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Vladimir G. Maz'ya
On the Occasion of His 65th Anniversary

Vladimir Gilelevich Maz'ya is an outstanding mathematician, the author of numerous papers which have won international acclaim. He has obtained fundamental results and devised new methods in different branches of functional analysis, the theory of partial differential equations and their applications.

V. G. Maz'ya was born on December 31, 1937 in Leningrad. His father was killed in World War II in December 1941, and all four of his grandparents died during the blockade of Leningrad. He was brought up by his mother, who worked as an accountant. They lived on her miserable salary in a nine square meter room of a big communal apartment.

In 1955 V. G. Maz'ya entered the Department of Mathematics and Mechanics (*Mathmech*) of Leningrad University. His first publication, "On the criterion of de la Vallée–Poussin", was in ordinary differential equations and appeared in a rotaprinted collection of student papers when he was in his third year of undergraduate studies. In the following year his note on the Dirichlet problem for second order elliptic equations was published in Doklady.

At the end of his undergraduate studies at *Mathmech*, V. G. Maz'ya obtained the position of junior research fellow at the Research Institute of Mathematics and Mechanics of Leningrad State University (RIMM LSU). Two years later he successfully defended his PhD thesis on "Classes of sets and embedding theorems for function spaces." For the research work contained in the thesis, he was granted the Leningrad mathematical society's prize for young scientists. For three years V. G. Maz'ya was the director of the Mathematical School for high school students at *Mathmech* which he himself had created.

V. G. Maz'ya had no formal scientific adviser, neither for his "diploma paper" (master's thesis) nor for his PhD: he chose the problems considered in his work by himself. However, during his undergraduate years his long-lasting friendship with S. G. Mikhlin began, and the latter influenced the development of his mathematical style more than anyone else.

In 1961 V. G. Maz'ya obtained the position of senior research fellow at the RIMM LSU and stayed in it until 1986. In 1965 he successfully defended his Doctorate thesis (habilitation) on "The Dirichlet and Neumann problems in domains with nonregular boundaries" at Leningrad University. From 1968 to 1978 V. G. Maz'ya worked part time in the Leningrad Ship-building Institute, where he was granted the title of full professor in 1976. In 1986 he transferred to the Leningrad Section of the Engineering Research Institute of the USSR Academy of Sciences, where he created and directed the Laboratory of mathematical models in mechanics. In the nineteen eighties V. G. Maz'ya headed the Leningrad seminar in the Marine Hydrodynamics. During the same period, the Consultation Center in Mathematics for engineers that he created functioned successfully.

Since 1990 V. G. Maz'ya lives in Sweden and is a professor at Linköping University. In 1990 he became doctor *honoris causa* of Rostock University and won the Humboldt prize in 1999. He was elected corresponding fellow of the Royal Society of Edinbourgh in 2000 and full member of the Swedish Royal Academy of Sciences in 2002. V. G. Maz'ya was an invited speaker at the International Congress of Mathematicians in Beijing at the section of Partial Differential Equations in 2002. In 1993 a conference in his honor, called "Sobolev spaces and related topics in analysis", was organized in Kyoto. In connection with his 60th anniversary two international conferences were conducted in 1998: one (in functional analysis and partial differential equations) in Rostock, the other (on the method of boundary elements) at the École Polytechnique in Paris.

V. G. Maz'ya is an editorial board member of eight mathematical journals published in Germany, the USA, Sweden, France, India, and the Netherlands.

The results of his 45 years of research are reflected in 15 books (see the list below) and 370 articles. Further, in our brief description of these results, we will make use of the first volume of the *The Maz'ya Anniversary Collection* published by Birkhäuser, where a survey of his books and papers written before 1998 appeared.

Equivalence of isoperimetric and integral inequalities. In 1959, being a fourth year university student, V. G. Maz'ya gave two talks at the V. I. Smirnov seminar, where he presented necessary and sufficient conditions for the validity of various integral inequalities of Sobolev type. He discovered that these inequalities are equivalent to certain isoperimetric and isocapacitary inequalities. These results became part of his PhD thesis and were published in 1960–1961. As early as in 1966 V. G. Maz'ya stated that his proofs did not use any specific properties of Euclidean space and could be carried over to Riemannian manifolds. Later it turned out that his method gives a unified approach to the parabolicity conditions for a manifold, inequalities for eigenvalues, and estimates of the kernel of inverse operator for the heat equation. Already in 1962, Maz'ya obtained a lower estimate for the first eigenvalue of the Laplace operator, which was sharper than the well-known Cheeger inequality (1970). Isocapacitary tests allowed him to obtain necessary and sufficient conditions for various spectral properties of the Schrödinger operator (1962, 1964). These results are intimately connected with the so-called strong capacitary inequalities, which he discovered in 1972.

