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Vladimir Gilelevich Maz'ya (on his 70th birthday)

Vladimir Gilelevich Maz'ya, the prominent mathematician and author of numerous publications and fundamental results in various fields of analysis and mathematical physics, celebrated his 70th birthday on 31 December 2007.

V. G. Maz'ya was born in Leningrad in 1937. His father was killed at the front in 1941, and both his grandfathers and grandmothers died during the siege of Leningrad. His mother raised Vladimir all by herself. The two lived together on her meagre salary of an accountant, sharing a nine-square-meter room in a large communal flat.

Vladimir finished secondary school with a gold medal, and in his last school years he was a repeated winner of city olympiads in mathematics and physics.

In 1955 Maz'ya entered the Faculty of Mathematics and Mechanics of Leningrad State University (LSU). His first papers were published quite early: the first (on the Dirichlet problem for second-order elliptic equations) appeared in *Doklady Akad. Nauk SSSR* in 1959 when he was a fourth-year student. In the same year he gave two talks at the seminar of V. I. Smirnov, on necessary and sufficient conditions for the validity of integral inequalities of Sobolev type. The results were published in *Doklady* in 1960. For this work he became the first winner of the Prize for Young Mathematicians, established in 1962 by the Leningrad Mathematical Society.

After graduating from the university, Maz'ya obtained a position of junior research fellow at the Research Institute of Mathematics and Mechanics of LSU. In 1961 he organized a mathematical school for high school students at the Faculty of Mathematics and Mechanics and became its first director. He defended his Ph.D. thesis entitled "Classes of sets and embedding theorems for function spaces" in 1962 at Moscow State University. It was based on ideas from his talks at Smirnov's seminar. In their reviews, the opponents and the external reviewer noted that the level of the work far exceeded the requirements of the Higher Certification Commission for Ph.D. theses, and his work was recognized as outstanding at the thesis defence in the Academic Council of Moscow State University.

Maz'ya had no formal research advisor either for his master's thesis or for his Ph.D. thesis: he himself chose his research topics, and he solved the problems by

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himself. However, already in his undergraduate years he became acquainted with S. G. Mikhlin, and their relationship turned into a long-lasting friendship and had a great influence on shaping the mathematical style of Maz'ya.

In 1961 he obtained a position of senior research fellow at the Research Institute of Mathematics and Mechanics of LSU, staying there until 1986. In 1965 he defended his D.Sc. thesis, "Dirichlet and Neumann problems in domains with non-regular boundaries", at Leningrad State University. From 1968 to 1978 he lectured at the Leningrad Shipbuilding Institute, where he was made a professor in 1976. In 1986 he left the university for the Leningrad Division of the Institute of Engineering Studies of the Academy of Sciences of the USSR, where he created and headed the Laboratory of Mathematical Models in Mechanics. At the same time he also founded the influential Consultation Centre in Mathematics for Engineers, serving as its head for several years. Other activities of his during this period include a successful city seminar on hydrodynamics.

In 1990 Maz'ya moved to Sweden and became a professor at Linköping University. The same year he received an honorary doctorate from the University of Rostock. In 1999 he received the Humboldt Prize. The next year he was elected a corresponding member of the Royal Society of Edinburgh, and in 2002 he became a full member of the Royal Swedish Academy of Sciences. A conference in his honour, on Sobolev spaces, was organized in Kyoto in 1993. On the occasion of the his 60th birthday two international conferences took place in 1998: one (on functional analysis and differential equations) at the University of Rostock, and the other (on the method of boundary elements) at the École Polytechnique in Paris. In 2002 he was an invited speaker at the International Congress of Mathematicians in Beijing, in the section on partial differential equations. He is a member of the editorial boards of several mathematical journals published in Germany, the USA, Israel, the Netherlands, India, and France. In recent years he has been a professor at the University of Liverpool (England) and at Ohio State University (USA).

The results of Maz'ya's 50 years of research are reflected in his more than 20 books and more than 420 articles. Below we give a brief description of his results, using the article [1], where a survey is given of his publications up to 2002.

