Author's correction to pages 595-596 in "Sobolev Spaces with applications to elliptic partial differential equations", 2011, Springer

by Vladimir Maz'ya

The proof of sufficiency in Corollary 11.10.2/2 is erroneous. Starting from line 9 on p. 595 till the end of Section 11.10.2 the text should be replaced by the following:

Corollary 2. (i) *If $q \geq 1$ and the following two conditions hold*

$$\sup_{x\in\mathbb{R}^n,\rho>0} \rho^{(1-n)q}\big(\mu(B(x,\rho),\mathbb{R}^n\backslash B(x,\rho)) + \mu(\mathbb{R}^n\backslash B(x,\rho), B(x,\rho))\big) < \infty, \quad (11.10.20)$$

$$\sup_{x\in\mathbb{R}^n,\rho>0,S\subset B(x,\rho)} \rho^{(1-n)q}\big(\mu(S,B(x,\rho)\backslash S)+\mu(B(x,\rho)\backslash S,S)\big) < \infty, \quad (11.10.20')$$

where S is any Borel subset of $B(x,\rho)$, then the inequality

$$\Big(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |u(x)-u(y)|^q \mu(dx,dy)\Big)^{1/q} \leq C\, \|\nabla u\|_{L_1(\mathbb{R}^n)} \quad (11.10.21)$$

holds for all $u \in C^\infty(\mathbb{R}^n)$ and

$$C^q \leq c^q \sup_{x\in\mathbb{R}^n,\rho>0} \rho^{(1-n)q}\big(\mu(B(x,\rho),\mathbb{R}^n\backslash B(x,\rho)) + \mu(\mathbb{R}^n\backslash B(x,\rho), B(x,\rho))\big)$$

$$+c^q \sup_{x\in\mathbb{R}^n,\rho>0,S\subset B(x,\rho)} \rho^{(1-n)q}\big(\mu(S,B(x,\rho)\backslash S) + \mu(B(x,\rho)\backslash S,S)\big), \quad (11.10.22)$$

where c depends only on n.

(ii) *If* (11.10.21) *holds for all $u \in C^\infty(\mathbb{R}^n)$, then*

$$C^q \geq \omega_n^{-q} \sup_{x\in\mathbb{R}^n,\rho>0} \rho^{(1-n)q}\big(\mu(B(x,\rho),\mathbb{R}^n\backslash B(x,\rho)) + \mu(\mathbb{R}^n\backslash B(x,\rho), B(x,\rho))\big).$$

Proof. We note that $C^\infty(\mathbb{R}^n)$ in the formulation can be replaced by $C_0^\infty(\mathbb{R}^n)$ because the finiteness of $\|\nabla u\|_{L_1(\mathbb{R}^n)}$ implies the existence of a constant $c(u)$ such that $u + c(u) \in \mathring{L}_1^1(\mathbb{R}^n)$.

(i) Let us fix a compact set F in $\mathbb{R}^n$ and introduce the measure $\nu_F(E)$ of an arbitrary Borel set E by

$$\nu_F(E) = \mu(E\backslash F, F) + \mu(F, E\backslash F).$$

The conditions (11.10.20) and (11.10.20′) give for any ball $B(x, \rho)$

$$\mu(B(x,\rho)\backslash F, F) \le \mu(B(x,\rho), \mathbb{R}^n\backslash B(x,\rho)) + \mu(B(x,\rho)\backslash F, F\cap B(x,\rho)) \le const.\rho^{(n-1)q}.$$

Analogously,

$$\mu(F, B(x,\rho)\backslash F) \le \mu(\mathbb{R}^n\backslash B(x,\rho), B(x,\rho)) + \mu(F\cap B(x,\rho), B(x,\rho)\backslash F) \le const.\rho^{(n-1)q}.$$

Hence,

$$\nu_F(B(x,\rho)) \le const.\rho^{(n-1)q}.$$

Now, Theorem 1.4.2 implies

$$\nu_F(g)^{1/q} \le const.s(\partial g)$$

for any open set g with compact closure and smooth boundary. Therefore, if $g \cap F = \emptyset$, we have

$$\big(\mu(F, g) + \mu(g, F)\big)^{1/q} \le const.s(\partial g).$$

We can replace here F by $\mathbb{R}^n\backslash g$, i.e. (11.10.15) with $\Omega = \mathbb{R}^n$ holds. The result follows from Theorem 11.10.2.

The assertion (ii) stems from (11.10.7) by setting $g = B(x, \rho)$.

The same correction should be made on page 97 in my earlier paper "Integral and isocapacitary inequalities", Amer. Math. Soc. Transl. (2) Vol. 226, 2009, p. 85 - 107.