MAI0130 - Linear Optimization

Meeting 2

Pontus Söderbäck
pontus.soderback@liu.se

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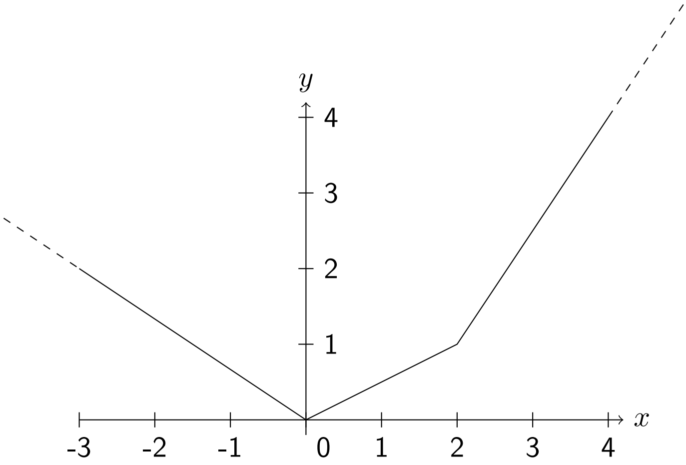
Outline

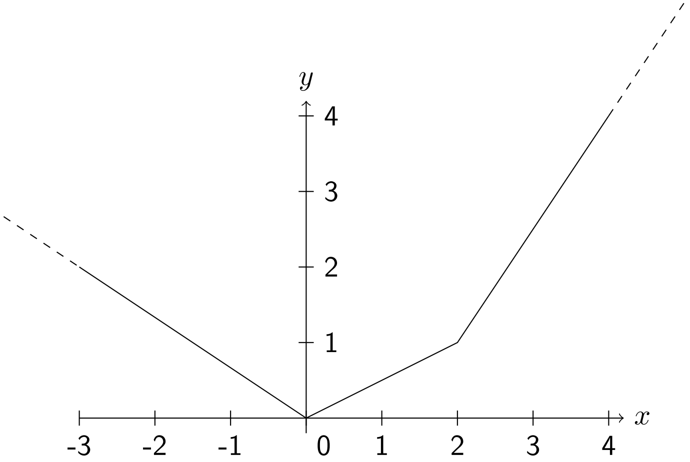
Piecewise Linear (PL) Functions

PL Functions in one Variable



$$I_1 = (-\infty, 0],\ I_2 = (0, 2],\ I_3 = (2, \infty),$$

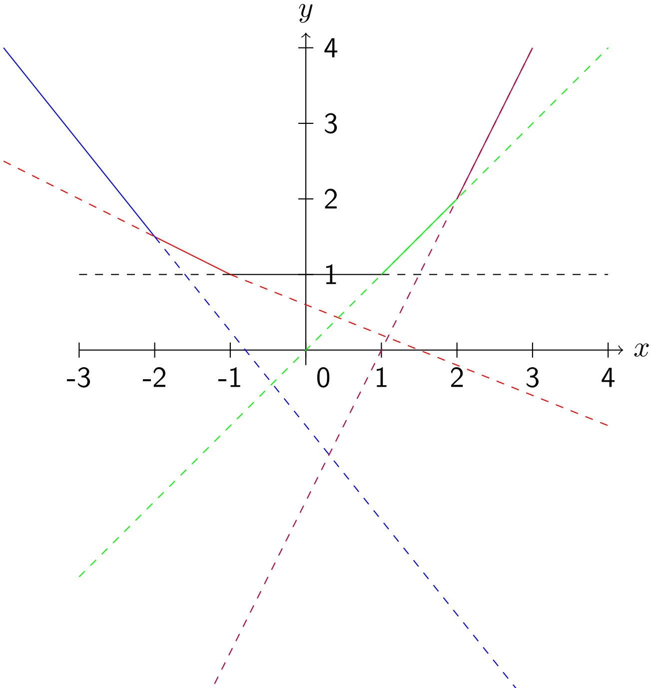
where $I_1 \cup I_2 \cup I_3 = \mathbb{R}$.