The early work of V. G. Maz'ya was the origin of his deep studies of various aspects of the theory of Sobolev spaces, which strongly influenced that theory. At the present time his methods are penetrating the rapidly developing theory of differentiable functions on metric spaces. The results obtained by V. G. Maz'ya before 1985 are collected in his monograph [5]. In the 1980s and 90s V. G. Maz'ya and S. V. Poborchi developed a theory of Sobolev spaces on domains singularly depending on parameters. In a recent paper, published in *Acta Mathematica* (2002), V. G. Maz'ya and I. E. Verbitsky solved the old problem of finding a criterion of form-boundedness for the Schrödinger operator with arbitrary potential. In his new paper, written jointly with V. A. Kondratiev and M. A. Shubin, necessary and sufficient conditions for the spectrum of the Schrödinger operator with magnetic potential to be positive and discrete were obtained. At the same time, the paper generalized a classical discreteness of spectrum criterion due to A. M. Molchanov in the absence of a magnetic field. Quite recently Maz'ya and Shubin completely characterized the so-called negligible sets in Molchanov's criterion thus solving a problem posed by I. M. Gelfand in 1953.

Theory of capacities and nonlinear potentials. A typical trait of several V. G. Maz'ya's works was the systematic use of the notion of capacity of a set. For example, in 1963 he defined the so-called polyharmonic capacity and, with its help, found necessary and sufficient conditions on domains ensuring the unique solvability of the Dirichlet problem in the energy space for elliptic operators of arbitrary order. In 1970 V. G. Maz'ya and V. P. Khavin introduced the so-called nonlinear potentials and studied their properties. At the present time, nonlinear potential theory, which may be regarded as a generalization of the classical linear theory, is a vast field of study, which has produced answers to important questions in the theory of functions, especially concerning the so-called exceptional sets.

Boundary behavior of solutions to elliptic equations. From the 1960s to the present day, V. G. Maz'ya often returned to the problem of finding necessary and sufficient conditions for the preservation of properties of boundary value problems with smooth coefficients in smooth domains when the smoothness requirements are omitted. He is also interested in how the usual properties change when the quality of the boundary or the smoothness of the coefficients deteriorates. Already his first papers on Sobolev spaces yielded criteria for the solvability in energy spaces of the Dirichlet and Neumann problems for elliptic second order equations with measurable bounded coefficients. He introduced and used the so-called isoperimetric function in order to obtain exact conditions on the boundary of the domain ensuring various estimates of the solutions to these problems. In 1972, V. G. Maz'ya generalized the Beurling theorem on the boundary minimum principle for positive harmonic functions in a half-plane for second order elliptic equations in n-dimensional domains. In 1972 he found necessary and sufficient conditions for the coincidence of the Martin topology with the Euclidean one for harmonic functions in axially symmetric domains whose boundary consists of two mutually tangent components.

The subject matter of the regularity of a boundary point in the sense of Wiener often attracted V. G. Maz'ya's attention. Thus, in 1962 he found an estimate for the modulus of continuity of a harmonic function formulated in terms of the Wiener integral, which later had important applications to the qualitative theory of linear and nonlinear elliptic equations. In 1970, he obtained a sufficient condition for regularity (in the sense of Wiener) of a boundary point for a certain class of quasi-linear second order elliptic operators containing the p-Laplacian. The sufficiency of this condition was established by T. Kilpeläinen and J. Malý in 1994. Note that until recently practically nothing was known about the regularity of a boundary point for equations of order higher than two. In 2002 V. G. Maz'ya generalized the Wiener test to higher order elliptic equations. This result was the topic of his report at the International Congress of Mathematicians in Beijing.

Counterexamples related to Hilbert's 19th problem. According to Hilbert's conjecture, the solutions of regular variational problems of the first order with analytic coefficients must be analytic. Attempts to prove this conjecture were undertaken by many mathematicians, and in the second half of the 20th century such a proof was obtained in sufficient generality. It was hoped that this statement remains true for variational problems of order higher than one. However, in 1968 V. G. Maz'ya proved that this is not the case. He constructed higher order quasilinear elliptic equations with analytic coefficients and variational problems with analytic integrands whose solutions are nonsmooth.

