ALGORITHMS FOR SOLVING LPs

ULEDI NGULO

Linkoping University

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The Primal and Dual Degeneracy of a Basic Vector for an LP in Standard Form

$$\begin{aligned} \text{minimize} \quad & z(x) = cx \\ \text{s. to} \quad & Ax = b \qquad (1) \\ & x \geq 0 \end{aligned}$$

$$A \in \mathbb{R}^{m\times n};\ c \in \mathbb{R}^n;\ b \in \mathbb{R}^m; x \in \mathbb{R}^n$$

- Primal non-degenerate If every entry in the basic values vector $B^{-1}b$ is nonzero.
- Primal degenerate if at least one entry in the basic values vector $B^{-1}b$ is zero.

- Dual non-degenerate If none of the nonbasic dual slacks $\overline{c_j}$ have 0−value at its dual basic solution ($\overline{c_j} = c_j - c_B B^{-1} A_{.j}$ is nonzero for every nonbasic variables x_j).
- Dual degenerate If at least one of the nonbasic dual slacks $\overline{c_j}$ has 0−value at its dual basic solution ($\overline{c_j} = c_j - c_B B^{-1} A_{.j}$ is zero for at least one of the nonbasic variables x_j).

Tableau 1

BV	x_1	x_2	x_3	x_4	x_5	$-z$	RHS
x_1	1	0	1	−1	1	0	3
x_2	0	1	−1	1	1	0	4
$-z$	0	0	−2	3	2	1	−10

$x_j \geq 0$ for all j, minimize z.

Tableau 2

BV	x_1	x_2	x_3	x_4	x_5	$-z$	RHS
x_1	1	0	1	−1	1	0	5
x_2	0	1	−1	1	1	0	0
$-z$	0	0	2	3	2	1	−10

$x_j \geq 0$ for all j, minimize z.

Sufficient Conditions for Checking the Uniqueness of Primal and Dual Optimum Solutions

Theorem

Consider the LP in standard form and let x_B be an optimum basic vector for it. Let x_D be the vector of nonbasic variables(i.e., those not x_B). Let the basic, nonbasic partition of the canonical tableau wrt x_B be("BV" is abbreviation for "basic vector")

Canonical tableau

BV	x_B	x_D	$-z$	Updated RHS
x_B	I	$\bar{D}$	0	$\bar{b}$
$-z$	0	$\bar{c}_D$	1	$-\bar{z}$

$x_j \geq 0$ for all j, min z

Theorem(Cont'd)

If $\overline{c_D} > 0$(i.e., all nonbasic relative cost coefficients are positive or x_B is dual nondegenerate), then $\overline{x} = (x_B, x_D) = (\overline{b}, 0)$ is the unique primal optimum solution for this LP.
If $\overline{b} > 0$(all updated RHS constants>0,or x_B is primal nondegenerate), then the dual optimum solution is unique for this problem.

Procedure to check if the BFS corresponding to an optimum basic vector x_B is the unique optimum solution

- **Example 1:** consider the following LP in standard form, for which the optimum canonical tableau *wrt* the basic vector (x_1, x_2, x_3) is given.

BV	x_1	x_2	x_3	x_4	x_5	x_6	x_7	$-z$	$\bar{b}$	Ratio
x_1	1	0	0	−1	1	1	2	0	0	
x_2	0	1	0	$\boxed{1}$	−1	2	1	0	2	2
x_3	0	0	1	2	2	4	3	0	6	3
$-z$	0	0	0	0	0	10	20	1	−100	$\theta = 2$

$x_j \geq 0$ for all j, min z

- The BFS $\overline{x} = (0, 2, 6, 0, 0, 0, 0)^T$ is an optimum solution with optimum objective value 100. .
- $\overline{c}_4, \overline{c}_5$ are zero and $\overline{c}_6, \overline{c}_7$ are positive. so, $x_6 = x_7 = 0$, then any feasible solution of the above LP is an optimum solution.

- x_4 enters the basic vector with nondegenerate pivot step.