Isoperimetric and integral inequalities. As a fourth-year university student, Maz'ya discovered that integral inequalities of Sobolev type are equivalent to certain isoperimetric and capacitary inequalities for subsets of the domain where a function is defined. These results (which later became a part of his Ph.D. thesis) were published in 1960–61. The method of proof enabled him to obtain sharp constants in integral inequalities. In particular, the exact constant in the Gagliardo inequality

$$\|u\|_{L_{n/(n-1)}(\mathbb{R}^n)} \leqslant C\|\nabla u\|_{L_1(\mathbb{R}^n)}, \qquad u \in C_0^\infty(\mathbb{R}^n),$$

proved to be equal to that in the classical isoperimetric inequality:[1] $C = n^{-1}v_n^{-1/n}$, where v_n is the volume of the unit ball in $\mathbb{R}^n$. Moreover, as Maz'ya emphasized in 1966, his proofs did not use any specific properties of Euclidean space and could be carried over to Riemannian manifolds. The capacitary criteria for Sobolev-type

[1]The exact constant in the Gagliardo inequality was obtained simultaneously and independently by G. Federer and W. H. Fleming.

estimates are based on a remarkable inequality proved by Maz'ya (1964, 1972) which later became known as the strong type capacitary inequality. He has obtained some generalizations of this inequality in his recent papers (2005, 2006). He also found (2003) that embeddings in fractional Besov spaces or Riesz potential spaces are equivalent to isoperimetric inequalities of a new type. The early papers of Maz'ya mentioned above became the starting point for a study of different aspects of the theory of Sobolev spaces and strongly influenced the development of this theory. At the present time the methods in those papers are being successfully used, for example, in the study of Sobolev spaces on metric spaces.

The results obtained up to 1985 are collected in [2], perhaps the most well-known book by Maz'ya.[2]

Properties of the Schrödinger operator. Using capacitary criteria, Maz'ya obtained necessary and sufficient conditions for the validity of various spectral properties of the Schrödinger operator (1962, 1964). Maz'ya and I. E. Verbitsky [I. È. Verbitskii] (2002) found a description of the potential energy operator V (in the class of complex-valued distributions) for which the Schrödinger operator $H = -\Delta + V$ maps the energy space into its dual. Maz'ya, V. A. Kondrat'ev, and M. A. Shubin (2004) found necessary and sufficient conditions for the spectrum of the Schrödinger operator with a magnetic potential to be positive and discrete. This result generalizes the classical criterion of A. M. Molchanov for discreteness of the spectrum in the absence of a magnetic field.

Maz'ya and Shubin characterized the so-called negligible sets in Molchanov's criterion (2005), thus solving a problem posed by I. M. Gel'fand in 1953. A refinement of this result in the simplest case of the Laplace operator with the Dirichlet boundary condition in a domain of n-dimensional Euclidean space (namely, an explicit indication of the constants in a two-sided estimate of the lower bound of the spectrum in terms of the inverse square of the capacitary internal diameter) is given in the paper[3] by Maz'ya and Shubin entitled "Can one see the fundamental frequency of a drum?" and giving a partial answer to this question, which is in a sense dual to the famous question of Mark Kac, "Can one hear the shape of a drum?"

Theory of capacities and non-linear potentials. A typical trait of several of Maz'ya's works is the systematic use of the notion of capacity of a set. In 1963 he defined the polyharmonic capacity and with its help found necessary and sufficient conditions for unique solvability of the Dirichlet problem in the energy space for elliptic equations of arbitrary order. In 1970 Maz'ya and V. P. Khavin introduced non-linear potentials and studied their properties. At present the theory of non-linear potentials (which may be regarded as a generalization of the classical linear theory) is a vast and actively developing field. The theory has helped produce answers to many questions in the theory of functions, especially concerning the exceptional sets.

Boundary behaviour of solutions of elliptic equations. In his research, Maz'ya repeatedly turned to the problem of regularity of a boundary point in the sense of Wiener. In 1962 he found an estimate for the modulus of continuity of

[2] The Springer publishing company is preparing a new revised edition.

[3] *Lett. Math. Phys.* **74** (2005), 135–151.

a harmonic function, formulated in terms of the Wiener integral and having important applications in the qualitative theory of linear and non-linear elliptic equations. In 1970 he obtained a condition for regularity (in the sense of Wiener) of a boundary point for a certain class of quasi-linear second-order elliptic operators including the p-Laplacian. We remark that before 2002 almost nothing was known about the regularity of a boundary point for equations of order higher than two. It was Maz'ya who in 2002 generalized the Wiener test to higher-order elliptic equations. This result was the subject of his talk at the International Congress of Mathematicians in Beijing.

Counterexamples related to Hilbert's 19th problem. According to Hilbert's conjecture, the solutions of regular variational problems of first order with analytic coefficients must be analytic. Attempts to prove this conjecture were undertaken by many mathematicians, and in the second half of the 20th century such a proof was obtained in sufficient generality. One could suppose that this statement remains true for variational problems of order higher than one. However, in 1968 Maz'ya proved that this is not the case. He constructed higher-order quasi-linear elliptic equations with analytic coefficients whose solutions are not smooth.