Oblique derivative problem and degenerate elliptic pseudo-differential equations. By the end of the 1960s, the oblique derivative problem, first stated by Poincaré in connection with the theory of tides, was well studied in the two-dimensional case. It was studied just as well in the multidimensional case, provided that the direction field of the derivatives is transversal to the boundary at each point, so that ellipticity is nowhere violated. The case of degenerate ellipticity turned out to be considerably more difficult. In the 1960s a series of papers appeared in which the degenerate problem with oblique derivative was considered under the condition that the vector field is tangent to the boundary along a submanifold of codimension one to which it is not tangent. In 1970 V. G. Maz'ya carried out a deep investigation of the problem assuming that the boundary contains submanifolds $\Gamma_1 \supset \Gamma_2 \supset \cdots \supset \Gamma_s$ such that the vector field is tangent to Γ_k at points of Γ_{k+1} and is transversal to Γ_s. Using a new method, V. G. Maz'ya established the unique solvability of this modified problem, whose setting includes an additional Dirichlet condition on the entry set of the vector field and the possibility of discontinuities of the solution at points of the exit set. Until the present time this is the only known result concerning the oblique derivative problem in the generic situation in the sense of V. I. Arnold.

General degenerate elliptic pseudo-differential operators on compact manifolds without boundary were studied in detail by V. G. Maz'ya jointly with B. P. Paneah at the end of the 1960s. They introduced a classification of degeneracy types and, for each type, gave a comprehensive study of uniqueness and existence conditions, as well as of smoothness and asymptotics near the degeneracy manifold.

Estimates for general differential operators. In the 1970s V. G. Maz'ya and I. V. Gel'man investigated various inequalities for differential and pseudo-differential operators in a half-space. They obtained results of conclusive character without imposing any a priori conditions on the type of operators considered. The monograph [4] contains necessary and sufficient conditions for the validity of such inequalities.

The method of boundary integral equations. In the 1970s Yu. D. Burago and V. G. Maz'ya made a significant contribution to the theory of boundary integral equations, developing the theory of harmonic single layer and double layer potentials in the spaces C and C^* on very general surfaces [1]. This problem was proposed by F. Riesz and B. Szökefalvy-Nagy. Together with G. I. Kresin, V. G. Maz'ya found essential norms of elastic and hydrodynamic double layer potentials in the space of continuous vector functions (1979, 1995). He is the author of a new method for the study of boundary integral equations in domains with piecewise smooth boundary, based on the preliminary investigation of the properties of certain auxiliary boundary value problems (1981). This method allowed to establish the main theorems on the solvability of classical boundary integral equations on piecewise smooth surfaces and to find the asymptotics of their solutions near singularities of the boundary. Thus, N. V. Grachev and V. G. Maz'ya (1991) solved the classical problem on the unique solvability of the boundary integral equation of the Dirichlet problem for the Laplace operator in the space C for any polyhedron. Before the work of V. G. Maz'ya and A. A. Soloviev (1990), boundary integral equations on curves with peaks were never studied. By using V. G. Maz'ya's method just mentioned above, they developed a conclusive theory of the logarithmic potential, recently carried over to the integral equations of elasticity theory on curves with peaks directed inwards as well as outwards with respect to the domain (2001).

Theory of water waves. Working at the Leningrad Ship-building Institute in the 1970s, V. G. Maz'ya became interested in a wide range of problems in the theory of surface waves. In two articles published in 1973, together with B. R. Vainberg he studied the main boundary value problems of this theory. In 1977 V. G. Maz'ya solved the problem stated by F. John on the solvability conditions for the stationary boundary value problem on the oscillations of a layer of liquid in the presence of an immersed body. The cycle of papers by V. G. Maz'ya and his colleagues in the linear theory of water waves is reflected in the monograph [15].

Approximation by analytic and harmonic functions. Cauchy problem for the Laplace operator. Around 1965, V. P. Khavin drew V. G. Maz'ya's attention to approximation problems in L_p by harmonic and analytic functions. In the process of their joint work, exact results on the possibility of such approximations and on uniqueness sets for analytic functions in the disk were obtained.