Interval	Slope of $\theta(\lambda)$ in interval	Value of $\theta(\lambda)$
$\lambda \leq 0$	$-\frac{2}{3}$	$-\frac{2}{3}\lambda$
$0 \leq \lambda < 2$	$\frac{1}{2}$	$\frac{1}{2}\lambda$
$2 \leq \lambda$	$\frac{3}{2}$	$1 + \frac{3}{2}(\lambda - 2)$

PL Functions in Several Variable

- Intervals is replaced with convex polyhedral regions, $K_1, \ldots, K_r$, s.t. $\mathbb{R}^n = K_1 \cup \cdots \cup K_r$. We have $S \subset \{1, \ldots, r\}$.
- $f(x)$ is a PL function iff
  - $f(x)$ can be written as an affine function, $f^i(x) = c_0^i + \mathbf{c^i}^{\mathrm{T}} x$ on every region
  - In every point $x \in \bigcap_{i \in S} K_i$, the different functions have the same value: $f_i(x) \forall i \in S$.



y
x
4
3
2
1
-3
-2
-1
0
1
2
3
4

Result (2.2)

Let $\theta(\lambda)$ be a PL function of a single variable $\lambda \in \mathbb{R}$. Let $\lambda_1, \dots, \lambda_r$ be the various breakpoints in increasing order where its slope changes. $\theta(\lambda)$ is convex iff at each breakpoint λ_t; its slope to the right of λ_t is strictly greater than its slope to the left of λ_t; that is, iff its slopes are monotonic increasing with the variable.

Theorem (2.5)

Let $K_1 \cup \ldots \cup K_r$ be a partition of $\mathbb{R}^n$ into convex polyhedral regions, and $f(x)$ a PL function. Then $f(x)$ is convex iff for each $t = 1$ to r, and for all $x \in K_t$

$$c_o^t + (c^t)^{\mathrm{T}} x = \max\{c_o^p + (c^p)^{\mathrm{T}} x : p = 1, \ldots, r\}.$$

In effect, this says that $f(x)$ is convex iff for each $x \in \mathbb{R}^n$

$$f(x) = \max\{c_o^p + (c^p)^{\mathrm{T}} x : p = 1, \ldots, r\}.$$

Optimizing PL Functions Subject to Linear Constraints

Optimizing general PL function

"PL function subject to linear constraints is a hard problem for which there are no known efficient algorithms"

Special cases:

- Minimizing a PL convex function, or equivalently
- Maximizing a PL concave function

"subject to linear constraints can be transformed into LPs by introducing additional variables, and solved by efficient algorithms available for LPs."

We will study different ways of transforming into LP.

Separable function

Definition

A real-value function $z(x)$ *of decision variables* $x = (x_1, \dots, x_n)^{\mathrm{T}}$ *is separable if it can be expressed as a sum of n different functions,* $z(x) = z_1(x_1) + \dots + z_n(x_n)$ *where each functions is dependent on one variable.*

Result

If each function, $z_j(x_j),\ \forall j = 1, \dots, n$ *is a convex PL function then is* $z(x)$ *also a convex PL function.*

Result

The negative of a concave function is convex. Maximizing a concave function is the same as minimizing its negative, which is a convex function

Minimizing a Separable PL Convex Function Subject to Linear Constraints

We want to solve:

$$\begin{array}{ll} \min\limits_{x} & z(x) = z_1(x_1) + \ldots + z_n(x) \\ \text{s.t.} & \\ & \begin{array}{rcl} Ax & = & b \\ x & \geq & 0, \end{array} \end{array}$$

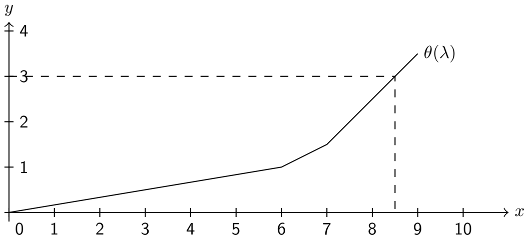
where $z_i(x)$ are convex PL functions. We **do not** have algorithms for solving this type of problem.

