Linear Optimization

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Overview

1. THE DECOMPOSITION PRINCIPLE
   - INTRODUCTION
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Introduction

- When we have a very large scale problems and /or special structured linear programming problems
- Example having many thousands of rows and unlimited number of columns.
- In such problems some methods must be applied to convert the problem into one or more small problem of manageable size.
- Application of decomposition principle (Dantzig Wolfe, Benders and Lagrangian relaxation techniques) can do this.

Introduction Cont..

- The problem can have manageable size, but with some constraints posses a special structure that permits efficient handling
- We can partition the problem into two sets
- General or complicating constraints and special structure and Constraints having special structure
- The linear program over general constraints is called the MASTER problem and with special constraints is called SUBPROBLEM
- The Master Problem passes down a continually a revised set of cost coefficients (or prices) to the Subproblem and receives from subproblem a new column/columns based on cost coefficients
- This procedure is called a column generation method or a price directive algorithm

The Decomposition Algorithm.

- Consider the linear problem where X is a polyhedral set representing special structured constraints, A is na $m \times n$ matrix, and b is an $m-$vector:

$$\begin{aligned} &\text{Minimize } cx \\ &\text{subject to } Ax = b \\ &x \in X \end{aligned}$$

- By assuming X is nonempty and bounded, the any point $x \in X$ can be represented as a convex combination of the finite number of extreme points of X

The Decomposition Algorithm Cont..

- Denoting these points by $x_1, x_2, ..., x_t$ any $x \in X$ can be represented as

$$x = \sum_{j=1}^{t} \lambda_j x_j$$

$$\sum_{j=1}^{t} \lambda_j = 1$$

$$\lambda_j \geq 0, \quad j = 1, ..., t$$

The Decomposition Algorithm Cont.

- This can be transformed into MASTER problem in the variables $\lambda_1, \lambda_2, ..., \lambda_t$

$$\text{Minimize } \sum_{j=1}^{t} (cx_j)\lambda_j x_j$$

Subject to

$$\sum_{j=1}^{t} (Ax_j)\lambda_j = b \tag{1}$$

$$\sum_{j=1}^{t} \lambda_j = 1 \tag{2}$$

$$\lambda_j \geq 0, \quad j = 1, ..., t \tag{3}$$

- t is a number of extreme point and is usually very large
- Explicit enumeration and solving extreme points is an onerous task

INITIALIZATION STEP

- Find an initial basic feasible solution for the system defined by equation 1,2 and 3. Let the basis be B and from the following *master array* where $(w, \alpha) = \hat{c}_B B^{-1}$

BASIS INVERSE	RHS
$(\mathbf{w}, \alpha)$	$\hat{\mathbf{c}}_B \bar{\mathbf{b}}$
$\mathbf{B}^{-1}$	$\bar{\mathbf{b}}$

MAIN STEP

1. Solve the following subproblem:

$$\text{Maximize } (wA - c)x + \alpha$$
$$\text{Subject to } x \in X$$

Let x_k be an optimal basic feasible solution with objective value $z_k - \hat{c}_k$. If $z_k - \hat{c}_k = 0$ stop; the basic feasible solution of the last master step is an optimal solution to the overall problem. Otherwise, go to step 2

2. Let

$$y_k = \begin{bmatrix} Ax_k \\ 1 \end{bmatrix}$$

and adjoin the updated column

$$\begin{bmatrix} z_k - \hat{c}_k \\ y_k \end{bmatrix}$$

to the master array. Pivot at y_{rk} where the index r is determined as follows:

$$\frac{\overline{b}_r}{y_{rk}} = \min_{1 \leq i \leq m+1} \left\{ \frac{\overline{b}}{y_{ik}} : y_{ik} > 0 \right\}$$

This updates the dual variables, the basis inverse and the right hand side. Repeat step 1 using the resulting updates master array.

Calculation and Use of the Lower Bound

- The decomposition algorithm stops when $z_k - \hat{c}_k = 0$.
- Due to very large number of variables $(\lambda_1, \lambda_2, ..., \lambda_t)$, continuing until this condition is satisfied may be time consuming for large problem.
- Since the decomposition algorithm generates feasible points having non worsen objective values via the master problem
- we have the sequence of non increasing upper bounds
- Hence we may stop when the difference between the objective value of the current feasible point and the available lower bound is within an acceptable tolerance
- This may not give the true optimal solution but may guarantee a good feasible solution, within any desired accuracy from the optimum.

Calculation and Use of the Lower Bound cont...