In 1974 V. G. Maz'ya and V. P. Khavin published an article in the journal *Trudy MMO* containing unimprovable uniqueness conditions for harmonic functions, with a given majorant for the gradient, vanishing at a boundary point. Thus, the problem set by S. M. Mergelyan was solved. In the same paper two other problems proposed by Mergelyan on the normality of certain families of harmonic functions and on the stability of solutions to the Cauchy problem for the Laplace operator were also solved.

Multipliers in pairs of spaces of differentiable functions. Beginning with 1979, V. G. Maz'ya and his wife T. O. Shaposhnikova develop the theory of multipliers in various spaces of differentiable functions. Their book [6] is the only monograph on this topic. For multipliers, various theorems on the spectrum, on traces and extensions, on implicit functions, and two-sided estimates for the essential norm were established. Classes of mappings (characterized in terms of multipliers and preserving Sobolev spaces) and classes of nonsmooth manifolds on which these spaces are invariantly defined were found. A calculus of singular integral operators with symbols in the space of multipliers was constructed. Deep applications to elliptic boundary value problems in domains with nonsmooth boundaries were given. Among other applications of the theory of multipliers, let us note recent results on the solvability of integral equations on nonsmooth manifolds. In the last few years V. G. Maz'ya and T. O. Shaposhnikova characterized the maximal Banach subalgebras in various spaces of multipliers and described the embeddings of such subalgebras (2000).

Characteristic Cauchy problem for hyperbolic equations. Jointly with B. R. Vainberg, V. G. Maz'ya established the unique solvability of the Cauchy problem for hyperbolic operators of arbitrary even order $2m$ (1981). In contrast with earlier work by Gårding, Kotake, Leray (1964) and Kondratiev (1974), no assumptions on the characteristic subset of the initial surface are imposed, and this set may have a positive measure. In this case, on this subset one can eliminate the time derivative of order $2m-1$ from the Cauchy data. It was shown that this leads to a well-posed problem in appropriately chosen function spaces.

Maximum principle for parabolic and elliptic systems. In a series of articles initiated in 1984, V. G. Maz'ya and G. I. Kresin obtained necessary and sufficient conditions in algebraic terms for the validity of the classical maximum modulus principle for second order elliptic and parabolic systems. Together with J. Rossmann, V. G. Maz'ya showed that the Miranda–Agmon maximum principle is valid for any strongly elliptic operator of arbitrary order in a plane domain bounded by a piecewise smooth boundary without peaks. A similar result was obtained in the three-dimensional case. Moreover, it was shown that starting with dimension four, this principle fails for certain domains with conical boundary singularities.

Boundary value problems in domains with piecewise smooth boundary. V. G. Maz'ya began his research in this area in the early 1960s. His first publications contained deep results related to elliptic second order equations. For example, studying self-adjointness conditions of the Laplace operator with zero Dirichlet data on contours of class C^1 (but not C^2), he discovered the surprising instability effect of the index under affine coordinate changes.

After the appearance of V. A. Kondratiev's fundamental paper (1967) on general elliptic boundary value problems in domains with conic singularities, V. G. Maz'ya began active work in this field with the collaboration of B. A. Plamenevskii and later J. Rossmann and V. A. Kozlov. In their papers a systematic theory of boundary value problems in domains with piecewise smooth boundaries was constructed including various estimates, asymptotic representations of solutions, and solvability theorems. In particular, V. G. Maz'ya and B. A. Plamenevskii proposed a method for computing the coefficients in the asymptotics of the solutions near boundary singularities, a method which plays an important role in fracture mechanics. The monographs [9] and [10] are devoted to boundary value problems in domains with point singularities.

Singularities of solutions to nonlinear differential equations. Singularities of solutions may be due to the nonsmoothness of the coefficients as well as to the nonlinearities. The essence of the new method for studying singularities proposed by V. A. Kozlov and V. G. Maz'ya (1998–2000) consists in reducing, for a large class of boundary value problems, the study of asymptotics of solutions, near a point or at infinity, to the investigation of the asymptotic behavior of trajectories of a finite-dimensional dynamical system. The linear version of this theory is developed in [12].