BV	x_1	x_2	x_3	x_4	x_5	x_6	x_7	$-z$	$\bar{b}$	Ratio
x_1	1	1	0	0	0	3	3	0	2	
x_4	0	1	0	1	−1	2	1	0	2	
x_3	0	−2	1	0	4	0	1	0	2	
$-z$	0	0	0	0	0	10	20	1	−100	

This gives us an alternate optimum BFS $\widehat{x} = (2, 0, 2, 2, 0, 0, 0)$ *wrt* new basic vector (x_1, x_4, x_3).

The Optimum Face for an LP

Definition

The optimum face of any LP is the set of its optimum solutions.

Consider an LP in standard form

$$\begin{aligned} \text{minimize} \quad & z(x) = cx \\ \text{s. to} \quad & Ax = b \\ & x \geq 0 \end{aligned} \tag{2}$$

Let $x \in K$ where K is a convex polyhedron and given x^* as any optimum solution for this LP, then optimum face is the set of feasible solutions of the following system of constraints.

$$\begin{aligned} Ax &= b \\ cx &= cx^* \\ x &\geq 0. \end{aligned}$$

Mathematical Equivalence of LP to the Problem of Finding a Feasible Solution of a System of Linear Constraints Involving Inequalities

Consider a Primal problem

$$\begin{aligned} \text{minimize} \quad & f\xi \\ \text{s. to} \quad & F\xi = h \\ & G\xi \geq g. \end{aligned} \tag{3}$$

Let π, μ be dual vectors, then its dual problem is

$$\begin{aligned} \text{maximize} \quad & \pi h + \mu g \\ \text{s. to} \quad & \pi F + \mu G = f \\ & \mu \geq 0. \end{aligned} \tag{4}$$

- If $\xi, (\pi, \mu)$ are primal, dual feasible solutions, then by weak duality we get, $f\xi - \pi h - \mu g \geq 0$.
- Any feasible solution satisfying the system containing both primal and dual constraints must satisfy $f\xi - \pi h - \mu g \leq 0$ as an equation.
- By duality theorem the solution will be a primal, dual pair of optimum solutions. So, instead of finding an optimum solution for (3), is equivalent to find the feasible solution to the system of linear constraints

$$\begin{aligned} F\xi &= h, \\ \pi F + \mu G &= f, \\ G\xi &\geq g, \\ \mu \geq 0; \quad -f\xi + \pi h + \mu g &\geq 0. \end{aligned}$$

Marginal Values and the Dual Optimum Solution

Consider an LP in standard form

$$\begin{aligned} \text{minimize} \quad & z(x) = cx \\ \text{s. to} \quad & Ax = b \\ & x \geq 0 \end{aligned} \tag{5}$$

- Marginal values are defined as rates of change of the optimum objective value in this LP per unit change in the RHS constants from their current values.
- Mathematically, MVs is $\frac{\partial f(b)}{\partial b_i} = \lim_{\varepsilon \to 0} \frac{f(b_1,\ldots,b_{i-1},b_i+\varepsilon,b_{i+1},\ldots,b_m) - f(b)}{\varepsilon}$

Theorem

If the LP (5) has a primal nondegenerate optimum BFS, then MVs wrt b_i exist for all i, and the unique optimum dual solution is the vector of MVs of (5).

Primal Revised Simplex Algorithm Using the Explicit Basis Inverse

Consider an LP problem in standard form

$$\begin{aligned} \text{minimize} \quad & z(x) = cx \\ \text{s. to} \quad & Ax = b \\ & x \geq 0 \end{aligned} \tag{6}$$

Original Tableau

x_1	$\dots$	x_j	$\dots$	x_n	$-z$	RHS
a_{11}	$\dots$	a_{1j}	$\dots$	a_{1n}	0	b_1
$\vdots$		$\vdots$		$\vdots$	$\vdots$	$\vdots$
a_{m1}	$\dots$	a_{mj}	$\dots$	a_{mn}	0	b_m
c_1	$\dots$	c_j	$\dots$	c_n	1	0

$x_j \geq 0$ for all j, minimize z

- The extended basis corresponding to $(x_B, -z)$ is

$$\mathcal{B} = \begin{pmatrix} \mathbf{B} & \mathbf{0} \\ c_{\mathbf{B}} & \mathbf{1} \end{pmatrix} \tag{7}$$

- The inverse tableau corresponding to $(x_B, -z)$ is

$$\mathcal{B}^{-1} = \begin{pmatrix} \mathbf{B} & \mathbf{0} \\ c_{\mathbf{B}} & \mathbf{1} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{B}^{-1} & \mathbf{0} \\ -\pi & \mathbf{1} \end{pmatrix} \tag{8}$$

where $\pi = c_{\mathbf{B}}\mathbf{B}^{-1}$ as dual basic vector.

