Criterion for the L^p-dissipativity of second order differential operators with complex coefficients

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Abstract. We prove that the algebraic condition $|p-2|\,|\langle \mathscr{I}m\, \mathscr{A}\, \xi,\xi\rangle| \leqslant 2\sqrt{p-1}\,\langle \mathscr{R}e\, \mathscr{A}\, \xi,\xi\rangle$ (for any $\xi \in \mathbb{R}^n$) is necessary and sufficient for the L^p-dissipativity of the Dirichlet problem for the differential operator $\nabla^t(\mathscr{A}\,\nabla)$, where $\mathscr{A}$ is a matrix whose entries are complex measures and whose imaginary part is symmetric. This result is new even for smooth coefficients, when it implies a criterion for the L^p-contractivity of the corresponding semigroup. We consider also the operator $\nabla^t(\mathscr{A}\,\nabla)+\mathbf{b}\nabla+a$, where the coefficients are smooth and $\mathscr{I}m\,\mathscr{A}$ may be not symmetric. We show that the previous algebraic condition is necessary and sufficient for the L^p-quasi-dissipativity of this operator. The same condition is necessary and sufficient for the L^p-quasi-contractivity of the corresponding semigroup. We give a necessary and sufficient condition for the L^p-dissipativity in $\mathbb{R}^n$ of the operator $\nabla^t(\mathscr{A}\,\nabla)+\mathbf{b}\nabla+a$ with constant coefficients.

Résumé. On montre que la condition algébrique $|p-2|\,|\langle \mathscr{I}m\, \mathscr{A}\, \xi,\xi\rangle| \leqslant 2\sqrt{p-1}\,\langle \mathscr{R}e\, \mathscr{A}\, \xi,\xi\rangle$ (pour tout $\xi \in \mathbb{R}^n$) est nécessaire et suffisante pour la dissipativité L^p du problème de Dirichlet pour l'opérateur différentiel $\nabla^t(\mathscr{A}\,\nabla)$, où $\mathscr{A}$ est une matrice dont les coefficients sont des mesures complexes et dont la partie imaginaire est symétrique. Ce résultat est nouveau même pour des coefficients réguliers, quand il implique un critère pour la contractivité L^p du semigroupe correspondant. On considère aussi l'opérateur $\nabla^t(\mathscr{A}\,\nabla)+\mathbf{b}\nabla+a$, où les coefficients sont réguliers et $\mathscr{I}m\,\mathscr{A}$ n'est pas nécessairement symétrique. On

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montre que la condition algébrique précédente est nécessaire et suffisante pour la quasi-dissipativité L^p de cet opérateur. La même condition est nécessaire et suffisante pour la quasi-contractivité L^p du semi-groupe correspondant. On donne une condition nécessaire et suffisante pour la dissipativité L^p dans $\mathbb{R}^n$ de l'opérateur $\nabla^t(\mathscr{A}\,\nabla) + \mathbf{b}\nabla + a$ avec des coefficients constants.

1 Introduction

Various aspects of the L^p-theory of semigroups generated by linear differential operators were studied in [4, 6, 2, 23, 7, 11, 21, 8, 9, 15, 19, 14, 13, 5, 10, 22, 16] *et al.* In particular, it has been known for years that scalar second order elliptic operators with real coefficients may generate contractive semigroups in L^p [18].

Necessary and sufficient conditions for the L^∞-contractivity for general second order strongly elliptic systems with smooth coefficients were given in [12], where scalar second order elliptic operators *with complex coefficients* were handled as a particular case. Such operators generating L^∞-contractive semigroups were later characterized in [3] under the assumption that the coefficients are measurable and bounded.

In the present paper we find an algebraic necessary and sufficient condition for the L^p-dissipativity of the Dirichlet problem for the differential operator

$$A = \nabla^t(\mathscr{A}\,\nabla)$$

where $\mathscr{A}$ is a matrix whose entries are complex measures and whose imaginary part is symmetric. Namely in Section 3, after giving the definition of L^p-dissipativity of the corresponding form

$$\mathscr{L}(u,v) = \int_\Omega \langle \mathscr{A}\,\nabla u, \nabla v\rangle \ ,$$

we prove that $\mathscr{L}$ is L^p-dissipative if and only if

$$|p-2|\,|\langle \mathscr{I}m\,\mathscr{A}\,\xi,\xi\rangle| \leqslant 2\sqrt{p-1}\,\langle \mathscr{R}e\,\mathscr{A}\,\xi,\xi\rangle \tag{1.1}$$

for any $\xi \in \mathbb{R}^n$. This result is new even for smooth coefficients. An example shows that the statement is not true if $\mathscr{I}m\,\mathscr{A}$ is not symmetric.

It is impossible, in general, to obtain a similar algebraic characterization for the operator with lower order terms

$$Au = \nabla^t(\mathscr{A}\,\nabla u) + \mathbf{b}\nabla u + \nabla^t(\mathbf{c}u) + au. \tag{1.2}$$

In fact, consider for example the operator

$$Au = \Delta u + a(x)u$$

in a bounded domain $\Omega \subset \mathbb{R}^n$. Denote by λ_1 the first eigenvalue of the Dirichlet problem for Laplace equation in Ω. A sufficient condition for A to be L^2-dissipative is $\mathscr{R}e\, a \leqslant \lambda_1$ and we cannot give an algebraic characterization of λ_1. However in Section 4 we give a necessary and sufficient condition for the L^p-dissipativity of operator (1.2) in $\mathbb{R}^n$ for the particular case of constant coefficients.

In Section 5 we consider operator (1.2) with smooth coefficients without the requirement of simmetricity of $\mathscr{I}m\,\mathscr{A}$. After showing that the concept of L^p-dissipativity of the form $\mathscr{L}$ is equivalent to the usual L^p-dissipativity of the operator A, we prove that the algebraic condition (1.1) is, in general, necessary and sufficient for the L^p-quasi-dissipativity, i.e. for the L^p-dissipativity of $A - \omega I$ for a suitable $\omega > 0$.

In other words the range of the exponent p admissible for the L^p-quasi-dissipativity is given by the inequalities

$$2 + 2\lambda(\lambda - \sqrt{\lambda^2+1}) \leqslant p \leqslant 2 + 2\lambda(\lambda + \sqrt{\lambda^2+1}),$$

where

$$\lambda = \inf_{(\xi,x)\in\mathcal{M}} \frac{\langle \mathscr{R}e\,\mathscr{A}(x)\xi,\xi\rangle}{|\langle \mathscr{I}m\,\mathscr{A}(x)\xi,\xi\rangle|}$$

and $\mathcal{M} = \{(\xi,x) \in \mathbb{R}^n \times \Omega \mid \langle \mathscr{I}m\,\mathscr{A}(x)\xi,\xi\rangle \neq 0\}$.

Finally we show that (1.1) is necessary and sufficient for the L^p-quasi-contractivity of the semigroup generated by the Dirichlet problem for the operator (1.2).

2 Preliminaries

Let Ω be an open set in $\mathbb{R}^n$. By $C_0(\Omega)$ we denote the space of complex valued continuous functions having compact support in Ω. Let $C_0^1(\Omega)$ consist of all the functions in $C_0(\Omega)$ having continuos partial derivatives of the first order. The inner product either in $\mathbb{C}^n$ or in $\mathbb{C}$ is denoted by $\langle\cdot,\cdot\rangle$ and, as usual, the bar denotes complex conjugation.

In what follows, $\mathscr{A}$ is a $n\times n$ matrix function with complex valued entries $a^{hk}\in(C_0(\Omega))^*$, $\mathscr{A}^t$ is its transposed matrix and $\mathscr{A}^*$ is its adjoint matrix, i.e. $\mathscr{A}^*=\overline{\mathscr{A}}^t$.

Let $\mathbf{b}=(b_1,\ldots,b_n)$ and $\mathbf{c}=(c_1,\ldots,c_n)$ stand for complex valued vectors with $b_j,c_j\in(C_0(\Omega))^*$. By a we mean a complex valued scalar distribution in $(C_0^1(\Omega))^*$.

We denote by $\mathscr{L}(u,v)$ the sesquilinear form

$$\mathscr{L}(u,v)=\int_\Omega(\langle\mathscr{A}\,\nabla u,\nabla v\rangle-\langle\mathbf{b}\nabla u,v\rangle+\langle u,\overline{\mathbf{c}}\nabla v\rangle-a\langle u,v\rangle)$$

defined on $C_0^1(\Omega)\times C_0^1(\Omega)$.

If $p\in(1,\infty)$, p' denotes its conjugate exponent $p/(p-1)$.

Definition 1 Let $1<p<\infty$. The form $\mathscr{L}$ is called L^p*-dissipative* if for all $u\in C_0^1(\Omega)$

$$\mathscr{Re}\,\mathscr{L}(u,|u|^{p-2}u)\geqslant 0 \qquad \text{if } p\geqslant 2; \tag{2.1}$$

$$\mathscr{Re}\,\mathscr{L}(|u|^{p'-2}u,u)\geqslant 0 \qquad \text{if } 1<p<2 \tag{2.2}$$

(we use here that $|u|^{q-2}u\in C_0^1(\Omega)$ for $q\geqslant 2$ and $u\in C_0^1(\Omega)$).

The form $\mathscr{L}$ is related to the operator

$$Au=\nabla^t(\mathscr{A}\,\nabla u)+\mathbf{b}\nabla u+\nabla^t(\mathbf{c}u)+au. \tag{2.3}$$

where ∇^t denotes the divergence operator. The operator A acts from $C_0^1(\Omega)$ to $(C_0^1(\Omega))^*$ through the relation

$$\mathscr{L}(u,v)=\int_\Omega\langle Au,v\rangle$$

for any $u,v\in C_0^1(\Omega)$.

We start with the following Lemma

Lemma 1 *The form $\mathscr{L}$ is L^p-dissipative if and only if for all $v \in C_0^1(\Omega)$*

$$\mathscr{R}e \int_\Omega \Big[\langle \mathscr{A}\, \nabla v, \nabla v\rangle - (1-2/p)\langle (\mathscr{A}-\mathscr{A}^*)\nabla(|v|), |v|^{-1}\overline{v}\nabla v\rangle - \\ (1-2/p)^2 \langle \mathscr{A}\, \nabla(|v|), \nabla(|v|)\rangle\Big] + \int_\Omega \langle \mathscr{I}m(\mathbf{b}+\mathbf{c}), \mathscr{I}m(\overline{v}\nabla v)\rangle + \\ \int_\Omega \mathscr{R}e(\nabla^t(\mathbf{b}/p - \mathbf{c}/p') - a)|v|^2 \geqslant 0. \tag{2.4}$$

Here and in the sequel the integrand is extended by zero on the set where v vanishes.

Proof.