- Consider the following subproblem:

$$\begin{aligned} &\text{Maximize} \;\; (wA - c)x + \alpha \\ &\text{subject to} \;\; x \in X, \end{aligned}$$

  where w is the dual vector passed from the master step.
- Let the optimal objective value of the foregoing subproblem be $z_k - \hat{c}_k$.
- Let x be any feasible solution to the overall problem, that is $Ax = b$ and $x \in X$. By definition of $z_k - \hat{c}_k$, and because $x \in X$, we have

$$(wA - c)x + \alpha \leq z_k - \hat{c}_k$$

- $Ax = b$ implies that

$$cx \geq wAx - (z_k - \hat{c}_k) + \alpha = wb + \alpha - (z_k - \hat{c}_k) = \hat{c}_B \overline{b} - z_k - \hat{c}_k.$$

Calculation and Use of the Lower Bound cont...

- Since this is true for each $x \in X$ with $Ax = b$, then

$$\min_{Ax=b; x\in X} \quad cx \geq \hat{c}_B\overline{b} - (z_k - \hat{c}_k)$$

- In other words, $\hat{c}_B\overline{b} - (z_k - \hat{c}_k)$ is a lower bound on the optimal objective value for the overall problem.
- Note $\hat{c}_B\overline{b}$ is the current best upper bound
- However, the lower bounds generated need not be monotone and we need to maintain the best (greatest) lower bound

Numerical Example.

- Consider the following problem

$$
\begin{array}{lrcrcrcrcl}
\text{Minimize} & -2x_1 & - & x_2 & - & x_3 & + & x_4 & & \\
\text{subject to} & x_1 & & & + & x_3 & & & \leq & 2 \\
 & x_1 & + & x_2 & & & + & 2x_4 & \leq & 3 \\
 & x_1 & & & & & & & \leq & 2 \\
 & x_1 & + & 2x_2 & & & & & \leq & 5 \\
 & & & & - & x_3 & + & x_4 & \leq & 2 \\
 & & & & & 2x_3 & + & x_4 & \leq & 6 \\
 & x_1, & & x_2, & & x_3, & & x_4 & \geq & 0.
\end{array}
$$

- Note the third and fourth constraints involve only x_1 and x_2, whereas the fifth and sixth constraints involve only x_3 and x_4.
- If we let X consist of the last four constraints in addition to the nonegativity restrictions
- Then minimizing a linear function over X becomes a simple process since the subproblem can be decomposed into two subproblems.

Numerical Example cont...

- Therefore, we shall handle the first two constraints as $Ax \leq b$ where

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 2 \end{bmatrix}$$

and

$$b = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

- And then remaining constraints as $x \in X$
- Note that any point (x_1, x_2, x_3, x_4) in X must have its first two components and its last two components in respective sets X_1 and X_2 that are depicted in the figure 7.1

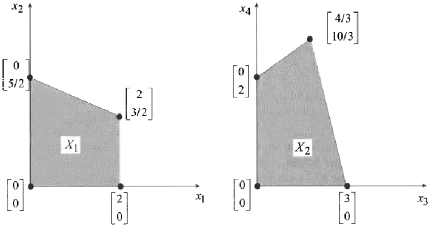

Figure 7.1. Representation of X by two sets.

Initialization Step

- The problem is reformulated as follows, where $x_1, x_2, ..., x_t$ are the extreme points of X, $\hat{c}_j = cx$ for $j = 1, .., t$ and $s \geq 0$ is the slack vector

Numerical Example cont...

$$\begin{aligned} \text{Minimize} \quad & \sum_{j=1}^{t} \hat{c}_j \lambda_j \\ \text{subject to} \quad & \sum_{j=1}^{t} (\mathbf{A}\mathbf{x}_j)\lambda_j + \mathbf{s} = \mathbf{b} \\ & \sum_{j=1}^{t} \lambda_j = 1 \\ & \lambda_j \geq 0, \qquad j = 1,\ldots,t \\ & \mathbf{s} \geq \mathbf{0}. \end{aligned}$$

Let the starting basis consist of $\mathbf{s}$ and λ_1 where $\mathbf{x}_1 = (0, 0, 0, 0)$ is an extreme point of X with $\mathbf{c}\mathbf{x}_1 = 0$. Therefore,

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The vector $(\mathbf{w}, \alpha) = \hat{\mathbf{c}}_B \mathbf{B}^{-1} = \mathbf{0}\mathbf{B}^{-1} = \mathbf{0}$, and $\bar{\mathbf{b}} = \mathbf{B}^{-1}\begin{bmatrix} \mathbf{b} \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ 1 \end{bmatrix}$. This gives the following tableau, where the first three columns give (w_1, w_2, α) in row 0 and $\mathbf{B}^{-1}$ in the remaining rows:

	BASIS INVERSE			RHS
z	0	0	0	0
s_1	1	0	0	2
s_2	0	1	0	3
λ_1	0	0	1	1

Numerical Example cont...

