$
$$\mathbf{w} \text{ unrestricted } , \mathbf{v} \geq \mathbf{0})$$

$$= \max_{w}(\mathbf{wx} + \max \mathbf{c} - \mathbf{wa}$$
$$\mathbf{x} \in X)$$

Differece

- Danzig wolfe is column generation
- Benders is row generation

Lagrangian relaxation

Lagrangian dual

$$\max(\theta(\mathbf{w}) : \mathbf{w} \text{ unrestricted}$$

Lagrangian subproblem

$$\theta(\mathbf{w}) = \mathbf{wb} + \min((\mathbf{c} - \mathbf{wA})\mathbf{x} : \mathbf{x} \in X)$$

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