Sufficiency. Let us prove the sufficiency for $p \geqslant 2$. Suppose (2.4) holds, take $u \in C_0^1(\Omega)$ and set

$$v = |u|^{\frac{p-2}{2}} u.$$

Since $p \geqslant 2$ we have $v \in C_0^1(\Omega)$. Moreover, $u = |v|^{\frac{2-p}{p}} v$ and therefore

$$\langle \mathscr{A}\, \nabla u, \nabla(|u|^{p-2}u)\rangle = \langle \mathscr{A}\, \nabla(|v|^{\frac{2-p}{p}} v), \nabla(|v|^{\frac{p-2}{p}} v)\rangle = \\ \Big\langle \mathscr{A}\, (\nabla v - (1-2/p)|v|^{-1} v \nabla|v|), \nabla v + (1-2/p)|v|^{-1} v\nabla |v|\Big\rangle = \\ \langle \mathscr{A}\, \nabla v, \nabla v\rangle - (1-2/p)\left(\langle |v|^{-1} v\, \mathscr{A}\, \nabla |v|, \nabla v\rangle - \langle \mathscr{A}\, \nabla v, |v|^{-1} v \nabla |v|\rangle\right) - \\ -(1-2/p)^2 \langle \mathscr{A}\, \nabla|v|, \nabla|v|\rangle$$

Since

$$\mathscr{R}e(\langle v\, \mathscr{A}\, \nabla|v|, \nabla v\rangle - \langle \mathscr{A}\, \nabla v, v\nabla|v|\rangle) = \\ \mathscr{R}e(v\langle \mathscr{A}\, \nabla|v|, \nabla v\rangle - \overline{\langle v\, \mathscr{A}^*\, \nabla|v|, \nabla v\rangle}) = \mathscr{R}e(\langle v(\mathscr{A}-\mathscr{A}^*)\nabla|v|, \nabla v\rangle)$$

we have

$$\mathscr{R}e\langle \mathscr{A}\, \nabla u, \nabla(|u|^{p-2}u)\rangle = \mathscr{R}e\Big[\langle \mathscr{A}\, \nabla v, \nabla v\rangle - \\ (1-2/p)\langle (\mathscr{A}-\mathscr{A}^*)\nabla(|v|), |v|^{-1}\overline{v}\nabla v\rangle - (1-2/p)^2\langle \mathscr{A}\, \nabla(|v|), \nabla(|v|)\rangle\Big].$$

Moreover, we have

$$\langle \mathbf{b}\nabla u, |u|^{p-2}u\rangle = (1-2/p)\, |v|\, \mathbf{b}\nabla|v| + \overline{v}\, \mathbf{b}\nabla v$$

and then

$$\mathscr{R}e\langle \mathbf{b}\nabla u, |u|^{p-2}u\rangle = 2\, \mathscr{R}e(\mathbf{b}/p)\, \mathscr{R}e(\overline{v}\nabla v) - (\mathscr{I}m\, \mathbf{b})\, \mathscr{I}m(\overline{v}\nabla v) = \\ \mathscr{R}e(\mathbf{b}/p)\nabla(|v|^2) - (\mathscr{I}m\, \mathbf{b})\, \mathscr{I}m(\overline{v}\nabla v).$$

An integration by parts gives

$$\int_\Omega \mathscr{R}e\langle \mathbf{b}\nabla u, |u|^{p-2}u\rangle = -\int_\Omega \mathscr{R}e(\nabla^t(\mathbf{b}/p))|v|^2 - \int_\Omega \langle \mathscr{I}m\, \mathbf{b}, \mathscr{I}m(\overline{v}\nabla v)\rangle\,. \tag{2.5}$$

In the same way we find

$$\begin{gathered}\mathscr{R}e\langle u, \overline{\mathbf{c}}\nabla(|u|^{p-2}u)\rangle = \mathscr{R}e\left((1-2/p)\,|v|\mathbf{c}\nabla|v| + v\,\mathbf{c}\nabla\overline{v}\right) = \\ 2\,\mathscr{R}e(\mathbf{c}/p')\,\mathscr{R}e(\overline{v}\nabla v) + (\mathscr{I}m\,\mathbf{c})\,\mathscr{I}m(\overline{v}\nabla v) = \\ \mathscr{R}e(\mathbf{c}/p')\nabla(|v|^2) + (\mathscr{I}m\,\mathbf{c})\,\mathscr{I}m(\overline{v}\nabla v)\end{gathered}$$

and then

$$\int_\Omega \mathscr{R}e\langle u, \overline{\mathbf{c}}\nabla(|u|^{p-2}u)\rangle = -\int_\Omega \mathscr{R}e(\nabla^t(\mathbf{c}/p')|v|^2 + \int_\Omega \langle \mathscr{I}m\,\mathbf{c}, \mathscr{I}m(\overline{v}\nabla v)\rangle. \tag{2.6}$$

Finally, since we have also

$$\mathscr{R}e(a\langle u, |u|^{p-2}u\rangle = (\mathscr{R}e\,a)|u|^p = (\mathscr{R}e\,a)|v|^2,$$

the left-hand side in (2.4) is equal to $\mathscr{R}e\,\mathscr{L}(u, |u|^{p-2}u)$ and (2.1) follows from (2.4).

Let us suppose that $1 < p < 2$. Now (2.2) can be written as

$$\begin{gathered}\mathscr{R}e\int_\Omega (\langle \mathscr{A}^*\,\nabla u, \nabla(|u|^{p'-2}u)\rangle + \langle \overline{\mathbf{c}}\nabla u, |u|^{p'-2}u\rangle - \langle \nabla u, \mathbf{b}\nabla(|u|^{p'-2}u)\rangle - \\ -a\langle u, |u|^{p'-2}u\rangle)\,\geqslant 0.\end{gathered} \tag{2.7}$$

We know that this is true if

$$\begin{gathered}\mathscr{R}e\int_\Omega \Big[\langle \mathscr{A}^*\,\nabla v, \nabla v\rangle - (1-2/p')\langle(\mathscr{A}^* - \mathscr{A})\nabla(|v|), |v|^{-1}\overline{v}\nabla v\rangle - \\ -(1-2/p')^2\langle \mathscr{A}^*\,\nabla(|v|), \nabla(|v|)\rangle\Big] + \\ +\int_\Omega \langle \mathscr{I}m(-\overline{\mathbf{c}} - \overline{\mathbf{b}}), \mathscr{I}m(\overline{v}\nabla v)\rangle + \\ \int_\Omega \mathscr{R}e\left[\nabla^t\left((-\overline{\mathbf{c}})/p' - (-\overline{\mathbf{b}})/p\right) - a\right]|v|^2 \geqslant 0\end{gathered} \tag{2.8}$$

for any $v \in C_0^1(\Omega)$. This condition is exactly (2.4) and the sufficiency is proved also for $1 < p < 2$.

Necessity. Let us suppose (2.1) holds. Let $v \in C_0^1(\Omega)$ and set

$$g_\varepsilon = (|v|^2 + \varepsilon^2)^{\frac{1}{2}}, \quad u_\varepsilon = g_\varepsilon^{\frac{2}{p}-1} v. \tag{2.9}$$

We have

$$\langle \mathscr{A}\, \nabla u_\varepsilon, \nabla(|u_\varepsilon|^{p-2} u_\varepsilon)\rangle = \\ |u_\varepsilon|^{p-2} \langle \mathscr{A}\, \nabla u_\varepsilon, \nabla u_\varepsilon\rangle + (p-2)|u_\varepsilon|^{p-3}\langle \mathscr{A}\, \nabla u_\varepsilon, u_\varepsilon \nabla |u_\varepsilon|\rangle$$

A direct computation shows that

$$|u_\varepsilon|^{p-2}\langle \nabla u_\varepsilon, \nabla u_\varepsilon\rangle = (1-2/p)^2\, g_\varepsilon^{-(p+2)} |v|^{p+2} \langle \nabla |v|, \nabla |v|\rangle - \\ (1-2/p)\, g_\varepsilon^{-p} |v|^{p-1} (\langle v \nabla |v|, \nabla v\rangle + \langle \nabla v, v\, \nabla |v|\rangle) + g_\varepsilon^{2-p} |v|^{p-2} \langle \nabla v, \nabla v\rangle\,,$$

$$|u_\varepsilon|^{p-3}\langle \nabla u_\varepsilon, u_\varepsilon \nabla |u_\varepsilon|\rangle = \\ \left[(1-2/p)^2\, g_\varepsilon^{-(p+2)} |v|^{p+2} - (1-2/p)\, g_\varepsilon^{-p} |v|^p\right] \langle \nabla |v|, \nabla |v|\rangle + \\ \left[-\,(1-2/p)\, g_\varepsilon^{-p} |v|^{p-1} + g_\varepsilon^{-p+2} |v|^{p-3}\right] \langle \nabla v, v \nabla |v|\rangle.$$

Observing that g_ε tends to $|v|$ as $\varepsilon \to 0$ and referring to Lebesgue's dominated convergence theorem we find

$$\lim_{\varepsilon\to 0} \int_\Omega \langle \mathscr{A}\, \nabla u_\varepsilon, \nabla(|u_\varepsilon|^{p-2} u_\varepsilon)\rangle = \\ \int_\Omega \langle \mathscr{A}\, \nabla v, \nabla v\rangle - \\ (1-2/p) \int_\Omega \frac{1}{|v|} \left(\langle v\, \mathscr{A}\, \nabla |v|, \nabla v\rangle - \langle \mathscr{A}\, \nabla v, v\, \nabla |v|\rangle\right) - \\ -\,(1-2/p)^2 \int_\Omega \langle \mathscr{A}\, \nabla |v|, \nabla |v|\rangle \;\; . \tag{2.10}$$

Similar computations show that

$$\langle \mathbf{b} \nabla u_\varepsilon, |u_\varepsilon|^{p-2} u_\varepsilon\rangle = -(1-2/p) g_\varepsilon^{-p} |v|^{p+1} \mathbf{b} \nabla |v| + g_\varepsilon^{2-p} |v|^{p-2} \overline{v} \mathbf{b} \nabla v$$

$$\langle u_\varepsilon, \overline{\mathbf{c}} \nabla(|u_\varepsilon|^{p-2} u_\varepsilon)\rangle = g_\varepsilon^{2-p} |v|^{p-2} \mathbf{c} \Big[(1-p)\,(1-2/p)\, g_\varepsilon^{-2} |v|^3 \nabla |v| + \\ +(p-2)|v| \nabla |v| + v \nabla \overline{v}\Big]$$

$$a\langle u_\varepsilon, |u_\varepsilon|^{p-2} u_\varepsilon\rangle = a g_\varepsilon^{2-p} |v|^p$$

from which follows

$$\lim_{\varepsilon\to 0}\int_\Omega \langle \mathbf{b}\nabla u_\varepsilon, |u_\varepsilon|^{p-2}u_\varepsilon\rangle = \int_\Omega (-(1-2/p)\,|v|\,\mathbf{b}\nabla|v| + \overline{v}\,\mathbf{b}\nabla v) \tag{2.11}$$

$$\lim_{\varepsilon\to 0}\int_\Omega \langle u_\varepsilon, \overline{\mathbf{c}}\nabla(|u_\varepsilon|^{p-2}u_\varepsilon)\rangle = \int_\Omega ((1-2/p)\,|v|\mathbf{c}\nabla|v| + v\,\mathbf{c}\nabla\overline{v}) \tag{2.12}$$

$$\lim_{\varepsilon\to 0}\int_\Omega a\langle u_\varepsilon, |u_\varepsilon|^{p-2}u_\varepsilon\rangle = \int_\Omega a|v|^2 \tag{2.13}$$

From (2.10)–(2.13) we obtain that

$$\lim_{\varepsilon\to 0}\mathscr{R}e\,\mathscr{L}(u_\varepsilon, |u_\varepsilon|^{p-2}u_\varepsilon)$$

exists and is equal to the left-hand side of (2.4). This shows that (2.1) implies (2.4) and so the necessity is proved for $p\geqslant 2$.