LP problem of convex PL function

Let $\theta(\lambda),\ \lambda \in \mathbb{R}^1$ be a convex PL function. Then the value $\theta(\bar{\lambda})$ is the value as the solution for the problem

$$\begin{array}{ll} \min & h = c_1\mu_1 + \cdots + c_r\mu_r \\ \text{s.t.} & \\ & \begin{array}{rcl} \mu_1 + \cdots + \mu_r & = & \bar{\lambda} \\ 0 \le \mu_t & \le & \lambda_t - \lambda_{t-1}, t = 1, \ldots, r \end{array} \end{array}$$

$$\bar{\lambda} = 8.5$$



$$
\begin{array}{ll}
\min\limits_{x} & z(x) = z_1(x_1) + \ldots + z_n(x) = \sum_{j=1}^{n} z_j(x_j) = \sum_{j}^{n} \sum_{k=1}^{r_j} c_j^k x_j^k \\
\text{s.t.} & \\
& \begin{array}{rcl}
\sum_{k=1}^{r_j} x_j^k & = & x_j,\ j = 1, \ldots, n \\
Ax & = & b \\
x & \geq & 0, \\
0 & \leq & x_j^k \leq l_j^k \quad \forall 1 \leq j \leq n, 1 \leq k \leq r_j
\end{array}
\end{array}
$$

Min-max, Max-min Problems

We want to solve:

$$\begin{array}{rl} \max & z(x) = \min\{c_0^1 + c^1 x, \ldots, c_0^r + c^r x\} \\ \text{s.t.} & \\ & \begin{array}{rcl} Ax & = & b \\ x & \geq & 0, \end{array} \end{array}$$

Reformulating:

$$\begin{array}{rl} \max & x_{n+1} \\ \text{s.t.} & \\ & \begin{array}{rcl} x_{n+1} & \leq & c_0^1 + c^1 x \\ x_{n+1} & \leq & c_0^2 + c^2 x \\ & \vdots & \\ x_{n+1} & \leq & c_0^r + c^r x \\ Ax & = & b \\ x & \geq & 0, \end{array} \end{array}$$

Example of max-min problem

"Production of products P1 and P2, with resources RM1, RM2 and RM3. The produced quantity of the products is denoted by x_1 and x_2"

$$
\begin{array}{lrcl}
\max & \multicolumn{3}{l}{p(x) = \min\{15x_1 + 10x_2} \\
 & \multicolumn{3}{l}{10x_1 + 15x_2, 12x_1 + 12x_2\}} \\
\text{s.t.} & & & \\
 & 2x_1 + x_2 & \leq & 1500 \text{ (Supply of RM1)} \\
 & x_1 + x_2 & \leq & 1200 \text{ (Supply of RM2)} \\
 & x_1 & \leq & 500 \text{ (Supply of RM3)} \\
 & x_1, x_2 & \geq & 0
\end{array}
$$

$$
\begin{array}{lrcl}
\max & & p & \\
\text{s.t.} & & & \\
 & p & \leq & 15x_1 + 10x_2 \\
 & p & \leq & 10x_1 + 15x_2 \\
 & p & \leq & 12x_1 + 12x_2 \\
 & 2x_1 + x_2 & \leq & 1500 \\
 & x_1 + x_2 & \leq & 1200 \\
 & x_1 & \leq & 500 \\
 & x_1, x_2 & \geq & 0,
\end{array}
$$

Minimizing Positive Linear Combinations of Absolute Values of Affine Functions

We want to solve:

$$\begin{array}{lrcl} \min & & z(x) & \\ \text{s.t.} & Ax & \geq & b. \end{array}$$

where $z(x) = \omega_1 \left|c_0^1 + c^1 x\right| + \ldots + \omega_r \left|c_0^r + c^r x\right|$, where $\omega_i > 0$.

Solve in two steps:

1. Absolute value as linear function of two variables
2. Transform the problem.

Absolute value as linear function of two variables

We consider the *affine* function $c_0^k + (c^k)^{\mathrm{T}} x$ and the value in point $\bar{x} \in \mathbb{R}^n$ is denoted by $\beta = c_0^k + (c^k)^{\mathrm{T}} \bar{x}$. Then we formulate the problem

$$\begin{array}{rrcl} \min & u + v & & \\ \text{s.t.} & u - v & = & \beta \\ & u, v & \geq & 0 \end{array}$$

We claim that the value of the objective function in the optimal point is $|\beta|$.

Proof (one case)

Consider $\beta \leq 0$, then the solution can be written as $(\alpha, |\beta| + \alpha)$, where $\alpha \geq 0$. The value of the objective function $2\alpha + |\beta|$ and the optimal is $\alpha = 0$, i.e. the point $(0, |\beta|)$ is the optimal.

