MAI0130 - Linear Optimization

Meeting 7

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Outline - Importance of the Dual Simplex

Notation

We denote the primal and dual solution by x and π respectively and the optimal solution is denote by $\bar{x}$ and $\bar{\pi}$.

We remember that the primal feasibility is

$$x_B = B^{-1}b \geq 0 \tag{1}$$

and the dual feasibility as

$$c - c_B B^{-1} A \geq 0 \tag{2}$$

We will refer to these two equations.

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Importance of the Dual Simplex Algorithm

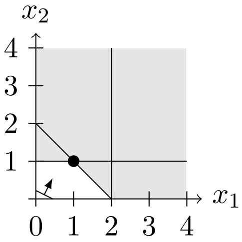
Importance of the Dual Simplex Algorithm

- Find new optimum when RHS changes
- Have to add an inequality constraint

$$\begin{array}{ll} \min & z = cx \\ \text{s.t.} & \\ & \begin{array}{rcl} Ax & = & b \\ x & \geq & 0 \end{array} \end{array}$$

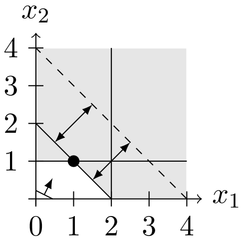
Change in the RHS Vector

$$
\begin{array}{lrcl}
\min & & z = cx & \\
\text{s.t.} & & & \\
& Ax & = & b \\
& x & \geq & 0
\end{array}
$$



Change in the RHS Vector

$$\begin{array}{lrcl} \min & z = cx & & \\ \text{s.t.} & & & \\ & Ax & = & b' \\ & x & \geq & 0 \end{array}$$



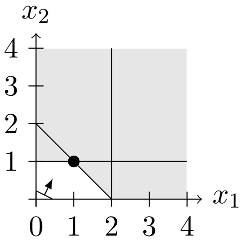
Change in the RHS Vector

We have 2 cases - "same" optimum and new optimum.

If $B^{-1}b' \geq 0$ then x_B "continues" to be an optimum, i.e. $(x_B, x_D) = (B^{-1}b', 0)$ and the same dual solution.

If $B^{-1}b' \not\geq 0$ then x_B then we have no longer **primal** feasibility but we have still **dual** feasibility. We have $(x_B, -z) = (B^{-1}b', -c_B B^{-1}b')$ and from these can we start a dual simplex method.

Change in the RHS Vector



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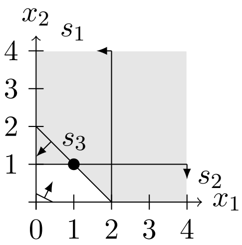
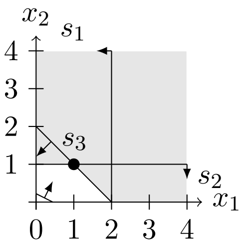


x_1	x_2	s_1	s_2	s_3	$-z$	b
1	0	1	0	0	0	2
0	1	0	1	0	0	1
1	1	0	0	1	0	2
1	2	0	0	0	1	0

Change in the RHS Vector



x_1	x_2	s_1	s_2	s_3	$-z$	b
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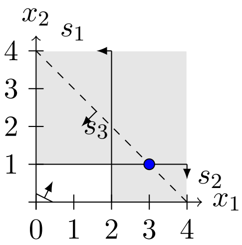
1	1	0	0	1	0	2
1	2	0	0	0	1	0

optimal solution is $(x_1, x_2, s_1, s_2, s_3) = (1, 1, 1, 0, 0)$

$$B^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \tag{3}$$

$$B^{-1} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

Change in the RHS Vector



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$$B^{-1} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad B^{-1} \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$$

Adding New Constraints to the Model

Original tableau

x_B	x_D	$-z$	RHS
B	D	0	b
c_B	c_D	1	0

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Add r new inequality constraints

x_B	x_D	x_E	$-z$	RHS
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F	G	I	0	g
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with the basic variables (x_B, x_E).

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with the basic variables (x_B, x_E).

We can then derive the

$$\begin{pmatrix} B & 0 \\ F & I \end{pmatrix}^{-1} = \begin{pmatrix} B^{-1} & 0 \\ -FB^{-1} & I \end{pmatrix} \tag{4}$$

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with the basic variables (x_B, x_E).