Let us assume $1<p<2$. Since (2.2) can be written as (2.7), replacing $\mathscr{A}$, $\mathbf{b}$, $\overline{\mathbf{c}}$ by $\mathscr{A}^*$, $-\overline{\mathbf{c}}$, $-\mathbf{b}$ respectively in formulas (2.10)–(2.13) we find that

$$\lim_{\varepsilon\to 0}\mathscr{R}e\,\mathscr{L}(|u_\varepsilon|^{p'-2}u_\varepsilon, u_\varepsilon)$$

exists and is equal to the left-hand side of (2.8). Thus (2.2) implies (2.4). □

Corollary 1 *If the form $\mathscr{L}$ is L^p-dissipative, we have*

$$\langle \mathscr{R}e\,\mathscr{A}\,\xi,\xi\rangle \geqslant 0 \tag{2.14}$$

for any $\xi\in\mathbb{R}^n$.

Proof. Given a function v, let us set

$$X=\mathscr{R}e(|v|^{-1}\overline{v}\,\nabla v),\quad Y=\mathscr{I}m(|v|^{-1}\overline{v}\,\nabla v),$$

on the set $\{x\in\Omega \mid v\neq 0\}$. We have

$$\mathscr{R}e\langle \mathscr{A}\,\nabla v,\nabla v\rangle = \mathscr{R}e\left\langle \mathscr{A}(|v|^{-1}\overline{v}\,\nabla v), |v|^{-1}\overline{v}\,\nabla v\right\rangle =$$
$$\langle \mathscr{R}e\,\mathscr{A}\,X,X\rangle + \langle \mathscr{R}e\,\mathscr{A}\,Y,Y\rangle + \langle \mathscr{I}m(\mathscr{A}-\mathscr{A}^t)X,Y\rangle,$$
$$\mathscr{R}e\langle(\mathscr{A}-\mathscr{A}^*)\nabla(|v|),\nabla v\rangle|v|^{-1}v = \mathscr{R}e\langle(\mathscr{A}-\mathscr{A}^*)X, X+iY\rangle =$$
$$\langle \mathscr{I}m(\mathscr{A}-\mathscr{A}^*)X,Y\rangle,$$
$$\mathscr{R}e\langle \mathscr{A}\,\nabla|v|,\nabla|v|\rangle = \langle \mathscr{R}e\,\mathscr{A}\,X,X\rangle.$$

Since $\mathscr{L}$ is L^p-dissipative, (2.4) holds. Hence,

$$\begin{gathered}\int_\Omega \Big\{ \frac{4}{p\,p'} \langle \mathscr{R}e\, \mathscr{A}\, X, X\rangle + \langle \mathscr{R}e\, \mathscr{A}\, Y, Y\rangle + \\ 2\langle (p^{-1}\, \mathscr{I}m\, \mathscr{A} + p'^{-1}\, \mathscr{I}m\, \mathscr{A}^*)X, Y\rangle + \langle \mathscr{I}m(\mathbf{b}+\mathbf{c}), Y\rangle |v| + \\ \mathscr{R}e \left[\nabla^t \left(\mathbf{b}/p - \mathbf{c}/p'\right) - a\right] |v|^2 \Big\} \geqslant 0\end{gathered} \tag{2.15}$$

We define the function

$$v(x) = \varrho(x)\, e^{i\varphi(x)}$$

where ϱ and φ are real functions with $\varrho \in C^1_0(\Omega)$ and $\varphi \in C^1(\Omega)$. Since

$$|v|^{-1}\overline{v}\,\nabla v = |\varrho|^{-1}(\varrho\, e^{-i\varphi}\,(\nabla\varrho + i\varrho\nabla\varphi)\, e^{i\varphi}) = |\varrho|^{-1}\varrho\nabla\varrho + i|\varrho|\nabla\varphi$$

on the set $\{x \in \Omega \mid \varrho(x) \neq 0\}$, it follows from (2.15) that

$$\begin{gathered}\frac{4}{p\,p'} \int_\Omega \langle \mathscr{R}e\, \mathscr{A}\, \nabla\varrho, \nabla\varrho\rangle + \int_\Omega \varrho^2 \langle \mathscr{R}e\, \mathscr{A}\, \nabla\varphi, \nabla\varphi\rangle + \\ 2\int_\Omega \varrho\langle (p^{-1}\, \mathscr{I}m\, \mathscr{A} + p'^{-1}\, \mathscr{I}m\, \mathscr{A}^*)\nabla\varrho, \nabla\varphi\rangle + \\ \int_\Omega \varrho\langle \mathscr{I}m(\mathbf{b}+\mathbf{c}), \nabla\varphi\rangle + \int_\Omega \mathscr{R}e \left[\nabla^t \left(\mathbf{b}/p - \mathbf{c}/p'\right) - a\right] \varrho^2 \geqslant 0\end{gathered} \tag{2.16}$$

for any $\varrho \in C^1_0(\Omega)$, $\varphi \in C^1(\Omega)$.

We choose φ by the equality

$$\varphi = \frac{\mu}{2}\, \log(\varrho^2 + \varepsilon)$$

where $\mu \in \mathbb{R}$ and $\varepsilon > 0$. Then (2.16) takes the form

$$\begin{gathered}\frac{4}{p\,p'} \int_\Omega \langle \mathscr{R}e\, \mathscr{A}\, \nabla\varrho, \nabla\varrho\rangle + \mu^2 \int_\Omega \frac{\varrho^4}{(\varrho^2+\varepsilon)^2} \langle \mathscr{R}e\, \mathscr{A}\, \nabla\varrho, \nabla\varrho\rangle + \\ 2\mu \int_\Omega \frac{\varrho^2}{\varrho^2+\varepsilon} \langle (p^{-1}\, \mathscr{I}m\, \mathscr{A} + p'^{-1}\, \mathscr{I}m\, \mathscr{A}^*)\nabla\varrho, \nabla\varrho\rangle + \\ \mu \int_\Omega \frac{\varrho^3}{\varrho^2+\varepsilon} \langle \mathscr{I}m(\mathbf{b}+\mathbf{c}), \nabla\varrho\rangle + \int_\Omega \mathscr{R}e \left[\nabla^t \left(\mathbf{b}/p - \mathbf{c}/p'\right) - a\right] \varrho^2 \geqslant 0\end{gathered} \tag{2.17}$$

Letting $\varepsilon \to 0^+$ in (2.17) leads to

$$\begin{gathered}\frac{4}{p\,p'} \int_\Omega \langle \mathscr{R}e\, \mathscr{A}\, \nabla\varrho, \nabla\varrho\rangle + \mu^2 \int_\Omega \langle \mathscr{R}e\, \mathscr{A}\, \nabla\varrho, \nabla\varrho\rangle + \\ 2\mu \int_\Omega \langle (p^{-1}\, \mathscr{I}m\, \mathscr{A} + p'^{-1}\, \mathscr{I}m\, \mathscr{A}^*)\nabla\varrho, \nabla\varrho\rangle + \\ \mu \int_\Omega \varrho\langle \mathscr{I}m(\mathbf{b}+\mathbf{c}), \nabla\varrho\rangle + \int_\Omega \mathscr{R}e \left[\nabla^t \left(\mathbf{b}/p - \mathbf{c}/p'\right) - a\right] \varrho^2 \geqslant 0.\end{gathered} \tag{2.18}$$

Since this holds for any $\mu \in \mathbb{R}$, we have

$$\int_\Omega \langle \mathscr{R}e\, \mathscr{A}\, \nabla \varrho, \nabla \varrho \rangle \geqslant 0 \tag{2.19}$$

for any $\varrho \in C_0^1(\Omega)$.

Taking $\varrho(x) = \psi(x) \cos\langle \xi, x\rangle$ with a real $\psi \in C_0^1(\Omega)$ and $\xi \in \mathbb{R}^n$, we find

$$\int_\Omega \{ \langle \mathscr{R}e\, \mathscr{A}\, \nabla\psi, \nabla\psi\rangle \cos^2\langle \xi, x\rangle - [\langle \mathscr{R}e\, \mathscr{A}\, \xi, \nabla\psi\rangle + \\ \langle \mathscr{R}e\, \mathscr{A}\, \nabla\psi, \xi\rangle] \sin\langle \xi, x\rangle \cos\langle \xi, x\rangle + \langle \mathscr{R}e\, \mathscr{A}\, \xi, \xi\rangle \psi^2(x) \sin^2\langle \xi, x\rangle \} \geqslant 0.$$

On the other hand, taking $\varrho(x) = \psi(x) \sin\langle \xi, x\rangle$,

$$\int_\Omega \{ \langle \mathscr{R}e\, \mathscr{A}\, \nabla\psi, \nabla\psi\rangle \sin^2\langle \xi, x\rangle + [\langle \mathscr{R}e\, \mathscr{A}\, \xi, \nabla\psi\rangle + \\ \langle \mathscr{R}e\, \mathscr{A}\, \nabla\psi, \xi\rangle] \sin\langle \xi, x\rangle \cos\langle \xi, x\rangle + \langle \mathscr{R}e\, \mathscr{A}\, \xi, \xi\rangle \psi^2(x) \cos^2\langle \xi, x\rangle \} \geqslant 0.$$

The two inequalities we have obtained lead to

$$\int_\Omega \langle \mathscr{R}e\, \mathscr{A}\, \nabla\psi, \nabla\psi\rangle + \int_\Omega \langle \mathscr{R}e\, \mathscr{A}\, \xi, \xi\rangle \psi^2 \geqslant 0.$$

Because of the arbitrariness of ξ, we find

$$\int_\Omega \langle \mathscr{R}e\, \mathscr{A}\, \xi, \xi\rangle \psi^2 \geqslant 0.$$

On the other hand, any nonnegative function $v \in C_0(\Omega)$ can be approximated in the uniform norm in Ω by a sequence ψ_n^2, with $\psi_n \in C_0^\infty(\Omega)$, and then $\langle \mathscr{R}e\, \mathscr{A}\, \xi, \xi\rangle$ is a nonnegative measure. □