Estimates for general differential operators. In the 1970s Maz'ya together with I. V. Gel'man investigated various inequalities for differential and pseudodifferential operators in a half-space. They obtained definitive results without imposing any *a priori* conditions on the type of the operators under consideration. The book [3] contains necessary and sufficient conditions for the validity of such inequalities.

Boundary integral equations. In 1967 Yu. D. Burago and Maz'ya made a significant contribution to the theory of boundary integral equations by developing a theory of harmonic single-layer and double-layer potentials in the space C on rather general surfaces. In 1981 Maz'ya proposed a new method for the study of boundary integral equations, based on a preliminary investigation of the properties of certain auxiliary boundary-value problems. This method enabled one to prove the main theorems on solvability of classical boundary integral equations on piecewise smooth surfaces and to find the asymptotics of their solutions near singularities of the boundary (in 2005 Maz'ya and T. O. Shaposhnikova generalized this method to the case of Lipschitz surfaces). N. V. Grachev and Maz'ya (1991) solved the classical problem of unique solvability of a boundary integral equation in the Dirichlet problem for the Laplace operator in the space C on a polyhedral surface. Maz'ya and A. A. Solov'ev (1990) were the first to consider boundary integral equations on a curve with cusps. Modifying the method mentioned above, they developed a logarithmic potential theory which is applicable to integral equations in elasticity theory on a plane domain with inward or outward peaks on the boundary (2001).

Water waves. While working at the Leningrad Shipbuilding Institute, Maz'ya became interested in the theory of surface waves. He studied the main boundary-value problems of this theory in two joint articles with B. R. Vainberg (1973). In 1977 Maz'ya was the first to obtain a quite general uniqueness condition for a solution of the problem of oscillations of a body completely immersed in a liquid. This problem was stated by F. John as far back as 1950 in his paper on oscillations of a body partially immersed in a liquid. The papers by Maz'ya and his colleagues on this subject are reflected in the book [4].

Boundary-value problems in domains with piecewise smooth boundary. Maz'ya began his work in this area in the early 1960s. His first publications contained deep results related to second-order elliptic equations. For example, in studying self-adjointness conditions for the Laplace operator with zero Dirichlet data on contours of class C^1 (but not C^2), he discovered a surprising instability effect for the index under affine coordinate transformations. After the appearance of Kondrat'ev's well-known paper (1967) on elliptic boundary-value problems in domains with conic singularities, Maz'ya began working actively in this field in collaboration with B. A. Plamenevskii, and later with V. A. Kozlov and J. Rossmann. In their papers they constructed a theory of boundary-value problems in domains with piecewise smooth boundary, including various estimates, asymptotic representations of solutions, and solvability theorems. In particular, Maz'ya and Plamenevskii proposed a method for computing the coefficients in the asymptotics of solutions near boundary singularities, which has important applications in fracture mechanics. The books [5] and [6] are devoted to elliptic boundary-value problems in domains with point singularities.

Sobolev spaces on singularly perturbed domains. If the domain of a function depends on small or large parameters in such a way that the limit region degenerates, then properties of Sobolev spaces on these domains strongly depend on these parameters. Starting in the 1980s, Maz'ya and S. V. Poborchi [Poborchii] obtained sharp estimates of the norms of embedding, extension, and boundary-trace operators in dependence on the singular parameters. Such estimates are interesting not only in themselves but also because they turn out to be helpful in establishing the asymptotics of solutions of boundary-value problems for differential equations in singularly perturbed domains. Results in the theory of Sobolev spaces on such domains are described in the books [7] and [8].

Asymptotic theory of elliptic boundary-value problems in singularly perturbed domains. The two-volume monograph [9] written by Maz'ya, S. A. Nazarov, and Plamenevskii is devoted to a new general method for constructing asymptotic solutions of elliptic boundary-value problems in domains with small singular perturbations of the boundary (smoothed angles or edges, narrow cavities, and so on). The monograph [10] written by Maz'ya jointly with Kozlov and A. B. Movchan deals with the same set of problems concerning the asymptotic theory of multi-structures. One of the problems solved in the monograph is that of a smooth junction of three-dimensional and one-dimensional elements of elastic bodies. Recently, Maz'ya and Movchan obtained an analogue of Hadamard's asymptotic formula for the Green functions in singularly perturbed domains.