Transform the Problem

$$\begin{array}{rrcl} \max & & z(x) & \\ \text{s.t.} & Ax & \geq & b. \end{array}$$

where $z(x) = \omega_1 \left| c_0^1 + c^1 x \right| + \ldots + \omega_r \left| c_0^r + c^r x \right|$, where $\omega_i > 0$.

$$\begin{array}{rrcl} \max & \omega_1 \left[(u_1^+) + (u_1^-) \right] + \ldots + \omega_r \left[(u_r^+) + (u_r^-) \right] & & \\ \text{s.t.} & & & \\ & c_0^1 + c^1 x & = & u_1^+ - u_1^- \\ & & \vdots & \\ & c_0^r + c^r x & = & u_r^+ - u_r^- \\ & Ax & \geq & b \\ & (u_t^+), (u_t^-) & \geq & 0, \ t = 1, \ldots, r. \end{array}$$

Minimizing the Maximum of the Absolute Values of Several Affine Functions

We want to solve:

$$
\begin{array}{ll}
\min\limits_{x} & z(x) = \max\{\left|c_0^1 + c^1 x\right|, \ldots, |c_0^r + c^r x|\} \\
\text{s.t.} & \\
 & Ax \quad \geq \quad b.
\end{array}
$$

- Model 1: First transform the absolute value and then transform the max-operator
- Model 2: Rewrite problem and then transform the max-operator

$$\begin{array}{ll} \min\limits_{x} & z(x) = \max\{\left|c_0^1 + c^1 x\right|, \ldots, \left|c_0^r + c^r x\right|\} \\ \text{s.t.} & \\ & Ax \quad \geq \quad b. \end{array}$$

Model 1

$$\begin{array}{llcl} \min & z = \max\{u_1^+ + u_1^-, \ldots, u_r^+ + u_r^-\} & & \\ \text{s.t.} & c_0^1 + c^1 x & = & u_1^+ - u_1^- \\ & & \vdots & \\ & c_0^r + c^r x & = & u_r^+ - u_r^- \\ & Ax & \geq & b \\ & u_t^+, u_t^- & \geq & 0, t = 1, \ldots, r. \end{array}$$

$$\begin{array}{llcl} \min & z & & \\ \text{s.t.} & z & \geq & u_1^+ + u_1^-, \ \forall t = 1, \ldots, r \\ & c_0^1 + c^1 x & = & u_1^+ - u_1^- \\ & & \vdots & \\ & c_0^r + c^r x & = & u_r^+ - u_r^- \\ & Ax & \geq & b \\ & u_t^+, u_t^- & \geq & 0, t = 1, \ldots, r. \end{array}$$

Model 2

$$\begin{array}{ll} \min\limits_{x} & z(x) = \max\{c_0^1 + c^1 x, -c_0^1 - c^1 x, \ldots, \\ & \qquad c_0^r + c^r x, -c_0^r - c^r x\} \\ \text{s.t.} & \\ & Ax \quad \geq \quad b. \end{array}$$

$$\begin{array}{llcl} \min & z & & \\ \text{s.t.} & -z & \leq & c_0^t + c^t x, \ \forall t = 1, \ldots, r \\ & z & \geq & c_0^t + c^t x, \ \forall t = 1, \ldots, r \\ & Ax & \geq & b \end{array}$$

Model 2 has the advantage that only **one** new variable needs to be introduced.

Minimizing Positive Combinations of Excesses/Shortages

We also want to model excess and shortage or in more general we want to model function of the type $\max\{0, z(x)\} = (z(x))^+$ and $-\min\{0, z(x)\} = (z(x))^-$. To illustrate consider the following problem[1].

A company has two plants, P_1, P_2, which produce the same product. The plants have regular production capacity and cost, a_i and g_i respectively and the overtime (extra) capacity and cost b_i and h_i. The demand is d with a selling price p, but in the market a dealer can buy excess (over demand) to the price $0 \leq s \leq p$. The transportation cost from plant i to the market is c_i. The aim is to maximize the net profit.