Corollary 2 *If the form $\mathscr{L}$ is both L^p- and $L^{p'}$-dissipative, it is also L^r-dissipative for any r between p and p', i.e. for any r given by*

$$1/r = t/p + (1-t)/p' \qquad (0 \leqslant t \leqslant 1). \tag{2.20}$$

Proof. From the proof of Corollary 1 we know that (2.15) holds. In the same way, we find

$$\int_\Omega \Big\{ \frac{4}{p'\, p} \langle \mathscr{R}e\, \mathscr{A}\, X, X\rangle + \langle \mathscr{R}e\, \mathscr{A}\, Y, Y\rangle - \\ 2\langle (p'^{-1}\, \mathscr{I}m\, \mathscr{A} + p^{-1}\, \mathscr{I}m\, \mathscr{A}^*)X, Y\rangle + \langle \mathscr{I}m(\mathbf{b}+\mathbf{c}), Y\rangle |v| + \\ \mathscr{R}e \left[\nabla^t (\mathbf{b}/p' - \mathbf{c}/p) - a \right] |v|^2 \Big\} \geqslant 0. \tag{2.21}$$

We multiply (2.15) by t, (2.21) by $(1-t)$ and sum up. Since

$$t/p' + (1-t)/p = 1/r' \quad \text{and} \quad r\, r' \leqslant p\, p' \,,$$

we find, keeping in mind Corollary 1,

$$\begin{gathered}\int_\Omega \Big\{ \frac{4}{r\, r'} \langle \mathscr{R}e\, \mathscr{A}\, X, X\rangle + \langle \mathscr{R}e\, \mathscr{A}\, Y, Y\rangle - \\ 2\langle (r^{-1}\, \mathscr{I}m\, \mathscr{A} + r'^{-1}\, \mathscr{I}m\, \mathscr{A}^*) X, Y\rangle + \langle \mathscr{I}m(\mathbf{b}+\mathbf{c}), Y\rangle |v| + \\ + \mathscr{R}e \left[\nabla^t\, (\mathbf{b}/r - \mathbf{c}/r') - a\right] |v|^2 \Big\} \geqslant 0\end{gathered}$$

and $\mathscr{L}$ is L^r-dissipative by Lemma 1 . □

Corollary 3 *Suppose that either*

$$\mathscr{I}m\, \mathscr{A} = 0, \qquad \mathscr{R}e\, \nabla^t \mathbf{b} = \mathscr{R}e\, \nabla^t \mathbf{c} = 0 \tag{2.22}$$

or

$$\mathscr{I}m\, \mathscr{A} = \mathscr{I}m\, \mathscr{A}^t, \quad \mathscr{I}m(\mathbf{b}+\mathbf{c}) = 0, \quad \mathscr{R}e\, \nabla^t \mathbf{b} = \mathscr{R}e\, \nabla^t \mathbf{c} = 0. \tag{2.23}$$

If $\mathscr{L}$ is L^p-dissipative, it is also L^r-dissipative for any r given by (2.20)*.*

Proof. Assume that (2.22) holds. With the notation introduced in Corollary 1, inequality (2.4) reads as

$$\begin{gathered}\int_\Omega \Big(\frac{4}{p\, p'} \langle \mathscr{R}e\, \mathscr{A}\, X, X\rangle + \langle \mathscr{R}e\, \mathscr{A}\, Y, Y\rangle + \\ \langle \mathscr{I}m(\mathbf{b}+\mathbf{c}), Y\rangle |v| - \mathscr{R}e\, a |v|^2 \Big) \geqslant 0.\end{gathered}$$

Since the left-hand side does not change after replacing p by p', Lemma 1 gives the result.

Let (2.23) holds. Using the formula

$$\begin{gathered} p^{-1}\, \mathscr{I}m\, \mathscr{A} + p'^{-1}\, \mathscr{I}m\, \mathscr{A}^* = \\ p^{-1}\, \mathscr{I}m\, \mathscr{A} - p'^{-1}\, \mathscr{I}m\, \mathscr{A}^t = -(1-2/p)\, \mathscr{I}m\, \mathscr{A}\,, \end{gathered} \tag{2.24}$$

we obtain

$$\begin{gathered}\int_\Omega \Big(\frac{4}{p\, p'} \langle \mathscr{R}e\, \mathscr{A}\, x, x\rangle + \langle \mathscr{R}e\, \mathscr{A}\, Y, Y\rangle - \\ 2\,(1-2/p) \langle \mathscr{I}m\, \mathscr{A}\, X, Y\rangle - \mathscr{R}e\, a |v|^2 \Big) \geqslant 0.\end{gathered}$$

Replacing v by $\overline{v}$, we find

$$\int_\Omega \Big(\frac{4}{p\,p'}\langle \mathscr{R}e\,\mathscr{A}\,x,x\rangle + \langle \mathscr{R}e\,\mathscr{A}\,Y,Y\rangle + 2\,(1-2/p)\langle \mathscr{I}m\,\mathscr{A}\,X,Y\rangle - \mathscr{R}e\,a|v|^2\Big) \geqslant 0$$

and we have the $L^{p'}$-dissipativity by $1-2/p=-1+2/p'$. The reference to Corollary 2 completes the proof. □

We give now a sufficient condition for the L^p-dissipativity. This is a direct consequence of Lemma 1.

Corollary 4 *Let α,β two real constants. If*

$$\begin{gathered}\frac{4}{p\,p'}\langle \mathscr{R}e\,\mathscr{A}\,\xi,\xi\rangle + \langle \mathscr{R}e\,\mathscr{A}\,\eta,\eta\rangle + 2\langle (p^{-1}\,\mathscr{I}m\,\mathscr{A} + p'^{-1}\,\mathscr{I}m\,\mathscr{A}^*)\xi,\eta\rangle + \\ \langle \mathscr{I}m(\mathbf{b}+\mathbf{c}),\eta\rangle - 2\langle \mathscr{R}e(\alpha\mathbf{b}/p - \beta\mathbf{c}/p'),\xi\rangle + \\ \mathscr{R}e\,[\nabla^t\,((1-\alpha)\mathbf{b}/p - (1-\beta)\mathbf{c}/p') - a] \geqslant 0\end{gathered} \tag{2.25}$$

for any $\xi,\eta\in\mathbb{R}^n$, the form $\mathscr{L}$ is L^p-dissipative.

Proof. In the proof of Lemma 1 we have integrated by parts in (2.5) and (2.6). More generally, we have

$$\begin{gathered}2/p\int_\Omega \langle \mathscr{R}e\,\mathbf{b}, \mathscr{R}e(\overline{v}\nabla v)\rangle = 2\alpha/p\int_\Omega \langle \mathscr{R}e\,\mathbf{b}, \mathscr{R}e(\overline{v}\nabla v)\rangle - \\ (1-\alpha)/p\int_\Omega \mathscr{R}e(\nabla^t\mathbf{b})|v|^2\,; \\ 2/p'\int_\Omega \langle \mathscr{R}e\,\mathbf{c}, \mathscr{R}e(\overline{v}\nabla v)\rangle = 2\beta/p'\int_\Omega \langle \mathscr{R}e\,\mathbf{c}, \mathscr{R}e(\overline{v}\nabla v)\rangle - \\ (1-\beta)/p'\int_\Omega \mathscr{R}e(\nabla^t\mathbf{c})|v|^2\,.\end{gathered}$$

This leads to write conditions (2.4) in a slightly different form:

$$\mathscr{R}e\int_\Omega \Big[\langle \mathscr{A}\,\nabla v,\nabla v\rangle - (1-2/p)\langle (\mathscr{A}-\mathscr{A}^*)\nabla(|v|), |v|^{-1}\overline{v}\nabla v\rangle - (1-2/p)^2\langle \mathscr{A}\,\nabla(|v|),\nabla(|v|)\rangle\Big] + \int_\Omega \langle \mathscr{I}m(\mathbf{b}+\mathbf{c}), \mathscr{I}m(\overline{v}\nabla v)\rangle -$$

$$2\int_\Omega \langle \mathscr{R}e(\alpha\mathbf{b}/p-\beta\mathbf{c}/p'), \mathscr{R}e(\overline{v}\nabla v)\rangle +$$
$$\int_\Omega \mathscr{R}e(\nabla^t((1-\alpha)\mathbf{b}/p-(1-\beta)\mathbf{c}/p')-a)|v|^2 \geqslant 0.$$

By using the functions X and Y introduced in Corollary 1, the left-hand side of the last inequality can be written as

$$\int_\Omega Q(X,Y)$$

where Q denotes the polynomial (2.25). The result follows from Lemma 1. □

Generally speaking, conditions (2.25) are not necessary for L^p-dissipativity. We show this by the following example, where $\mathscr{I}m\,\mathscr{A}$ is not symmetric. Later we give another example showing that, even for symmetric matrices $\mathscr{I}m\,\mathscr{A}$, conditions (2.25) are not necessary for L^p-dissipativity (see Example 3). Nevertheless in the next section we show that the conditions are necessary for the L^p-dissipativity, provided the operator A has no lower order terms and the matrix $\mathscr{I}m\,\mathscr{A}$ is symmetric (see Theorem 1 and Remark 1).

Example 1 Let $n=2$ and

$$\mathscr{A} = \begin{pmatrix} 1 & i\gamma \\ -i\gamma & 1 \end{pmatrix}$$

where γ is a real constant, $\mathbf{b}=\mathbf{c}=a=0$. In this case polynomial (2.25) is given by

$$(\eta_1-\gamma\xi_2)^2+(\eta_2-\gamma\xi_1)^2-(\gamma^2-4/(pp'))|\xi|^2.$$

Taking $\gamma^2>4/(pp')$, condition (2.25) is not satisfied, while we have the L^p-dissipativity, because the corresponding operator A is the Laplacian.

3 The operator $\nabla^t(\mathscr{A}\,\nabla u)$

In this section we consider operator (2.3) without lower order terms:

$$Au=\nabla^t(\mathscr{A}\,\nabla u) \tag{3.1}$$

with the coefficients $a^{hk} \in (C_0(\Omega))^*$. The following Theorem contains an algebraic necessary and sufficient condition for the L^p-dissipativity.

This result is new even for smooth coefficients, when it implies a criterion for the L^p-contractivity of the corresponding semigroup (see Theorem 5 below).

Theorem 1 *Let the matrix $\mathscr{I}m\,\mathscr{A}$ be symmetric, i.e. $\mathscr{I}m\,\mathscr{A}^t = \mathscr{I}m\,\mathscr{A}$. The form*

$$\mathscr{L}(u,v) = \int_\Omega \langle \mathscr{A}\,\nabla u, \nabla v\rangle$$

is L^p-dissipative if and only if

$$|p-2|\,|\langle \mathscr{I}m\,\mathscr{A}\,\xi,\xi\rangle| \leqslant 2\sqrt{p-1}\,\langle \mathscr{R}e\,\mathscr{A}\,\xi,\xi\rangle \tag{3.2}$$

for any $\xi \in \mathbb{R}^n$, where $|\cdot|$ denotes the total variation.

Proof.

Sufficiency. In view of Corollary 4 the form $\mathscr{L}$ is L^p-dissipative if

$$\frac{4}{p\,p'}\langle \mathscr{R}e\,\mathscr{A}\,\xi,\xi\rangle + \langle \mathscr{R}e\,\mathscr{A}\,\eta,\eta\rangle - 2(1-2/p)\langle \mathscr{I}m\,\mathscr{A}\,\xi,\eta\rangle \geqslant 0 \tag{3.3}$$

for any $\xi, \eta \in \mathbb{R}^n$.