Singularities of solutions of non-linear differential equations. Singularities of solutions can arise from both non-linearity and non-smoothness of coefficients. Kozlov and Maz'ya (1998–2000) proposed a new method for studying singularities of solutions of a large class of boundary-value problems. The idea of the method is to reduce the study of the asymptotics of a solution near a point or at infinity to the study of the asymptotic behaviour of trajectories of a finite-dimensional dynamical system. A linear version of this theory is described in the book [11].

Numerical analysis. At the end of the 1980s Maz'ya proposed a new approximation method for functions, mainly oriented towards the numerical solution of operator equations. A characteristic feature of this method is that the approximation process does not necessarily converge when the grid size tends to zero. The main applications of the method are connected with the computation of multidimensional volume potentials and with semi-analytic methods for solving non-linear evolution equations. The method turned out to be especially effective for solving non-linear evolution equations with non-local operators, when traditional difference schemes cannot be applied. Maz'ya and G. Schmidt continue to develop the theory of such approximations. They also proposed so-called approximate wavelets, which turned out to be particularly convenient for constructing semi-analytic cubature formulae of high order used in the numerical solution of the pseudodifferential and integral equations of mathematical physics. This research was presented in the recent book [12].

Multipliers in pairs of spaces of differentiable functions. In 1979 Maz'ya and his wife T. O. Shaposhnikova began a study of multipliers in pairs of various spaces of differentiable functions. The results were collected in their joint book [13]. For multipliers they established various theorems on the spectrum, on traces and extensions, on implicit functions, and on two-sided estimates for the essential norm. They found classes of maps described in terms of multipliers and the preservation of Sobolev spaces. They constructed a calculus of singular integral operators with symbols in the spaces of multipliers, and they gave applications to elliptic boundary-value problems in domains with non-smooth boundary. Maz'ya and Shaposhnikova described maximal Banach subalgebras in various spaces of multipliers (2000), and characterized the multipliers in pairs of Besov spaces (2004) and the traces of multipliers in pairs of weighted Sobolev spaces (2005).

Maximum principle for elliptic and parabolic systems. In a series of papers starting from 1984, Maz'ya and G. I. Kresin obtained a necessary and sufficient condition formulated in algebraic terms for the validity of the classical maximum modulus principle for second-order elliptic and parabolic systems. Maz'ya and Rossmann (1992) showed that the Miranda–Agmon maximum principle is valid for any strongly elliptic operator of arbitrary order in a plane domain bounded by a piecewise smooth boundary without peaks. A similar result holds in the three-dimensional case. However, beginning with dimension four this principle fails for certain domains with conical vertices.

Pointwise interpolation inequalities for derivatives. The idea of obtaining information about intermediate derivatives from properties of higher-order derivatives and the function itself is classical. Maz'ya and Shaposhnikova (1999–2002) obtained interpolation inequalities for derivatives of integer and fractional order, in which the role of the norms is played by the values of certain operators (acting on the function) at an arbitrary point of the domain. They found interesting applications of these inequalities to estimates of Gagliardo–Nirenberg type, to operators acting in Sobolev spaces of fractional order, and to the theory of multipliers.

Scientific biography of Hadamard. For twelve years, Maz'ya and Shaposhnikova worked on the first book about the life and work of the great French mathematician Jacques Hadamard [14]. This book contains unique information gathered by the authors in numerous archives and libraries, as well as an analysis of Hadamard's huge scientific heritage. In 2003 the French Academy of Sciences awarded the authors a special prize for this book.

Estimates for analytic functions with a bounded real part. Sharp pointwise estimates for functions analytic in a disk, where the maximum of the real part of the function on the circle serves as a majorant, are a classical subject of analysis and play a special role in the theory of analytic functions. They have applications in analytic number theory and mathematical physics. In the monograph [15] (based on the joint research of the authors in recent years), Maz'ya and Kresin consider diverse inequalities of such a type from a unified viewpoint.

The scientific interests of Vladimir Gilelevich Maz'ya are not limited to the topics listed above. Some more are: approximation by analytic and harmonic functions, the oblique derivative problem, degenerate elliptic pseudodifferential operators, uniqueness theorems for Lamé systems with data on a part of the boundary, methods for solving ill-posed boundary-value problems. The list can be extended.

We cannot fail to note the astonishing productivity of Maz'ya as well as the breadth and depth of his mathematical results. We wish Vladimir Gilelevich good health and further achievements.

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