- In general, we have the iverse tableau.

Inverse tableau wrt x_B

BV	Inverse Tableau		Basic values
x_B	B^{-1}	0	$\bar{b}$
$-z$	$-\pi$	1	$-\bar{z}$

Steps in an iteration of the primal simplex algorithm when $(x_B, -z)$ is the primal feasible basic vector

1. *Compute relative cost coefficients of nonbasic variables*
2. *Check the optimality criterion*
3. *Select the entering variable*
4. *Compute the updated column of the entering variable*
5. *Check the unboundedness criterion*
6. *Minimum ratio test to determine the dropping basic variable, and pivot step to update the inverse tableau*

Example

Original tableau

x_1	x_2	x_3	x_4	x_5	x_6	x_7	$-z$	b
1	0	0	0	-1	1	1	0	2
0	1	0	0	1	-1	1	0	1
0	0	1	0	2	20	1	0	5
0	0	0	1	0	-1	1	0	0
0	0	1	1	-1	29	-8	1	0

$x_j \geq 0$ for all j, minimize z

- The primal BFS corresponding to $x_{\mathbf{B}}$ is $\overline{x} = (2, 1, 5, 0, 0, 0, 0)^T$

First inverse tableau

Basic var.	Inverse tableau					Basic values	PC x_5	Ratios	
x_1	1	0	0	0	0	2	−1		
x_2	0	1	0	0	0	1	$\boxed{1}$	1/1	PR
x_3	0	0	1	0	0	5	2	5/2	
x_4	0	0	0	1	0	0	0		
$-z$	0	0	−1	−1	1	−5	−3	Min. $= \theta = 1$	

PC pivot column, *PR* pivot row

- The cost coefficient for nonbasic variables x_5, x_6, x_7 is the vector $(\overline{c}_5, \overline{c}_6, \overline{c}_7) = (-3, 10, -10)^T$
- The solution $x(\lambda) = (2+\lambda, 1-\lambda, 5-2\lambda, 0, \lambda, 0, 0)^T$, $z(\lambda) = 5 - 3\lambda$.
- The minimum ratio is $\theta =$min$\{1/1, 5/2\} = 1$, then $\lambda \leq 1$. suppose $\lambda = 1$, then we drop x_2 form the present basic variable

Second inverse tableau

Basic var.	Inverse tableau					Basic values	PC x_7	Ratios	
x_1	1	1	0	0	0	3	2	3/2	
x_5	0	1	0	0	0	1	1	1/1	
x_3	0	−2	1	0	0	3	−1		
x_4	0	0	0	1	0	0	$\boxed{1}$	0/1	PR
$-z$	0	3	−1	−1	1	−2	−7	Min. $= \theta = 0$	

PC pivot column, *PR* pivot row

- The new BFS is $\widehat{x} = (3, 0, 3, 0, 1, 0, 0)^T, \widehat{z} = 2$.
- The cost coefficient for nonbasic variables x_2, x_6, x_7 is the vector $(\overline{c}_2, \overline{c}_6, \overline{c}_7) = (3, 7, -7)^T$.

Third inverse tableau

Basic var.	Inverse tableau					Basic values
x_1	1	1	0	−2	0	3
x_5	0	1	0	−1	0	1
x_3	0	−2	1	1	0	3
x_7	0	0	0	1	0	0
$-z$	0	3	−1	6	1	−2

- The cost coefficient for nonbasic variables x_2, x_4, x_6 is the vector $(\overline{c}_2, \overline{c}_4, \overline{c}_6) = (3, 7, 0)^T$. All are ≥ 0, the optimality criterion is satisfied.
- The present BFS $\widehat{x} = (3, 0, 3, 0, 1, 0, 0)^T$ in an Optimum solution, $\widehat{z} = 2$ and $\widehat{\pi} = (0, -3, 1, -6)^T$ is the dual solution.