Iteration 1

SUBPROBLEM

Solve the following subproblem:

$$\begin{array}{ll} \text{Maximize} & (\mathbf{wA}-\mathbf{c})\mathbf{x}+\alpha \\ \text{subject to} & \mathbf{x}\in X. \end{array}$$

Here, $(w_1, w_2) = (0, 0)$ from the foregoing array. Therefore, the subproblem is as follows:

$$\begin{array}{ll} \text{Maximize} & 2x_1+x_2+x_3-x_4+0 \\ \text{subject to} & \mathbf{x}\in X. \end{array}$$

This problem is separable in the vectors (x_1, x_2) and (x_3, x_4) and can be solved geometrically. Using Figure 7.1, it is easily verified that the optimal solution is $\mathbf{x}_2 = (2, 3/2, 3, 0)$ with objective value $z_2 - \hat{c}_2 = 17/2$. Since $z_2 - \hat{c}_2 = 17/2 > 0$,

Numerical Example cont...

then λ_2 corresponding to $\mathbf{x}_2$ is introduced into the basis. The lower bound equals $\hat{\mathbf{c}}_B\bar{\mathbf{b}} - (z_2 - \hat{c}_2) = 0 - 17/2$. Recall that the best objective value so far is 0.

MASTER STEP

$z_2 - \hat{c}_2 = 17/2$, and

$$\mathbf{A}\mathbf{x}_2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 3/2 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 7/2 \end{bmatrix}.$$

Accordingly,

$$\begin{bmatrix} \mathbf{A}\mathbf{x}_2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 7/2 \\ 1 \end{bmatrix}$$

is updated by premultiplying by $\mathbf{B}^{-1}$. This yields

$$\mathbf{y}_2 = \mathbf{B}^{-1} \begin{bmatrix} 5 \\ 7/2 \\ 1 \end{bmatrix} = \mathbf{I} \begin{bmatrix} 5 \\ 7/2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 7/2 \\ 1 \end{bmatrix}.$$

Numerical Example cont...

We therefore insert the column

$$\begin{bmatrix} z_2 - \hat{c}_2 \\ \mathbf{y}_2 \end{bmatrix} = \begin{bmatrix} 17/2 \\ 5 \\ 7/2 \\ 1 \end{bmatrix}$$

into the foregoing array and pivot. This leads to the following two tableaux (the λ_2 column is not displayed after pivoting):

	BASIS INVERSE			RHS	λ_2
z	0	0	0	0	17/2
s_1	1	0	0	2	(5)
s_2	0	1	0	3	7/2
λ_1	0	0	1	1	1

	BASIS INVERSE			RHS
z	−17/10	0	0	−17/5
λ_2	1/5	0	0	2/5
s_2	−7/10	1	0	8/5
λ_1	−1/5	0	1	3/5

The best–known feasible solution to the overall problem is given by

Numerical Example cont...

$$\begin{aligned} \mathbf{x} &= \lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2 \\ &= (3/5)(0,0,0,0) + (2/5)(2,3/2,3,0) = (4/5,3/5,6/5,0). \end{aligned}$$

The current objective value is $-17/5$. Also, $(w_1, w_2, \alpha) = (-17/10, 0, 0)$.

Iteration 2

Since $w_1 < 0$, s_1 is not eligible to enter the basis.

SUBPROBLEM

Solve the following problem:

$$\begin{array}{ll} \text{Maximize} & (\mathbf{wA} - \mathbf{c})\mathbf{x} + \alpha \\ \text{subject to} & \mathbf{x} \in X, \end{array}$$

where

$$\mathbf{wA} - \mathbf{c} = (-17/10, 0)\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 2 \end{bmatrix} - (-2,-1,-1,1) = (3/10, 1, -7/10, -1).$$

Numerical Example cont...

Therefore, the subproblem is as follows:

$$\begin{array}{ll} \text{Maximize} & (3/10)x_1 + x_2 - (7/10)x_3 - x_4 + 0 \\ \text{subject to} & \mathbf{x} \in X. \end{array}$$

This problem decomposes into two problems involving (x_1, x_2) and (x_3, x_4). Using Figure 7.1, the optimal solution is $\mathbf{x}_3 = (0, 5/2, 0, 0)$ with objective value $z_3 - \hat{c}_3$ =5/2. Since $z_3 - \hat{c}_3 > 0$, then λ_3 is introduced into the basis.

