Revised Simplex Variants of the Primal and Dual Simplex Methods and Sensitivity Analysis

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1. Revised Simplex Methods Using the Product Form of the Inverse
2. Finding the Optimum Face of an LP(Alternate Optimum Solutions)
3. The Dual Simplex Algorithm

Pivot Matrices

- Suppose $D = (d_{ij})$ is a matrix of $p \times q$, and consider performing a GJ pivot step on D, there will be a square matrix P of order $p \times p$ called the *pivot matrix* corresponding to this pivot step satisfying the property that $\overline{D} = PD$.
- The pivot matrix P will be the same as the unit matrix I of order p except for one column known as its *eta*(η)*column*. let $d_{rs} \neq 0$.

$$
\begin{array}{c|c|c}
\text{PC} & \eta\text{-col.} & \text{Pivot matrix } P \\
\hline
\begin{array}{c} d_{1s} \\ \vdots \\ d_{r-1,s} \\ \boxed{d_{rs}} \\ d_{r+1,s} \\ \vdots \\ d_{ms} \end{array}
&
\begin{array}{c} -d_{1s}/d_{rs} \\ \vdots \\ -d_{r-1,s}/d_{rs} \\ 1/d_{rs} \\ -d_{r+1,s}/d_{rs} \\ \vdots \\ -d_{ms}/d_{rs} \end{array}
&
\begin{pmatrix}
1 \ldots 0 & -d_{1s}/d_{rs} & 0 \ldots 0 \\
\vdots \ \ddots \ \vdots & \vdots & \vdots \quad \vdots \\
0 \ldots 1 & -d_{r-1,s}/d_{rs} & 0 \ldots 0 \\
0 \ldots 0 & 1/d_{rs} & 0 \ldots 0 \\
0 \ldots 0 & -d_{r+1,s}/d_{rs} & 1 \ldots 0 \\
\vdots \quad \vdots & \vdots & \vdots \ \ddots \ \vdots \\
0 \ldots 0 & -d_{ms}/d_{rs} & 0 \ldots 1
\end{pmatrix}
\end{array}.
$$

Example

Let D be the matrix $(A : b)$in the system of linear equations in the top tableau.

x_1	x_2	x_3	x_4	RHS b
7	0	-1	1	10
4	-6	$\boxed{2}$	2	4
5	-2	1	0	15
9	-3	0	2	12
2	-3	1	1	2
3	1	0	-1	13

Example (cont'd)

- Performing a GJ pivot step in the column of x_3 and the pivot element is boxed. The pivot matrix corresponding to this pivot operation is

$$P = \begin{pmatrix} 1 & 1/2 & 0 \\ 0 & 1/2 & 0 \\ 0 & -1/2 & 1 \end{pmatrix}.$$

- The tableau obtained at the bottom of the original tableau say $\overline{D}$ is equivalent to PD (i.e., $\overline{D} = PD$).

A general Iteration in the Revised Simplex Method Using the Product Form of the Inverse

- Suppose P_0 is the initial inverse tableau. P_0
  (i) differs from the unit matrix in the last row only if we begin with phase II and
  (ii) differs from the unit matrix in the two last rows only if we begin with phase I.
- P_0 is also like a pivot matrix and it can be generated by storing only those one or two rows.
- *Current string of pivot matrices* This is the ordering of the pivot matrices in the storage that the newest is always join the string on the left side of all those generated above.

A general Iteration in the Revised Simplex Method Using the Product Form of the Inverse

- In the various steps in this iteration of the revised simplex method, the inverse tableau is used only twice:
  (i) To get its last row to compute the relative cost coefficients of the nonbasic variables.
  (ii) To compute the pivot column which is the updated column of the entering variable selected.
- Let the string of pivot matrices at this stage be $P_r, P_{r-1}, ..., P_1, P_0$, then the inverse tableau formula is $P_r P_{r-1} ... P_1 P_0$.
- The last row of the inverse tableau $=$(last row of the unit matrix of the same order as the inverse tableau)(inverse tableau) $= (0, ..., 0, 1)(P_r P_{r-1} ... P_1 P_0)$. (This is left-to-right string multiplication)

A general Iteration in the Revised Simplex Method Using the Product Form of the Inverse