Revised primal simplex method(Phase I,II) with Explicit Basis Inverse

Let the original tableau be

Original Tableau

x_1	$\dots$	x_j	$\dots$	x_n	$-z$	RHS
a_{11}	$\dots$	a_{1j}	$\dots$	a_{1n}	0	b_1
$\vdots$		$\vdots$		$\vdots$	$\vdots$	$\vdots$
a_{m1}	$\dots$	a_{mj}	$\dots$	a_{mn}	0	b_m
c_1	$\dots$	c_j	$\dots$	c_n	1	0

$x_j \geq 0$ for all j, minimize z

- Search for a unit basic vector in the original tableau. If a full unit basic vector,$x_{\mathbf{B}}$, is found, then it will be an initial feasible solution and we apply revised simplex algorithm.

- If a full unit basic vector is not attained in the original tableau implies that we don't have feasible solution.
- The simplex method divides the task of solving the problem into two phases.
- Phase 1 focuses on finding a BFS for the problem, ignoring the original objective function.
- The artificial variables are added to the rows that do not have basic variables.
- Then, we minimize the phase I objective function w starting with the unit basic vector $x^1_{\mathbf{B}}$. with the artificial variables introduced.
- If the sum of all artificial variables $w = 0$, then we drop all artificial variables and the associated objective function we go to phase II.

Phase I original tableau

Original			Artificial					
x_1	$\dots$	x_n	x_{n+1}	$\dots$	x_{n+m-r}	$-z$	$-w$	RHS
a_{11}	$\dots$	a_{1n}				0	0	b_1
$\vdots$		$\vdots$		Missing unit		$\vdots$	$\vdots$	$\vdots$
a_{m1}	$\dots$	a_{mn}		vectors		0	0	b_m
c_1	$\dots$	c_n	0	$\dots$	0	1	0	0
0	$\dots$	0	1	$\dots$	1	0	1	0

All variables ≥ 0, minimize w

Phase I Inverse Tableau wrt x_B

BV	Inverse tableau			Basic values
x_B	B^{-1}	0	0	$\bar{b}$
$-z$	$-\pi$	1	0	$-\bar{z}$
$-w$	$-\sigma$	0	1	$-\bar{w}$

- Its relative cost coefficient is given by

$$\overline{d}_j = \begin{pmatrix} -\sigma, & 0, & 1 \end{pmatrix} \begin{pmatrix} A_{.j} \\ c_j \\ d_j \end{pmatrix}$$

- During phase I, the only artificial variables left in the original tableau are those which are still basic variables.
- Phase I termination condition is When the relative cost coefficient $\overline{d}_j \geq 0$ for all original problem varaibles x_j.

How to find a feasible solution to a system of linear constraints

- if the system consists of linear equations only, then we apply the Gaussian elimination to find the feasible solution.
- If the system involves linear inequalities and/or bounds on the variables, we write it in std form and apply the phase I of the primal simplex method to find afeasible solution.

Infeasibility Analysis

Consider an LP problem in standard form

$$\begin{aligned} \text{minimize} \quad & z(x) = cx \\ \text{s. to} \quad & Ax = b \\ & x \geq 0 \end{aligned} \tag{9}$$

- Suppose the problem is infeasible, then it is required to be modified so that to be feasible.
- One way of modifying is making changes in the RHS constants b_i usually involves some expenses, typically proportional to the amount of change, and may be different rates for different i.
- We can modify $b = (b_i)$ by considering the final phase I solution.

- Consider the original tableau

x_1	x_2	x_3	x_4	x_5	$-z$	b
2	3	1	-1	0	0	10
1	2	-1	0	1	0	5
1	1	2	0	0	0	4
1	2	3	0	0	1	0

$x_j \geq 0$ for all j, minimize z

- The vector $b = (10, 5, 4)^T$ and in the final phase I solution obtained for this example, only artificial variable t_1, basic variable, has positive value of 1.

- So, changing the vector b to $(9, 5, 4)$, the becomes feasible.
- The final phase I inverse tableau for this modified problem is obtained from the original problem by changing the final value of the basic variable t_1 to 0.

Basic var.	Inverse tableau					Basic values
t_1	1	-1	-1	0	0	0
x_2	0	1	-1	0	0	1
x_1	0	-1	2	0	0	3
$-z$	0	-1	0	1	0	-5
$-w$	-1	1	1	0	1	0

THANK YOU!