We can then derive the

$$\begin{pmatrix} B & 0 \\ F & I \end{pmatrix}^{-1} = \begin{pmatrix} B^{-1} & 0 \\ -FB^{-1} & I \end{pmatrix} \tag{4}$$

and we can then check primal feasibility

$$\begin{pmatrix} \bar{b} \\ \bar{g} \end{pmatrix} = \begin{pmatrix} B^{-1} & 0 \\ -FB^{-1} & I \end{pmatrix} \begin{pmatrix} b \\ g \end{pmatrix} \geq 0 \tag{5}$$

Marginal Analysis

Example Introduction

"A company needs three products for internal use, P_1, P_2 and P_3. Four processes can be used for producing the products. There is a minimum requirement for each product, b_1, b_2, b_3 and a cost associated with each process, $c_1, \dots, c_4$. We want to minimize the cost." Denote the time we run process j by x_j and write the problem as

$$
\begin{array}{lrcl}
\min & z(x) = 28x_1 + 67x_2 + 12x_3 + 35x_4 & & \\
\text{s.t.} & & & \\
& x_1 + 2x_2 + x_4 & \geq & 17 \\
& 2x_1 + 5x_2 + x_3 + 2x_4 & \geq & 36 \\
& x_1 + x_2 + 3x_4 & \geq & 8 \\
& x_j & \geq & 0, \forall j = 1, \dots, 4
\end{array}
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Item	x_1	x_2	x_3	x_4	x_5	x_6	x_7	$-z$	b
P_1	1	2	0	1	-1	0	0	0	17
P_2	2	5	1	2	0	-1	0	0	36
P_3	1	1	0	3	0	0	-1	0	8
	28	67	12	35	0	0	0	1	0

This tableau can be solved by a revised simplex and we get this inverse tableu.

BV					Basic Values
x_1	5	-2	0	0	13
x_2	-2	1	0	0	2
x_7	3	-1	-1	0	7
$-z$	-6	-11	0	1	-498

Information from the Tableau

BV	Inv. Tableau				Basic Values
x_1	5	-2	0	0	13
x_2	-2	1	0	0	2
x_7	3	-1	-1	0	7
$-z$	-6	-11	0	1	-498

Minimum cost: $498.

Primal solution: $(13, 2, 0, 0, 0, 0, 7)$. The 7 means that we produce more of product 3 than we need.

Dual solution: $(6, 11, 0)$. These are the marginal costs.

Change in the Model

Remember: Dual solution: $(6, 11, 0)$.

Introduce a **fifth process** which produce $4P_1$ and $9P_2$. What is the break-even cost?

$$4 \cdot 6[\$/time] + 9 \cdot 11[\$/time] = 123[\$/time] \quad (6)$$

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Introduce a **new product** which needs 1,2 and 1 of product P_1, P_2 and P_3 respectively and a fix cost of \$50. What is the break even price?

$$1 \cdot 6 + 2 \cdot 11 + 1 \cdot 0 + 50 = 28 + 50 = \$78. \tag{7}$$

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Warning! A degenerate optimal primal solution may have corresponding dual solution**S**, which may give different positive and negative marginal values.

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- Introducing a new (Non-negative) Variable
- Ranging the Cost Coefficient (In-Out Coefficient) in a Non-basic Column Vector
- Ranging a Basic Cost Coefficient
- Ranging the RHS Constants

Introducing a new (Non-negative) Variable

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Optimality condition can be written as

$$\bar{c}_8 = c_8 - \bar{\pi} A_{.,8} \geq 0 \iff c_8 \geq \bar{\pi} A_{.,8}. \tag{8}$$

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We can calculate this as [1]

$$\bar{c}_8 = c_8 - (-6, 11, 0, 1)(4, 9, 0, c_8)^{\mathrm{T}} = c_8 - 123 \tag{9}$$

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Ranging the Cost Coefficient in a Non-basic Column Vector

How much can we change c_4 and not change the primal solution. The solution remains if $c_j \geq \bar{\pi} A_{.,j}$.

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$$

This now gives

$$
\bar{c}_4 = (-6, -11, 0, 1)(1, 2, 3, c_4)^{\mathrm{T}} = c_4 - 28 \implies c_4 \geq 28. \qquad (10)
$$

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This now gives

$$\bar{c}_4 = (-6, -11, 0, 1)(1, 2, 3, c_4)^{\mathrm{T}} = c_4 - 28 \implies c_4 \geq 28. \tag{10}$$

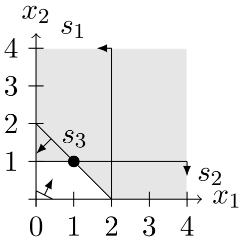
The same method can be used for the In-Out Coefficient.