The lower bound is $\hat{\mathbf{c}}_B \overline{\mathbf{b}} - (z_3 - \hat{c}_3) = -17/5 - 5/2 = -5.9$. (Recall that the best–known objective value so far is −3.4.)

MASTER STEP

$z_3 - \hat{c}_3$ = 5/2, and

$$\mathbf{A}\mathbf{x}_3 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 5/2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 5/2 \end{bmatrix},$$

$$\mathbf{y}_3 = \mathbf{B}^{-1} \begin{bmatrix} \mathbf{A}\mathbf{x}_3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/5 & 0 & 0 \\ -7/10 & 1 & 0 \\ -1/5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 5/2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 5/2 \\ 1 \end{bmatrix}.$$

Numerical Example cont...

We therefore insert the column $\begin{bmatrix} z_3 - \hat{c}_3 \\ \mathbf{y}_3 \end{bmatrix}$ into the foregoing array and pivot. This leads to the following two tableaux (the λ_3 column is not displayed after pivoting):

	BASIS INVERSE			RHS		λ_3
z	$-17/10$	0	0	$-17/5$		5/2
λ_2	1/5	0	0	2/5		0
s_2	$-7/10$	1	0	8/5		5/2
λ_1	$-1/5$	0	1	3/5		(1)

	BASIS INVERSE			RHS
z	$-6/5$	0	$-5/2$	$-49/10$
λ_2	1/5	0	0	2/5
s_2	$-1/5$	1	$-5/2$	1/10
λ_3	$-1/5$	0	1	3/5

The best–known feasible solution to the overall problem is given by

$$\begin{aligned} \mathbf{x} &= \lambda_2 \mathbf{x}_2 + \lambda_3 \mathbf{x}_3 \\ &= (2/5)(2,3/2,3,0) + (3/5)(0,5/2,0,0) = (4/5,21/10,6/5,0). \end{aligned}$$

The current objective value is −4.9. Also, $(w_1, w_2, \alpha) = (-6/5, 0, -5/2)$.

Numerical Example cont...

Iteration 3

Since $w_1 < 0$, s_1 is not eligible to enter the basis.

SUBPROBLEM

Solve the following subproblem:

$$\begin{array}{ll} \text{Maximize} & (\mathbf{wA}-\mathbf{c})\mathbf{x}+\alpha \\ \text{subject to} & \mathbf{x}\in X, \end{array}$$

where

$$\mathbf{wA}-\mathbf{c}=(-6/5,0)\begin{bmatrix}1 & 0 & 1 & 0\\ 1 & 1 & 0 & 2\end{bmatrix}-(-2,-1,-1,1)=(4/5,1,-1/5,-1).$$

Therefore, the subproblem is as follows:

$$\begin{array}{ll} \text{Maximize} & (4/5)x_1+x_2-(1/5)x_3-x_4-5/2 \\ \text{subject to} & \mathbf{x}\in X. \end{array}$$

Using Figure 7. 1, the optimal solution is $\mathbf{x}_4=(2,3/2,0,0)$ with objective value $z_4-\hat{c}_4=3/5$, and so, λ_4 is introduced into the basis.

Numerical Example cont...

The lower bound is given by $\hat{\mathbf{c}}_B\bar{\mathbf{b}} - (z_4 - \hat{c}_4) = -49/10 - 3/5 = -5.5$. Recall that the best–known objective value so far is −4.9. If we were interested only in an approximate solution, we could have stopped here with the feasible solution $\mathbf{x} = (4/5, 21/10, 6/5, 0)$, whose objective value is −4.9.

MASTER STEP

$z_4 - \hat{c}_4 = 3/5$, and

$$\mathbf{A}\mathbf{x}_4 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 3/2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 7/2 \end{bmatrix}.$$

The updated column $\mathbf{y}_4$ is given by

$$\mathbf{y}_4 = \mathbf{B}^{-1} \begin{bmatrix} \mathbf{A}\mathbf{x}_4 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/5 & 0 & 0 \\ -1/5 & 1 & -5/2 \\ -1/5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 7/2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/5 \\ 3/5 \\ 3/5 \end{bmatrix}.$$

Numerical Example cont...