By putting

$$\lambda = \frac{2\sqrt{p-1}}{p}\,\xi$$

we write (3.3) in the form

$$\langle \mathscr{R}e\,\mathscr{A}\,\lambda,\lambda\rangle + \langle \mathscr{R}e\,\mathscr{A}\,\eta,\eta\rangle - \frac{p-2}{\sqrt{p-1}}\langle \mathscr{I}m\,\mathscr{A}\,\lambda,\eta\rangle \geqslant 0.$$

Then (3.3) is equivalent to

$$\mathscr{S}(\xi,\eta) := \langle \mathscr{R}e\,\mathscr{A}\,\xi,\xi\rangle + \langle \mathscr{R}e\,\mathscr{A}\,\eta,\eta\rangle - \frac{p-2}{\sqrt{p-1}}\langle \mathscr{I}m\,\mathscr{A}\,\xi,\eta\rangle \geqslant 0$$

for any $\xi, \eta \in \mathbb{R}^n$.

For any nonnegative $\varphi \in C_0(\Omega)$, define

$$\lambda_\varphi = \min_{|\xi|^2+|\eta|^2=1} \int_\Omega \mathscr{S}(\xi,\eta)\,\varphi\,.$$

Let us fix ξ_0, η_0 such that $|\xi_0|^2 + |\eta_0|^2 = 1$ and

$$\lambda_\varphi = \int_\Omega \mathscr{S}(\xi_0, \eta_0)\, \varphi\, .$$

We have the algebraic system

$$\begin{cases} \displaystyle\int_\Omega \left(2\, \mathscr{R}e\, \mathscr{A}\, \xi_0 - \frac{p-2}{2\sqrt{p-1}}\ \mathscr{I}m(\mathscr{A} - \mathscr{A}^*)\eta_0 \right) \varphi\, = 2\, \lambda_\varphi\, \xi_0 \\ \displaystyle\int_\Omega \left(2\, \mathscr{R}e\, \mathscr{A}\, \eta_0 - \frac{p-2}{2\sqrt{p-1}}\ \mathscr{I}m(\mathscr{A} - \mathscr{A}^*)\xi_0 \right) \varphi\, = 2\, \lambda_\varphi\, \eta_0\, . \end{cases}$$

This implies

$$\int_\Omega \left(2\, \mathscr{R}e\, \mathscr{A}(\xi_0 - \eta_0) + \frac{p-2}{2\sqrt{p-1}}\ \mathscr{I}m(\mathscr{A} - \mathscr{A}^*)(\xi_0 - \eta_0) \right) \varphi\, = \\ 2\, \lambda_\varphi\, (\xi_0 - \eta_0)$$

and therefore

$$\int_\Omega \left(2\langle \mathscr{R}e\, \mathscr{A}(\xi_0 - \eta_0), \xi_0 - \eta_0 \rangle + \frac{p-2}{\sqrt{p-1}} \langle \mathscr{I}m\, \mathscr{A}(\xi_0 - \eta_0), \xi_0 - \eta_0 \rangle \right) \varphi\, = \\ 2\, \lambda_\varphi\, |\xi_0 - \eta_0|^2.$$

The left-hand side is nonnegative because of (3.2). Hence, if $\lambda_\varphi < 0$, we find $\xi_0 = \eta_0$. On the other hand we have

$$\lambda_\varphi = \int_\Omega \mathscr{S}(\xi_0, \xi_0)\, \varphi\, = \\ \int_\Omega \left(2\langle \mathscr{R}e\, \mathscr{A}\, \xi_0, \xi_0 \rangle - \frac{p-2}{\sqrt{p-1}} \langle \mathscr{I}m\, \mathscr{A}\, \xi_0, \xi_0 \rangle \right) \varphi\, \geqslant 0.$$

This shows that $\lambda_\varphi \geqslant 0$ for any nonnegative φ and the sufficiency is proved.

Necessity. We know from the proof of Corollary 1 that if $\mathscr{L}$ is L^p-dissipative, then (2.18) holds for any $\varrho \in C_0^1(\Omega)$, $\mu \in \mathbb{R}$. In the present case, keeping in mind (2.24), (2.18) can be written as

$$\int_\Omega \langle \mathscr{B}\, \nabla \varrho, \nabla \varrho \rangle \geqslant 0,$$

where

$$\mathscr{B} = \frac{4}{p\,p'}\,\mathscr{R}e\,\mathscr{A} + \mu^2\,\mathscr{R}e\,\mathscr{A} - 2\,\mu\,(1-2/p)\,\mathscr{I}m\,\mathscr{A}$$

In the proof of Corollary 1, we have also seen that from (2.19) for any $\varrho \in C_0^1(\Omega)$, (2.14) follows. In the same way, the last relation implies $\langle \mathscr{B}\,\xi, \xi\rangle \geqslant 0$, i.e.

$$\frac{4}{p\,p'}\,\langle \mathscr{R}e\,\mathscr{A}\,\xi, \xi\rangle + \mu^2 \langle \mathscr{R}e\,\mathscr{A}\,\xi, \xi\rangle - 2\,\mu\,(1-2/p)\langle \mathscr{I}m\,\mathscr{A}\,\xi, \xi\rangle \geqslant 0$$

for any $\xi \in \mathbb{R}^n$, $\mu \in \mathbb{R}$.

Because of the arbitrariness of μ we have

$$\int_\Omega \langle \mathscr{R}e\,\mathscr{A}\,\xi, \xi\rangle\,\varphi \geqslant 0$$

$$(1-2/p)^2 \left(\int_\Omega \langle \mathscr{I}m\,\mathscr{A}\,\xi, \xi\rangle\,\varphi\right)^2 \leqslant \frac{4}{p\,p'} \left(\int_\Omega \langle \mathscr{R}e\,\mathscr{A}\,\xi, \xi\rangle\,\varphi\right)^2$$

i.e.

$$|p-2|\,\left|\int_\Omega \langle \mathscr{I}m\,\mathscr{A}\,\xi, \xi\rangle\,\varphi\right| \leqslant 2\,\sqrt{p-1}\,\int_\Omega \langle \mathscr{R}e\,\mathscr{A}\,\xi, \xi\rangle\,\varphi$$

for any $\xi \in \mathbb{R}^n$ and for any nonnegative $\varphi \in C_0(\Omega)$.

We have

$$|p-2|\,\left|\int_\Omega \langle \mathscr{I}m\,\mathscr{A}\,\xi, \xi\rangle\varphi\right| \leqslant 2\sqrt{p-1}\,\int_\Omega \langle \mathscr{R}e\,\mathscr{A}\,\xi, \xi\rangle|\varphi|$$

for any $\varphi \in C_0(\Omega)$ and this implies (3.2), because

$$|p-2|\int_\Omega |\langle \mathscr{I}m\,\mathscr{A}\,\xi, \xi\rangle|\,g = |p-2| \sup_{\substack{\varphi\in C_0(\Omega)\\ |\varphi|\leqslant g}} \left|\int_\Omega \langle \mathscr{I}m\,\mathscr{A}\,\xi, \xi\rangle\varphi\right| \leqslant$$

$$2\sqrt{p-1}\sup_{\substack{\varphi\in C_0(\Omega)\\ |\varphi|\leqslant g}} \int_\Omega \langle \mathscr{R}e\,\mathscr{A}\,\xi, \xi\rangle|\varphi| \leqslant 2\sqrt{p-1}\int_\Omega \langle \mathscr{R}e\,\mathscr{A}\,\xi, \xi\rangle g$$

for any nonnegative $g \in C_0(\Omega)$. □

Remark 1 From the proof of Theorem 1 we see that condition (3.2) holds if and only if

$$\frac{4}{p\,p'}\langle \mathscr{R}e\,\mathscr{A}\,\xi,\xi\rangle + \langle \mathscr{R}e\,\mathscr{A}\,\eta,\eta\rangle - 2(1-2/p)\langle \mathscr{I}m\,\mathscr{A}\,\xi,\eta\rangle \geqslant 0$$

for any $\xi,\eta \in \mathbb{R}^n$. This means that conditions (2.25) are necessary and sufficient for the operators considered in Theorem 1.

Remark 2 Let us assume that either A has lower order terms or they are absent and $\mathscr{I}m\,\mathscr{A}$ is not symmetric. Using the same arguments as in Theorem 1, one could prove that (3.2) is still a necessary condition for A to be L^p-dissipative. However, in general, it is not sufficient. This is shown by the next example (see also Theorem 2 below for the particular case of constant coefficients).

Example 2 Let $n=2$ and let Ω be a bounded domain. Denote by σ a not identically vanishing real function in $C_0^2(\Omega)$ and let $\lambda \in \mathbb{R}$. Consider operator (3.1) with

$$\mathscr{A} = \begin{pmatrix} 1 & i\lambda\partial_1(\sigma^2) \\ -i\lambda\partial_1(\sigma^2) & 1 \end{pmatrix}$$

i.e.

$$Au = \partial_1(\partial_1 u + i\lambda\partial_1(\sigma^2)\,\partial_2 u) + \partial_2(-i\lambda\partial_1(\sigma^2)\,\partial_1 u + \partial_2 u),$$

where $\partial_i = \partial/\partial x_i$ $(i=1,2)$.

By definition, we have L^2-dissipativity if and only if

$$\mathscr{R}e \int_\Omega ((\partial_1 u + i\lambda\partial_1(\sigma^2)\,\partial_2 u)\partial_1\overline{u} + (-i\lambda\partial_1(\sigma^2)\,\partial_1 u + \partial_2 u)\partial_2\overline{u})\,dx \geqslant 0$$

for any $u \in C_0^1(\Omega)$, i.e. if and only if

$$\int_\Omega |\nabla u|^2 dx - 2\lambda \int_\Omega \partial_1(\sigma^2)\,\mathscr{I}m(\partial_1\overline{u}\,\partial_2 u)\,dx \geqslant 0$$

for any $u \in C_0^1(\Omega)$. Taking $u = \sigma\,\exp(itx_2)$ $(t \in \mathbb{R})$, we obtain, in particular,

$$t^2\int_\Omega \sigma^2 dx - t\lambda \int_\Omega (\partial_1(\sigma^2))^2 dx + \int_\Omega |\nabla\sigma|^2 dx \geqslant 0. \tag{3.4}$$

Since

$$\int_\Omega (\partial_1(\sigma^2))^2 dx > 0,$$

we can choose $\lambda \in \mathbb{R}$ so that (3.4) is impossible for all $t \in \mathbb{R}$. Thus A is not L^2-dissipative, although (3.2) is satisfied.

Since A can be written as

$$Au = \Delta u - i\lambda(\partial_{21}(\sigma^2)\,\partial_1 u - \partial_{11}(\sigma^2)\,\partial_2 u),$$

the same example shows that (3.2) is not sufficient for the L^2-dissipativity in the presence of lower order terms, even if $\mathscr{I}m\,\mathscr{A}$ is symmetric.