We introduce x_i as the tons shipped from plant i; $y_{i,1}$ is the tons produced in regular and $y_{i,2}$ is the tons produced in overtime. From these we can formulate the problem.

$$\begin{array}{ll}
\max & \left(\sum_{i=1}^{2} x_i\right) p - \left(\sum_{i=1}^{2} x_i - d\right)^+ (p - s) - (g_1 y_{11} + h_1 y_{12} + g_2 y_{21} + h_2 y_{22}) - \sum_{i=1}^{2} c_i x_i \\
\text{s.t.} & \\
& \begin{array}{rcl}
x_1 & = & y_{11} + y_{12} \\
x_2 & = & y_{21} + y_{22} \\
0 \leq & y_{i1} & \leq a_i, i = 1, 2 \\
0 \leq & y_{i2} & \leq b_i, i = 1, 2
\end{array}
\end{array}$$

[1]Which are a simplified version of Example 2.9 in Murty.

Rewriting the Problem

$$
\begin{array}{ll}
\max & \left(\sum_{i=1}^{2} x_i\right) p - \left(\sum_{i=1}^{2} x_i - d\right)^{+} (p - s) - (g_1 y_{11} + h_1 y_{12} + g_2 y_{21} + h_2 y_{22}) - \sum_{i=1}^{2} c_i x_i \\
\text{s.t.} & \\
& \begin{array}{rcl}
x_1 & = & y_{11} + y_{12} \\
x_2 & = & y_{21} + y_{22} \\
0 \leq & y_{i1} & \leq a_i, i = 1, 2 \\
0 \leq & y_{i2} & \leq b_i, i = 1, 2
\end{array}
\end{array}
$$

$$
\begin{array}{ll}
\max & \left(\sum_{i=1}^{2} x_i\right) p - u^{+} (p - s) \qquad - (g_1 y_{11} + h_1 y_{12} + g_2 y_{21} + h_2 y_{22}) - \sum_{i=1}^{2} c_i x_i \\
\text{s.t.} & \\
& \begin{array}{rcl}
x_1 & = & y_{11} + y_{12} \\
x_2 & = & y_{21} + y_{22} \\
0 \leq & y_{i1} & \leq a_i, i = 1, 2 \\
0 \leq & y_{i2} & \leq b_i, i = 1, 2 \\
\sum_{i=1}^{2} x_i & = & d + u^{+} - u^{-} \\
u^{+}, u^{-} & \geq & 0.
\end{array}
\end{array}
$$

Multiobjective LP Models

Multiobjective LP Models

"In most real-world decision-making problems there are usually several objective functions to be optimized simultaneously"

$$\begin{array}{rrcl} \min & z_1(x), \ldots, z_k(x) & & \\ \text{s.t.} & Ax & = & b \\ & Dx & \geq & d \\ & x & \geq & 0. \end{array}$$

Definition (Pareto optimal)

Pareto optimal[a] *solution is a feasible solution, $\bar{x}$ for which it is impossible to improve on of the objectives without diminish an other. A feasible solution which is not pareto optimal is called a* **dominated solution**.

[a]Also: vector minimum, nondominated solution, equilibrium solution, efficient solution

Practical Approaches to resolve the Problem of Optimality

Scale the different objectives to the same scale with an *exchange vector*

- Assign weights to the objective function and sum: $\sum_{i=1}^{k} \omega_i z_i(x)$, where $\sum_{i=1}^{k} \omega_i = 1$.
- Assign "realistic" values to the different objective functions, $g_r,\ r = 1, \ldots, k$.
    - Objective functions where a higher value is the goal - measure **shortage**
    - Objective functions where a lower value is the goal - measure **excess**
    - Objective functions where a fix value is the goal - measure both **shortage** and **excess**

Original

$$\begin{array}{llll} \min & z_1(x), \ldots, z_k(x) & & \\ \text{s.t.} & Ax & = & b \\ & Dx & \geq & d \\ & x & \geq & 0. \end{array}$$

Rewritten

$$\begin{array}{llll} \min & \sum_{i=1}^{k} \left(\alpha_r u_r^+ + \beta_r \mu_r^- \right) & & \\ \text{s.t.} & & & \\ & z_r(x) - g_r & = & u_r^+ - u_r^-, r = 1, \ldots, k \\ & u_r^+, u_r^- & \geq & 0,\ r = 1, \ldots, k \\ & Ax & = & b \\ & Dx & \geq & d \\ & x & \geq & 0 \end{array}$$