- Updating the entering column
  Let
  x_s = the entering variable
  $A_{.s}$ = the column vector in the original tableau of x_s.
  $\overline{A}_{.s}$ = the updated column of x_s.
  Therefore, $\overline{A}_{.s} = P_r P_{r-1} ... P_1 P_0 A_{.s}$
  This is left-to-right string multiplication

Transition from Phase I to Phase II

Consider solving the LP in standard form

$$\begin{aligned} \text{minimize} \quad z(x) &= cx \\ \text{s. to} \quad Ax &= b \\ x &\geq 0 \end{aligned} \tag{1}$$

- Solving this LP by the revised simplex method using the product form of the inverse(PFI), during phase I, pivot matrices will be of order $m+2$ while phase II will be of order $m+1$.
- At the end of phase I all $(m+2)$th row and column will be deleted from each pivot matrix in the string, then we move to phase II.
- Then the present string becomes the string corresponding to the present basic vector for the phase II problem, and you can begin phase II with it.

Transition from Phase I to Phase II

- This revised simplex method using PFI is quite difficult for hand computation to solve even a small size LPs. However, its computer implementations offer superior performance to solve LP models involving up to several hundreds of constraints.
- *Advantage of the PFI implementation:* When the eta column is sparse (i.e., if the fraction of nonzero entries in it is small) we say the revised simplex method using PFI continues its superior performance.
- The advantage of PFI will disappear if the number of current step(say r) over the number of constraints in the model being solved (say m) exceeds 2 (i.e., $r/m > 2$). At this stage, the method goes to an operation called *reinversion*.

Reinversions in the Revised Simplex Method Using PFI

- Let $x_{\mathbf{B}}$ = the present basic vector in the current iteration, with $\mathbf{B}, \mathcal{B}, \mathbf{c_B}$,the associated basis, augmented basis, basic vector, respectively.
- Let $\overline{x}, \overline{\pi}$ be the associated primal and dual basic solutions respectively.
- $\parallel Ax - b \parallel$ and $\parallel c_{\mathbf{B}} - \overline{\pi}\mathbf{B} \parallel$ can be used as measures to monitor the effect of round off, if the measures are small than a specified tolorances we continue with the iterations in the method otherwise we go to the reinversions.
- Reinversion involves getting rid of the present string of pivot matrices. from the present augmented basis $\mathcal{B}$, carry out the pivot operations needed to compute $\mathcal{B}^{-1}$.
- The new string of pivot matrices corresponding to the various pivot steps carried out to compute $\mathcal{B}^{-1}$ consists of $m + 1$ in phase I or $m + 2$ in phase II. With new string, resume the iterations of the revised simplex method for solving the LP model.

Finding the Optimum Face of an LP

- If some nonbasic relative cost coefficients *wrt* $x_{\mathbf{B}}$ are 0, by bringing the corresponding nonbasic variables into this basic vector, we can get alternate optimum BFSs for this LP.
- Consider the following LP

Original problem

x_1	x_2	x_3	x_4	x_5	x_6	x_7	$-z$	b
0	0	1	1	−1	−5	3	0	7
1	0	0	−1	−1	−3	−8	0	9
0	1	0	−1	−1	0	4	0	1
−1	−1	−1	10	6	8	1	1	0

$x_j \geq 0$ for all j, minimize z

an optimum basic vector is $x_{\mathbf{B}} = (x_3, x_1, x_2)$. item The corresponding inverse tableau is

Inverse tableau

Basic var.	Inverse tableau				Basic values
x_3	1	0	0	0	7
x_1	0	1	0	0	9
x_2	0	0	1	0	1
$-z$	1	1	1	1	17

- The primal BFS $\overline{x} = (9, 1, 7, 0, 0, 0)^T$ with $\overline{z} = -17$. So, any feasible solution in which $\overline{z} = -17$ is an optimum solution to this LP.
- The set of feasible solutions of the following system of constraints is optimum face of the LP above,

Optimum face representation

x_1	x_2	x_3	x_4	x_5	x_6	x_7	b
0	0	1	1	−1	−5	3	7
1	0	0	−1	−1	−3	−8	9
0	1	0	−1	−1	0	4	1
−1	−1	−1	10	6	8	1	−17

$x_j \geq 0$ for all j.

- The relative cost coefficients of the nonbasic variables are $(\overline{c}_4, \overline{c}_5, \overline{c}_6, \overline{c}_7) = (9, 3, 0, 0)$.