4 General equation with constant coefficients

In this section we characterize the L^p-dissipativity for a differential operator A, say

$$Au = \nabla^t(\mathscr{A}\,\nabla u) + \mathbf{b}\nabla u + au \tag{4.1}$$

with constant complex coefficients. Without loss of generality we assume that the matrix $\mathscr{A}$ is symmetric.

Theorem 2 *Let Ω be an open set in $\mathbb{R}^n$ which contains balls of arbitrarily large radius. The operator A is L^p-dissipative if and only if there exists a real constant vector V such that*

$$2\,\mathscr{R}e\,\mathscr{A}\,V + \mathscr{I}m\,\mathbf{b} = 0 \tag{4.2}$$

$$\mathscr{R}e\,a + \langle \mathscr{R}e\,\mathscr{A}\,V, V\rangle \leqslant 0 \tag{4.3}$$

and the inequality

$$|p-2|\,|\langle \mathscr{I}m\,\mathscr{A}\,\xi,\xi\rangle| \leqslant 2\sqrt{p-1}\,\langle \mathscr{R}e\,\mathscr{A}\,\xi,\xi\rangle \tag{4.4}$$

holds for any $\xi \in \mathbb{R}^n$.

Proof. First, let us prove the Theorem for the special case $\mathbf{b} = 0$, i.e. for the operator

$$A = \nabla^t(\mathscr{A}\,\nabla u) + au.$$

If A is L^p-dissipative, (2.4) holds for any $v \in C_0^1(\Omega)$. We find, by repeating the arguments used in the proof of Theorem 1, that

$$\begin{gathered}\frac{4}{p\,p'}\int_\Omega \langle \mathscr{R}e\,\mathscr{A}\,\nabla\varrho, \nabla\varrho\rangle\,dx + \mu^2\int_\Omega \langle \mathscr{R}e\,\mathscr{A}\,\nabla\varrho, \nabla\varrho\rangle\,dx - \\ 2\,\mu\,(1-2/p)\int_\Omega \langle \mathscr{I}m\,\mathscr{A}\,\nabla\varrho, \nabla\varrho\rangle\,dx - (\mathscr{R}e\,a)\int_\Omega \varrho^2 dx \geqslant 0\end{gathered} \tag{4.5}$$

for any $\varrho \in C_0^\infty(\Omega)$ and for any $\mu \in \mathbb{R}$. As in the proof of Theorem 1 this implies (4.4). On the other hand, we can find a sequence of balls contained in Ω with centres x_m and radii m. Set

$$\varrho_m(x) = m^{-n/2}\sigma\left((x-x_m)/m\right),$$

where $\sigma \in C_0^\infty(\mathbb{R}^n)$, $\operatorname{spt}\sigma \subset B_1(0)$ and

$$\int_{B_1(0)} \sigma^2(x)\, dx = 1.$$

Putting in (4.5) $\mu = 1$ and $\varrho = \varrho_m$, we obtain

$$\frac{4}{p\,p'}\int_{B_1(0)} \langle \mathscr{R}e\,\mathscr{A}\,\nabla\sigma, \nabla\sigma\rangle\, dy + \int_{B_1(0)} \langle \mathscr{R}e\,\mathscr{A}\,\nabla\sigma, \nabla\sigma\rangle\, dy\, - \\ 2\,(1-2/p)\int_{B_1(0)} \langle \mathscr{I}m\,\mathscr{A}\,\nabla\sigma, \nabla\sigma\rangle\, dy - m^2(\mathscr{R}e\, a) \geqslant 0$$

for any $m \in \mathbb{N}$. This implies $\mathscr{R}e\, a \leqslant 0$. Note that in this case the algebraic system (4.2) has always the trivial solution and that for any eigensolution V (if they exist) we have $\langle \mathscr{R}e\,\mathscr{A}\,V, V\rangle = 0$. Then (4.3) is satisfied.

Conversely, if (4.4) is satisfied, we have (see Remark 1)

$$\frac{4}{p\,p'}\langle \mathscr{R}e\,\mathscr{A}\,\xi,\xi\rangle + \langle \mathscr{R}e\,\mathscr{A}\,\eta,\eta\rangle - 2\,(1-2/p)\langle \mathscr{I}m\,\mathscr{A}\,\xi,\xi\rangle \geqslant 0$$

for any $\xi, \eta \in \mathbb{R}^n$. If also (4.3) is satisfied (i.e. if $\mathscr{R}e\, a \leqslant 0$), A is L^p-dissipative in view of Corollary 4.

Let us consider the operator in the general form (4.1). If A is L^p-dissipative, we find, by repeating the arguments employed in the proof of Theorem 1, that

$$\frac{4}{p\,p'}\int_\Omega \langle \mathscr{R}e\,\mathscr{A}\,\nabla\varrho, \nabla\varrho\rangle\, dx + \int_\Omega \varrho^2\langle \mathscr{R}e\,\mathscr{A}\,\nabla\varphi, \nabla\varphi\rangle\, dx - \\ 2\,(1-2/p)\int_\Omega \varrho\,\langle \mathscr{I}m\,\mathscr{A}\,\nabla\varrho, \nabla\varphi\rangle\, dx + \\ \int_\Omega \varrho^2\langle \mathscr{I}m\,\mathbf{b}, \nabla\varphi\rangle\, dx - \mathscr{R}e\, a\int_\Omega \varrho^2 dx \geqslant 0$$

for any $\varrho \in C_0^1(\Omega)$, $\varphi \in C^1(\Omega)$. By fixing ϱ and choosing $\varphi = t\langle \eta, x\rangle$ ($t \in \mathbb{R}$, $\eta \in \mathbb{R}^n$) we get

$$\frac{4}{p\,p'}\int_\Omega \langle \mathscr{R}e\,\mathscr{A}\,\nabla\varrho, \nabla\varrho\rangle\, dx + (t^2\langle \mathscr{R}e\,\mathscr{A}\,\eta,\eta\rangle + t\,\langle \mathscr{I}m\,\mathbf{b},\eta\rangle - \mathscr{R}e\, a)\int_\Omega \varrho^2\, dx \geqslant 0$$

for any $t \in \mathbb{R}$. This leads to

$$|\langle \mathscr{I}m\, \mathbf{b}, \eta\rangle|^2 \leqslant K\, \langle \mathscr{R}e\, \mathscr{A}\, \eta, \eta\rangle$$

for any $\eta \in \mathbb{R}^n$ and this inequality shows that system (4.2) is solvable. Let V be a solution of this system and let

$$z = e^{-i\langle V,x\rangle} u.$$

One checks directly that

$$Au = (\nabla^t(\mathscr{A}\, \nabla z) + \langle \mathbf{c}, \nabla z\rangle + \alpha z) e^{i\langle V,x\rangle}$$

where

$$\mathbf{c} = 2i\, \mathscr{A}\, V + \mathbf{b}, \quad \alpha = a + i\langle \mathbf{b}, V\rangle - \langle \mathscr{A}\, V, V\rangle.$$

Since we have

$$\int_\Omega \langle Au, u\rangle |u|^{p-2} dx = \int_\Omega \langle \nabla^t(\mathscr{A}\, \nabla z) + \langle \mathbf{c}, \nabla z\rangle + \alpha z, z\rangle |z|^{p-2} dx\,,$$

the L^p-dissipativity of A is equivalent to the L^p-dissipativity of the operator

$$\nabla^t(\mathscr{A}\, \nabla z) + \langle \mathbf{c}, \nabla z\rangle + \alpha z\,.$$

On the other hand Lemma 1 shows that, as far as the first order terms are concerned, the $\mathscr{R}e\, \mathbf{b}$ does not play any role. Since $\mathscr{I}m\, \mathbf{c} = \mathbf{0}$ because of (4.2), the L^p-dissipativity of A is equivalent to the L^p-dissipativity of the operator

$$\nabla^t(\mathscr{A}\, \nabla z) + \alpha z\,. \tag{4.6}$$

By what we have already proved above, the last operator is L^p-dissipative if and only if (4.4) is satisfied and $\mathscr{R}e\, \alpha \leqslant 0$. From (4.2) it follows that $\mathscr{R}e\, \alpha$ is equal to the left-hand side of (4.3).

Conversely, if there exists a solution V of (4.2), (4.3), and if (4.4) is satisfied, operator (4.6) is L^p-dissipative. Since this is equivalent to the L^p-dissipativity of A, the proof is complete. □

Corollary 5 *Let Ω be an open set in $\mathbb{R}^n$ which contains balls of arbitrarily large radius. Let us suppose that the matrix $\mathscr{R}e\, \mathscr{A}$ is not singular. The operator A is L^p-dissipative if and only if* (4.4) *holds and*

$$4\, \mathscr{R}e\, a \leqslant -\langle (\mathscr{R}e\, \mathscr{A})^{-1}\, \mathscr{I}m\, \mathbf{b}, \mathscr{I}m\, \mathbf{b}\rangle \tag{4.7}$$

Proof. If $\mathscr{R}e\,\mathscr{A}$ is not singular, the only vector V satisfying (4.2) is

$$V = -(1/2)(\mathscr{R}e\,\mathscr{A})^{-1}\,\mathscr{I}m\,\mathbf{b}$$

and (4.3) is satisfied if and only if (4.7) holds. The result follows from Theorem 2. □

Example 3 Let $n = 1$ and $\Omega = \mathbb{R}^1$. Consider the operator

$$\left(1 + 2\,\frac{\sqrt{p-1}}{p-2}\,i\right) u'' + 2iu' - u,$$

where $p \neq 2$ is fixed. Conditions (4.4) and (4.7) are satisfied and this operator is L^p-dissipative, in view of Corollary 5.

On the other hand, the polynomial considered in Corollary 4 is

$$Q(\xi,\eta) = \left(2\,\frac{\sqrt{p-1}}{p}\,\xi - \eta\right)^2 + 2\eta + 1$$

which is not nonnegative for any $\xi, \eta \in \mathbb{R}$. This shows that, in general, condition (2.25) is not necessary for the L^p-dissipativity, even if the matrix $\mathscr{I}m\,\mathscr{A}$ is symmetric.

5 Smooth coefficients

Let us consider the operator

$$Au = \nabla^t(\mathscr{A}\,\nabla u) + \mathbf{b}\nabla u + a\,u \tag{5.1}$$

with the coefficients $a^{hk}, b^h \in C^1(\overline{\Omega})$, $a \in C^0(\overline{\Omega})$. Here Ω is a bounded domain in $\mathbb{R}^n$, whose boundary is in the class $C^{2,\alpha}$ for some $\alpha \in [0, 1)$ (this regularity assumption could be weakened, but we prefer to avoid the technicalities related to such generalizations).

We consider A as an operator defined on the set

$$\mathscr{D}(A) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega). \tag{5.2}$$

Definition 2 The operator A is said to be L^p-dissipative if

$$\mathscr{R}e \int_\Omega \langle Au, u\rangle |u|^{p-2} dx \leqslant 0 \tag{5.3}$$

for any $u \in \mathscr{D}(A)$.