Asymptotic theory of elliptic boundary value problems in singularly perturbed domains. The two-volume monograph [7] written by V. G. Maz'ya with S. A. Nazarov and B. A. Plamenevskii is devoted to a new general method of asymptotic solution of elliptic boundary value problems in domains with small singular perturbations of the boundary (smoothed angles, conic vertices or edges, small apertures, narrow corridors, and so on). The papers written by V. G. Maz'ya jointly with V. A. Kozlov and A. B. Movchan deal with the same set of problems related to the asymptotic theory of multistructures. The main achievement in these papers is the solution of the problem of junctions of three-dimensional and one-dimensional elements of elastic bodies [13].

New iteration methods for solving ill-posed boundary value problems. In 1990, V. A. Kozlov and V. G. Maz'ya proposed new iteration methods for solving boundary value problems ill-posed in the sense of Hadamard. The general idea of these methods consists in solving, at each step of the iteration, a well-posed problem for the initial differential equation. The regularizing character of the algorithms is ensured only by the appropriate choice of the boundary conditions at each step of the iteration. The importance of these algorithms for applications is due to the fact that their numerical implementation only requires standard software packages.

Approximate approximations. At the end of the 1980s, V. G. Maz'ya proposed a new approximation method mainly oriented to the numerical solution of operator equations. Its characteristic trait is the accurate approximation in the precision interval within which the numerical problem is to be solved. When the grid size tends to zero, the approximation process does not necessarily converge. The main applications of this method are related with two of the hardest problems of numerical analysis, namely the computation of multidimensional volume potentials and the use of semi-analytic methods for nonlinear evolution equations. Maz'ya's method turned out to be especially useful for the solution of nonlinear evolution equations with nonlocal operators, when the traditional finite-difference schemes cannot be applied.

V. G. Maz'ya and G. Schmidt developed a theory of such approximations in detail. They also proposed approximate wavelet decompositions, which turned out to be particularly convenient for the construction of semi-analytic cubature formulas of high order used in the numerical solution of pseudo-differential and integral equations of mathematical physics.

Pointwise interpolation inequalities for derivatives and their applications. The idea of obtaining information about intermediate derivatives from the properties of higher order derivatives and the function itself is classical (E. Landau, 1913, A. N. Kolmogorov, 1938, and others). In the last few years V. G. Maz'ya and T. O. Shaposhnikova obtained interpolation inequalities for derivatives of integer and fractional order in which the role of the norms is played by the values of certain operators (acting on the function) at an arbitrary point (1999–2002). They found interesting applications of these inequalities to estimates of the Gagliardo–Nirenberg type, to composition operators in the Sobolev spaces of fractional order, to the theory of multipliers, etc.

Hadamard's scientific biography. For 12 years V. G. Maz'ya and T. O. Shaposhnikova worked on the first monograph about the life and work of the great French mathematician Jacques Hadamard [11]. This book contains unique information gathered by the authors in numerous archives and libraries, as well as an analysis of Hadamard's huge scientific heritage. At present French and Russian translations of the book are in preparation. In March 2003 the French Academy of Sciences granted a special prize to the authors for this book.

In the framework of this article, we can only mention the topics of Maz'ya's other research work: the description of the uniqueness sets of analytic functions, the Cosserat spectrum in linear elasticity, the asymptotics of ship waves, the asymptotics of solutions to Navier–Stokes equations in domains with thin channels, the Sapondjan–Babushka paradox in the theory of thin plates, uniqueness theorems for the Lamé system with data on a part of the boundary, the boundary point method for the numerical solution of the equations of potential theory, the Fredholm radius for operators of this theory, multidimensional singular integral equations, L_p-contraction of semigroups, homogenization of differential equations on curvilinear grids and of algebraic equations on crystal lattices. This list could have been continued.

The rarely occurring periods of free time of Vladimir and his wife Tatiana are filled with reading, music, and art. Their numerous friends remember with pleasure the wonderful evenings at the fireplace in their hospitable home in a suburb of Linköping.

However, it is his mathematical work which is V. G. Maz'ya's main preoccupation. One cannot but be impressed by his phenomenal research activity, the depth and breadth of his mathematical results.

We wish V. G. Maz'ya new brilliant achievements, and a long and happy life.

M. S. Agranovich, Yu. D. Burago, V. P. Khavin, V. A. Kondratiev, V. P. Maslov, S. M. Nikol'skii, Yu. G. Reshetnyak, M. A. Shubin, B. R. Vainberg, M. I. Vishik, L. R. Volevich

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[1]A complete list of V. G. Maz'ya's publications can be found on his home page http://www.mai.liu.se/˜vlmaz.