We therefore insert the column $\begin{pmatrix} z_4 - \hat{c}_4 \\ \mathbf{y}_4 \end{pmatrix}$ in the foregoing array and pivot. This leads to the following two tableaux (the λ_4 column is not displayed after pivoting):

	BASIS INVERSE			RHS	λ_4
z	$-6/5$	0	$-5/2$	$-49/10$	3/5
λ_2	1/5	0	0	2/5	2/5
s_2	$-1/5$	1	$-5/2$	1/10	(3/5)
λ_3	$-1/5$	0	1	3/5	3/5

	BASIS INVERSE			RHS
z	-1	-1	0	-5
λ_2	1/3	$-2/3$	5/3	1/3
λ_4	$-1/3$	5/3	$-25/6$	1/6
λ_3	0	-1	7/2	1/2

The best-known feasible solution to the overall problem is given by

$$\begin{aligned} \mathbf{x} &= \lambda_2\mathbf{x}_2 + \lambda_3\mathbf{x}_3 + \lambda_4\mathbf{x}_4 \\ &= (1/3)(2,3/2,3,0) + (1/2)(0,5/2,0,0) + (1/6)(2,3/2,0,0) = (1,2,1,0). \end{aligned}$$

The objective value is -5. Also, $(w_1, w_2, \alpha) = (-1, -1, 0)$.

Iteration 4

Since $w_1 < 0$ and $w_2 < 0$, neither s_1 nor s_2 is eligible to enter the basis.

Solve the following subproblem:

$$\begin{array}{ll} \text{Maximize} & (\mathbf{wA}-\mathbf{c})\mathbf{x}+\alpha \\ \text{subject to} & \mathbf{x}\in X, \end{array}$$

where

$$\mathbf{wA}-\mathbf{c}=(-1,-1)\begin{bmatrix}1 & 0 & 1 & 0\\ 1 & 1 & 0 & 2\end{bmatrix}-(-2,-1,-1,1)=(0,0,0,-3).$$

Therefore, the subproblem is as follows:

$$\begin{array}{ll} \text{Maximize} & 0x_1+0x_2+0x_3-3x_4+0 \\ \text{subject to} & \mathbf{x}\in X. \end{array}$$

Using Figure 7.1, an optimal solution is $\mathbf{x}_5=(0,0,0,0)$ with objective value $z_5-\hat{c}_5=0$, which is the termination criterion. Also, note that the lower bound is $\hat{\mathbf{c}}_B\overline{\mathbf{b}}-(z_5-\hat{c}_5)=-5-0=-5$, which is equal to the best (and therefore optimal) solution value known so far.

Numerical Example cont...

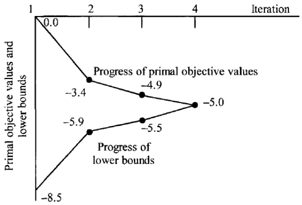

Figure 7.2. Progress of the primal objective values and the lower bounds.

Numerical Example cont...

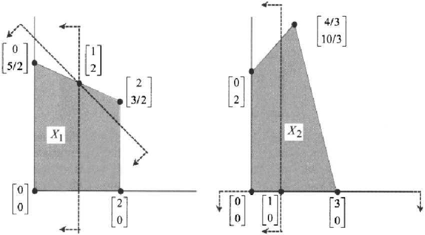

Figure 7.3. Illustration of the optimal solution.

Inequality Constraints

- Consider the following problem

$$\text{Minimize} \sum_{j=1}^{t} (cx_j)\lambda_j x_j$$

$$\text{Subject to} \quad \sum_{j=1}^{t} (Ax_j)\lambda_j \leq b$$

$$\sum_{j=1}^{t} \lambda_j = 1$$

$$\lambda_j \geq 0, \quad j = 1, ..., t$$

- If there is a convenient $x_1 \in X$ with $Ax_1 \leq b$, then the following basis is at hand, where the identity corresponds to the slack vector $s \geq 0$

Inequality Constraints Cont..

$$\mathbf{B} = \left[\begin{array}{c|c} \mathbf{I} & \mathbf{Ax}_1 \\ \hline \mathbf{0} & 1 \end{array}\right], \qquad \mathbf{B}^{-1} = \left[\begin{array}{c|c} \mathbf{I} & -\mathbf{Ax}_1 \\ \hline \mathbf{0} & 1 \end{array}\right].$$

The initial array is given by the following tableau:

	BASIS INVERSE		RHS
z	$\mathbf{0}$	$\mathbf{cx}_1$	$\mathbf{cx}_1$
s	$\mathbf{I}$	$-\mathbf{Ax}_1$	$\mathbf{b} - \mathbf{Ax}_1$
λ_1	$\mathbf{0}$	1	1

equality Constraints

In this case $m + 1$ artificial variables can be introduced to form the initial basis, and the two-phase or the big-M method can be applied