We show that the L^p-dissipativity of A is equivalent to the L^p-dissipativity of the sesquilinear form

$$\mathscr{L}(u,v) = \int_\Omega (\langle \mathscr{A}\, \nabla u, \nabla v\rangle - \langle \mathbf{b}\nabla u, v\rangle - a\langle u, v\rangle)$$

Lemma 2 *The form $\mathscr{L}$ is L^p-dissipative if and only if*

$$\begin{gathered}\mathscr{R}e \int_\Omega \Big[\langle \mathscr{A}\, \nabla v, \nabla v\rangle - (1-2/p)\langle(\mathscr{A} - \mathscr{A}^*)\nabla(|v|), |v|^{-1}\overline{v}\nabla v\rangle - \\ (1-2/p)^2 \langle \mathscr{A}\, \nabla(|v|), \nabla(|v|)\rangle\Big] dx + \\ \int_\Omega \langle \mathscr{I}m\, \mathbf{b}, \mathscr{I}m(\overline{v}\nabla v)\rangle dx + \int_\Omega \mathscr{R}e(\nabla^t(\mathbf{b}/p) - a)|v|^2 dx \geqslant 0\end{gathered} \tag{5.4}$$

for any $v \in H_0^1(\Omega)$.

Proof.

Sufficiency. We know from Lemma 1 that $\mathscr{L}$ is L^p-dissipative if and only if (5.4) holds for any $v \in C_0^1(\Omega)$. Since $C_0^1(\Omega) \subset H_0^1(\Omega)$, the sufficiency follows.

Necessity. Given $v \in H_0^1(\Omega)$, we can find a sequence $\{v_n\} \subset C_0^1(\Omega)$ such that $v_n \to v$ in $H_0^1(\Omega)$. Let us show that

$$\chi_{E_n} |v_n|^{-1} \overline{v}_n \nabla v_n \to \chi_E |v|^{-1} \overline{v} \nabla v \quad \text{in } L^2(\Omega) \tag{5.5}$$

where $E_n = \{x \in \Omega \mid v_n(x) \neq 0\}$, $E = \{x \in \Omega \mid v(x) \neq 0\}$. We may assume $v_n(x) \to v(x)$, $\nabla v_n(x) \to \nabla v(x)$ almost everywhere in Ω. We see that

$$\chi_{E_n} |v_n|^{-1} \overline{v}_n \nabla v_n \to \chi_E |v|^{-1} \overline{v} \nabla v \tag{5.6}$$

almost everywhere on the set $E \cup \{x \in \Omega \setminus E \mid \nabla v(x) = 0\}$. Since the set $\{x \in \Omega \setminus E \mid \nabla v(x) \neq 0\}$ has zero measure, we can say that (5.6) holds almost everywhere in Ω.

Moreover, since

$$\int_G |\chi_{E_n}|v_n|^{-1}\overline{v}_n\nabla v_n|^2dx \leqslant \int_G |\nabla v_n|^2dx$$

for any measurable set $G \subset \Omega$ and $\{\nabla v_n\}$ is convergent in $L^2(\Omega)$, the sequence $\{|\,\chi_{E_n}|v_n|^{-1}\overline{v}_n\nabla v_n - \chi_E|v|^{-1}\overline{v}\nabla v\,|^2\}$ has uniformly absolutely continuos integrals. Now we may appeal to Vitali's Theorem to obtain (5.5).

From this it follows that (5.4) for any $v \in H^1_0(\Omega)$ implies (5.4) for any $v \in C^1_0(\Omega)$. Lemma 1 shows that $\mathscr{L}$ is L^p-dissipative. $\square$

Lemma 3 *The form $\mathscr{L}$ is L^p-dissipative if and only if*

$$\mathscr{R}e\int_\Omega(\langle\mathscr{A}\,\nabla u, \nabla(|u|^{p-2}u)\rangle - \langle\mathbf{b}\nabla u, |u|^{p-2}u\rangle - a\,|u|^p)dx \geqslant 0 \tag{5.7}$$

for any $u \in \Xi$, where Ξ denotes the space $\{u \in C^2(\overline{\Omega}) \mid u|_{\partial\Omega} = 0\}$.

Proof.

Necessity. Since $\mathscr{L}$ is L^p-dissipative, (5.4) holds for any $v \in H^1_0(\Omega)$. Let $u \in \Xi$. We introduce the function

$$\varrho_\varepsilon(s) = \begin{cases} \varepsilon^{\frac{p-2}{2}} & \text{if } 0 \leqslant s \leqslant \varepsilon \\ s^{\frac{p-2}{2}} & \text{if } s > \varepsilon \end{cases}$$

Setting

$$v_\varepsilon = \varrho_\varepsilon(|u|)\,u$$

a direct computation shows that $u = \sigma_\varepsilon(|v_\varepsilon|)\,v_\varepsilon$ and $\varrho^2_\varepsilon(|u|)\,u = [\sigma_\varepsilon(|v_\varepsilon|)]^{-1}\,v_\varepsilon$, where

$$\sigma_\varepsilon(s) = \begin{cases} \varepsilon^{\frac{2-p}{2}} & \text{if } 0 \leqslant s \leqslant \varepsilon^{\frac{p}{2}} \\ s^{\frac{2-p}{p}} & \text{if } s > \varepsilon^{\frac{p}{2}}. \end{cases}$$

Therefore

$$\begin{gathered}\langle\mathscr{A}\,\nabla u, \nabla[\varrho^2_\varepsilon(|u|)\,u]\rangle = \langle\mathscr{A}\,\nabla[\sigma_\varepsilon(|v_\varepsilon|)\,v_\varepsilon], \nabla[(\sigma_\varepsilon(|v_\varepsilon|))^{-1}v_\varepsilon]\rangle = \\ \langle\mathscr{A}\,[\sigma_\varepsilon(|v_\varepsilon|)\,\nabla v_\varepsilon + \sigma'_\varepsilon(|v_\varepsilon|)\,v_\varepsilon\,\nabla|v_\varepsilon|]\,, \sigma_\varepsilon(|v_\varepsilon|)^{-1}\nabla v_\varepsilon - \\ \sigma'_\varepsilon(|v_\varepsilon|)\sigma_\varepsilon^{-2}(|v_\varepsilon|)v_\varepsilon\,\nabla|v_\varepsilon|\rangle = \\ \langle\mathscr{A}\,\nabla v_\varepsilon, \nabla v_\varepsilon\rangle + \sigma'_\varepsilon(|v_\varepsilon|)\sigma_\varepsilon(|v_\varepsilon|)^{-1}\left(\langle v_\varepsilon\,\mathscr{A}\,\nabla|v_\varepsilon|, \nabla v_\varepsilon\rangle - \langle\mathscr{A}\,\nabla v_\varepsilon, v_\varepsilon\,\nabla|v_\varepsilon|\rangle\right) - \\ -\sigma'_\varepsilon(|v_\varepsilon|)^2\sigma_\varepsilon(|v_\varepsilon|)^{-2}\langle v_\varepsilon\,\mathscr{A}\,\nabla|v_\varepsilon|, v_\varepsilon\nabla|v_\varepsilon|\rangle\,.\end{gathered}$$

Since

$$\frac{\sigma_\varepsilon'(|v_\varepsilon|)}{\sigma_\varepsilon(|v_\varepsilon|)} = \begin{cases} 0 & \text{if } 0 < |u| < \varepsilon \\ -(1-2/p)\ |v_\varepsilon|^{-1} & \text{if } |u| > \varepsilon \end{cases}$$

we may write

$$\int_\Omega \langle \mathscr{A}\,\nabla u, \nabla[\varrho_\varepsilon^2(|u|)\,u]\rangle\,dx = \int_\Omega \langle \mathscr{A}\,\nabla v_\varepsilon, \nabla v_\varepsilon\rangle\,dx -$$

$$-(1-2/p)\int_{E_\varepsilon} \frac{1}{|v_\varepsilon|}\left(\langle v_\varepsilon\,\mathscr{A}\,\nabla|v_\varepsilon|, \nabla v_\varepsilon\rangle - \langle \mathscr{A}\,\nabla v_\varepsilon, v_\varepsilon\,\nabla|v_\varepsilon|\rangle\right)\,dx -$$

$$-(1-2/p)^2\int_{E_\varepsilon} \langle \mathscr{A}\,\nabla|v_\varepsilon|, \partial_h\nabla|v_\varepsilon|\rangle\,dx$$

where $E_\varepsilon = \{x\in\Omega \mid |u(x)|>\varepsilon\}$. Then

$$\int_\Omega \langle \mathscr{A}\,\nabla u, \nabla[\varrho_\varepsilon^2(|u|)\,u]\rangle\,dx = \int_\Omega \langle \mathscr{A}\,\nabla v_\varepsilon, \nabla v_\varepsilon\rangle\,dx -$$

$$(1-2/p)\int_\Omega \frac{1}{|v_\varepsilon|}\left(\langle v_\varepsilon\,\mathscr{A}\,\nabla|v_\varepsilon|, \nabla v_\varepsilon\rangle - \langle \mathscr{A}\,\nabla v_\varepsilon, v_\varepsilon\,\nabla|v_\varepsilon|\rangle\right)\,dx -$$

$$(1-2/p)^2\int_\Omega \langle \mathscr{A}\,\nabla|v_\varepsilon|, \nabla|v_\varepsilon|\rangle\,dx + R(\varepsilon)$$

where

$$R(\varepsilon) = (1-2/p)\int_{\Omega\setminus E_\varepsilon} \frac{1}{|v_\varepsilon|}\left(v_\varepsilon\langle \mathscr{A}\,\nabla v_\varepsilon|, \nabla v_\varepsilon\rangle - \langle \mathscr{A}\,\nabla v_\varepsilon, v_\varepsilon\,\nabla|v_\varepsilon|\rangle\right)\,dx -$$

$$(1-2/p)^2\int_{\Omega\setminus E_\varepsilon} \langle \mathscr{A}\,\nabla|v_\varepsilon|, \nabla|v_\varepsilon|\rangle\,dx.$$

It is proved in [13] that if $u\in C^2(\overline{\Omega})$ and $u|_{\partial\Omega}=0$, then

$$\lim_{\varepsilon\to 0}\varepsilon^r\int_{\Omega\setminus E_\varepsilon}|\nabla u|^2dx = 0 \tag{5.8}$$

for any $r>-1$. Since

$$|\,\nabla|v_\varepsilon|\,| = \left|\mathscr{R}e\left(\frac{\overline{v}_\varepsilon\nabla v_\varepsilon}{|v_\varepsilon|}\,\chi_{E_0}\right)\right| \leqslant |\nabla v_\varepsilon| = \varepsilon^{\frac{p-2}{2}}|\nabla u|$$

in $E_0\setminus E_\varepsilon$, we obtain

$$\left|\int_{\Omega\setminus E_\varepsilon}\langle \mathscr{A}\,\nabla|v_\varepsilon|, \nabla|v_\varepsilon|\rangle\,dx\right| \leqslant K\,\varepsilon^{p-2}\int_{\Omega\setminus E_\varepsilon}|\nabla u|^2dx \to 0$$

as $\varepsilon \to 0$. We have also

$$|v_\varepsilon|^{-1}\,|\langle v_\varepsilon\,\mathscr{A}\,\nabla|v_\varepsilon|, \nabla v_\varepsilon\rangle - \langle \mathscr{A}\,\nabla v_\varepsilon, v_\varepsilon\,\nabla|v_\varepsilon|\rangle| \leqslant K\,\varepsilon^{p-2}|\nabla u|^2$$

and thus $R(\varepsilon) = o(1)$ as $\varepsilon \to 0$.