- Any BFS obtained by performing pivot steps with x_6, x_7 entering variables (since $\overline{c}_6 = \overline{c}_7 = 0$) is an optimum solution.
- Since $\overline{c}_4, \overline{c}_5 > 0$, then any feasible solution of the originall system in which $x_4 = x_5 = 0$ is an optimum solution of the LP. S, an optimum fase representation can be,as follows,

Second representation of the optimum face

x_1	x_2	x_3	x_4	x_5	x_6	x_7	b
0	0	1	1	−1	−5	3	7
1	0	0	−1	−1	−3	−8	9
0	1	0	−1	−1	0	4	1
0	0	0	1	0	0	0	0
0	0	0	0	1	0	0	0

$x_j \geq 0$ for all $j = 1$ to 5.

The Dual Simplex Algorithm

Consider an LP in standard form

$$\begin{aligned} \text{minimize} \quad & z(x) = cx \\ \text{s. to} \quad & Ax = b \\ & x \geq 0 \end{aligned} \tag{2}$$

Original tableau

x_B	x_D	$-z$	RHS
B	D	0	b
c_B	c_D	1	0

All $x_j \geq 0$, min z

- We find the basic vector $x_{\mathbf{B}}$ which satisfy the following properties:

1. Primal feasibility: $\bar{b} = B^{-1}b \geq 0$,
2. Dual feasibility: $\bar{c}_D = c_D - (c_B B^{-1})D \geq 0$.

- Example

Canonical tableau wrt (x_1, x_2, x_3)

BV	x_1	x_2	x_3	x_4	x_5	x_6	$-z$	RHS
x_1	1	0	0	−1	−2	2	0	−6
x_2	0	1	0	1	−1	−2	0	−2
x_3	0	0	1	1	1	−1	0	4
$-z$	0	0	0	3	8	10	1	−100

- To solve LPs like this we use dual simplex method which tries to attain primal feasibility while maintaining dual feasibility by performing a sequence of pivot steps changing the basic vector by one variable in each step.

Steps in a general iteration of the dual simplex algorithm

1. Check optimality (primal feasibility)
2. Select the pivot row.
3. Check primal infeasibility.
4. Dual simplex minimum ratio test.
5. Pivot step.
6. Change in objective value in this iteration.

Example Consider the following LP for which the canonical tableau *wrt* the basic vector $x_{\mathbf{B}} = (x_1, x_2, x_3)$ is given below.

Canonical tableau wrt (x_1, x_2, x_3)

BV	x_1	x_2	x_3	x_4	x_5	x_6	$-z$	RHS
x_1	1	0	0	−1	−2	2	0	−6
x_2	0	1	0	1	−1	−2	0	−2
x_3	0	0	1	1	1	−1	0	4
$-z$	0	0	0	3	8	10	1	−100

The various canonical tableau obtained are given below

				PC					
BV	x_1	x_2	x_3	x_4	x_5	x_6	$-z$	RHS	
x_1	1	0	0	$\boxed{-1}$	−2	2	0	−6	PR
x_2	0	1	0	1	−1	−2	0	−2	
x_3	0	0	1	1	1	−1	0	4	
$-z$	0	0	0	3	8	10	1	−100	
Ratios				3	4				

					PC				
x_4	−1	0	0	1	2	−2	0	6	
x_2	1	1	0	0	−3	0	0	−8	
x_3	1	0	1	0	$\boxed{-1}$	1	0	−2	PR
$-z$	3	0	0	0	2	16	1	−118	
Ratios					2				

			PC						
x_4	1	0	2	1	0	1	0	2	
x_2	−2	1	$\boxed{-3}$	0	0	−3	0	−2	PR
x_5	−1	0	−1	0	1	−1	0	2	
$-z$	5	0	2	0	0	18	1	−122	
Ratio	5/2		2/3			6			

Optimum canonical tableau

x_4	−1/3	2/3	0	1	0	−1	0	2/3
x_3	2/3	−1/3	1	0	0	1	0	2/3
x_5	−1/3	−1/3	0	0	1	0	0	4/3
$-z$	11/3	2/3	0	0	0	16	1	−370/3

- In the last tableau, all the updated RHS constants are>0, then is an optimum tableau and the method terminates.
- the optimal solution is $\overline{x} = (0,0,2/3,2/3,4/3,0)^T$,$z^* = 370/3$.

THANK YOU!