Ranging a Basic Cost Coefficient

We cannot use the same method as for non-basic since we get a change in the dual solution. We work with c_1 and write dual basic solution as $\pi(c_1) = c_b' B^{-1}$. Then we have the condition $\bar{c}_j(c_1) = c_j - \pi(c_1)A_{\cdot,j} \geq 0$ for all nonbasic variables x_j.

We can thus find a parametric description of $\pi(c_1)$ and then evaluate $\bar{c}_j(c_1)$ and derive condition of c_1.

Example of Ranging a Basic Cost Coefficient



x_1	x_2	s_1	s_2	s_3	$-z$	b
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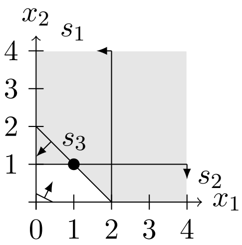

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$$B^{-1} = \begin{bmatrix} 0 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}, \ \bar{\pi} = c_B B^{-1} = (0, 1, 1) \tag{11}$$

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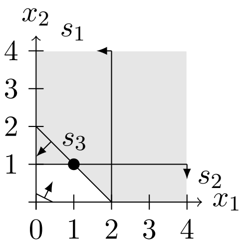

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c_1	2	0	0	0	1	0

$$B^{-1} = \begin{bmatrix} 0 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}, \ \bar{\pi} = c_B B^{-1} = (0, 2 - c_1, c_1) \tag{11}$$

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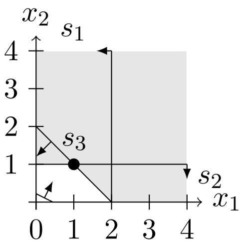

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c_1	2	0	0	0	1	0

$$B^{-1} = \begin{bmatrix} 0 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}, \; \bar{\pi} = c_B B^{-1} = (0, 2 - c_1, c_1) \tag{11}$$

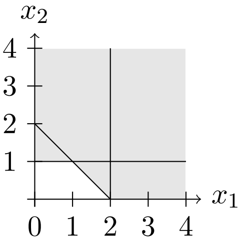
$$(0,0) - (0, 2 - c_1, c_1) \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = (c_1 - 2, -c_1) \overset{!}{\leq} 0 \implies \tag{12}$$

$$0 \leq c_1 \leq 2 \tag{13}$$

Example of Ranging a Basic Cost Coefficient

$$0 \leq c_1 \leq 2$$

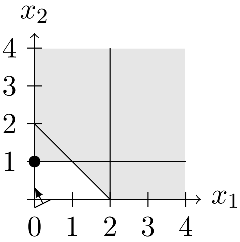
If we choose $c_1 = ?$



Example of Ranging a Basic Cost Coefficient

$$0 \leq c_1 \leq 2$$

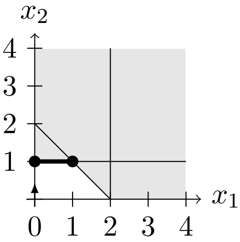
If we choose $c_1 = -1$



Example of Ranging a Basic Cost Coefficient

$$0 \leq c_1 \leq 2$$

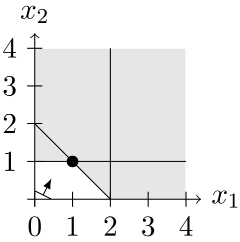
If we choose $c_1 = 0$



Example of Ranging a Basic Cost Coefficient

$$0 \leq c_1 \leq 2$$

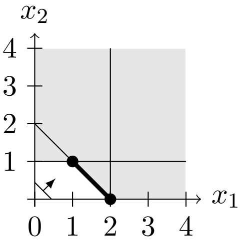
If we choose $c_1 = 1$



Example of Ranging a Basic Cost Coefficient

$$0 \leq c_1 \leq 2$$

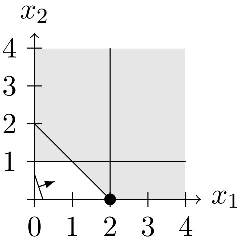
If we choose $c_1 = 2$



Example of Ranging a Basic Cost Coefficient

$$0 \leq c_1 \leq 2$$

If we choose $c_1 = 3$



Ranging the RHS Constants

A change in the RHS does not affect the dual feasibility but the primal. We have thus the condition

$$B^{-1}b' \geq 0, \tag{14}$$

where b' is the new RHS and from this can we derive conditions of the element in b'.