We have proved that

$$\begin{gathered}\mathscr{R}e\int_\Omega \langle \mathscr{A}\,\nabla u, \nabla[\varrho_\varepsilon^2(|u|)\,u]\rangle\,dx = \mathscr{R}e\Big[\int_\Omega \langle \mathscr{A}\,\nabla v_\varepsilon, \nabla v_\varepsilon\rangle\,dx - \\ (1-2/p)\int_\Omega \langle (\mathscr{A}-\mathscr{A}^*)\,\nabla|v_\varepsilon|, |v_\varepsilon|^{-1}\overline{v}_\varepsilon\nabla v_\varepsilon\rangle dx - \\ (1-2/p)^2\int_\Omega \langle \mathscr{A}\,\nabla|v_\varepsilon|, \nabla|v_\varepsilon|\rangle\,dx\Big] + o(1).\end{gathered} \tag{5.9}$$

By means of similar computations, we find by the identity

$$\begin{gathered}\int_\Omega \langle \mathbf{b}\nabla u, |u|^{p-2}u\rangle dx = \int_{\Omega\setminus E_\varepsilon} \langle \mathbf{b}\nabla u, |u|^{p-2}u\rangle dx - \\ (1-2/p)\int_{E_\varepsilon} \langle \mathbf{b}, |v_\varepsilon|\nabla(|v_\varepsilon|)\rangle dx + \int_{E_\varepsilon} \langle \mathbf{b}\nabla v_\varepsilon, v_\varepsilon\rangle dx\end{gathered}$$

that

$$\begin{gathered}\mathscr{R}e\int_\Omega \langle \mathbf{b}\nabla u, |u|^{p-2}u\rangle dx = \\ \int_\Omega \langle \mathscr{R}e(\mathbf{b}/p), \nabla(|v_\varepsilon|^2)\rangle dx - \int_\Omega \langle \mathscr{I}m\,\mathbf{b}, \mathscr{I}m(\overline{v}_\varepsilon\nabla v)\rangle dx + o(1).\end{gathered} \tag{5.10}$$

Moreover

$$\begin{gathered}\int_\Omega |u|^p dx = \int_{E_\varepsilon} |u|^p dx + \int_{\Omega\setminus E_\varepsilon} |u|^p dx = \\ \int_{E_\varepsilon} |v_\varepsilon|^2 dx + \int_{\Omega\setminus E_\varepsilon} |u|^p dx = \int_\Omega |v_\varepsilon|^2 dx + o(1).\end{gathered} \tag{5.11}$$

Equalities (5.9), (5.10) and (5.11) lead to

$$
\begin{gathered}
\mathscr{R}e \int_\Omega (\langle \mathscr{A}\, \nabla u, \nabla[\varrho_\varepsilon^2(|u|)\, u]\rangle - \langle \mathbf{b}\nabla u, |u|^{p-2}u\rangle - a|u|^p) dx = \\
\mathscr{R}e \Big[\int_\Omega \langle \mathscr{A}\, \nabla v_\varepsilon, \nabla v_\varepsilon\rangle\, dx - \\
-(1-2/p) \int_\Omega \langle (\mathscr{A} - \mathscr{A}^*)\, \nabla |v_\varepsilon|, \nabla v_\varepsilon\rangle) v_\varepsilon |v_\varepsilon|^{-1} dx - \\
-(1-2/p)^2 \int_\Omega \langle \mathscr{A}\, \nabla |v_\varepsilon|, \nabla |v_\varepsilon|\rangle\, dx \Big] + \\
\int_\Omega \mathscr{R}e(\nabla^t(\mathbf{b}/p)|v_\varepsilon|^2 dx + \int_\Omega \langle \mathscr{I}m\, \mathbf{b}, \mathscr{I}m(\overline{v}_\varepsilon \nabla v)\rangle dx - \\
\int_\Omega \mathscr{R}e\, a\, |v_\varepsilon|^2 dx + o(1).
\end{gathered}
\tag{5.12}
$$

As far as the left-hand side of (5.12) is concerned, we have

$$
\begin{gathered}
\int_\Omega \langle \mathscr{A}\, \nabla u, \nabla[\varrho_\varepsilon^2(|u|)\, u]\rangle\, dx = \\
\varepsilon^{p-2} \int_{\Omega \setminus E_\varepsilon} \langle \mathscr{A}\, \nabla u, \nabla u\rangle\, dx + \int_{E_\varepsilon} \langle \mathscr{A}\, \nabla u, \nabla(|u|^{p-2}u)\rangle\, dx.
\end{gathered}
$$

and then

$$
\begin{gathered}
\lim_{\varepsilon\to 0} \mathscr{R}e \int_\Omega (\langle \mathscr{A}\, \nabla u, \nabla[\varrho_\varepsilon^2(|u|)\, u]\rangle - \langle \mathbf{b}\nabla u, |u|^{p-2}u\rangle - a|u|^p) dx = \\
\int_\Omega \langle \nabla u, \nabla(|u|^{p-2}u)\rangle - \langle \mathbf{b}\nabla u, |u|^{p-2}u\rangle - a|u|^p) dx.
\end{gathered}
$$

Letting $\varepsilon \to 0$ in (5.12), we complete the proof of the necessity.

Sufficiency. Suppose that (5.7) holds. Let $v \in \Xi$ and let u_ε be defined by (2.9). We have $u_\varepsilon \in \Xi$ and arguing as in the necessity part of Lemma 1, we find (2.10), (2.11) and (2.13). These limit relations lead to (5.4) for any $v \in \Xi$ and thus (5.4) is true for any $v \in H_0^1(\Omega)$ (see the proof of Lemma 2). In view of Lemma 2, the form $\mathscr{L}$ is L^p-dissipative. □

Theorem 3 *The operator A is L^p-dissipative if and only if the form $\mathscr{L}$ is L^p-dissipative.*

Proof.

Necessity. Let $u \in \Xi$ and $g_\varepsilon = (|u|^2 + \varepsilon^2)^{\frac{1}{2}}$. Since $g_\varepsilon^{p-2}\overline{u} \in \Xi$ we have

$$-\int_\Omega \langle \nabla^t(\mathscr{A}\,\nabla u), u\rangle g_\varepsilon^{p-2} dx = \int_\Omega \langle \mathscr{A}\,\nabla u, \nabla(g_\varepsilon^{p-2} u)\rangle dx$$

and since

$$\partial_h(g_\varepsilon^{p-2}\overline{u}) = (p-2) g_\varepsilon^{p-4}\,\mathscr{R}e(\langle \partial_h u, u\rangle)\,\overline{u} + g_\varepsilon^{p-2}\partial_h \overline{u}$$

we have also

$$\partial_h(g_\varepsilon^{p-2}\overline{u}) =$$

$$\begin{cases} (p-2)|u|^{p-4}\,\mathscr{R}e(\langle \partial_h u, u\rangle)\,\overline{u} + |u|^{p-2}\partial_h\overline{u} = \partial_h(|u|^{p-2}\overline{u}) & \text{if } x \in F_0 \\ \varepsilon^{p-2}\partial_h\overline{u} & \text{if } x \in \Omega \setminus F_0. \end{cases}$$

We find, keeping in mind (5.8), that

$$\lim_{\varepsilon\to 0}\int_\Omega \langle \mathscr{A}\,\nabla u, \nabla(g_\varepsilon^{p-2} u)\rangle dx = \int_\Omega \langle \mathscr{A}\,\nabla u, \nabla(|u|^{p-2} u)\rangle dx\,.$$

On the other hand, using Lemma 3.3 in [14], we see that

$$\lim_{\varepsilon\to 0}\int_\Omega \langle \nabla^t(\mathscr{A}\,\nabla u), u\rangle g_\varepsilon^{p-2} dx = \int_\Omega \langle \nabla^t(\mathscr{A}\,\nabla u), u\rangle |u|^{p-2} dx.$$

Then

$$-\int_\Omega \langle \nabla^t(\mathscr{A}\,\nabla u), u\rangle |u|^{p-2} dx = \int_\Omega \langle \mathscr{A}\,\nabla u, \nabla(|u|^{p-2} u)\rangle dx \qquad (5.13)$$

for any $u \in \Xi$. Hence

$$-\int_\Omega \langle Au, u\rangle |u|^{p-2} dx =$$

$$\int_\Omega (\langle \mathscr{A}\,\nabla u, \nabla(|u|^{p-2} u)\rangle - \langle \mathbf{b}\nabla u, |u|^{p-2} u\rangle - a\,|u|^p) dx\,.$$

Therefore (5.7) holds. We can conclude now that the form $\mathscr{L}$ is L^p-dissipative, because of Lemma 3.

Sufficiency. Given $u \in \mathscr{D}(A)$, we can find a sequence $\{u_n\} \subset \Xi$ such that $u_n \to u$ in $W^{2,p}(\Omega)$. Keeping in mind (5.13), we have

$$- \int_\Omega \langle Au, u\rangle |u|^{p-2} dx = - \lim_{n\to\infty} \int_\Omega \langle Au_n, u_n\rangle |u_n|^{p-2} dx =$$

$$\lim_{n\to\infty} \int_\Omega \langle \mathscr{A}\,\nabla u_n, \nabla(|u_n|^{p-2}u_n)\rangle - \langle \mathbf{b}\nabla u_n, |u_n|^{p-2}u_n\rangle - a\,|u_n|^p) dx.$$

Since $\mathscr{L}$ is L^p-dissipative, (5.7) holds for any $u \in \Xi$ and (5.3) is true for any $u \in \mathscr{D}(A)$. □

Definition 3 We say that the operator A is L^p-quasi-dissipative if there exists $\omega \geqslant 0$ such that $A - \omega I$ is L^p-dissipative, i.e.

$$\mathscr{R}e \int_\Omega \langle Au, u\rangle |u|^{p-2} dx \leqslant \omega\, \|u\|_p^p$$

for any $u \in \mathscr{D}(A)$.

Lemma 4 *The operator* (5.1) *is L^p-quasi-dissipative if and only if there exists $\omega \geqslant 0$ such that*

$$\begin{gathered}\mathscr{R}e \int_\Omega \Big[\langle \mathscr{A}\,\nabla v, \nabla v\rangle - (1-2/p)\langle(\mathscr{A}-\mathscr{A}^*)\nabla(|v|), |v|^{-1}\overline{v}\nabla v\rangle - \\ (1-2/p)^2\langle \mathscr{A}\,\nabla(|v|), \nabla(|v|)\rangle\Big] dx + \int_\Omega \langle \mathscr{I}m\,\mathbf{b}, \mathscr{I}m(\overline{v}\nabla v)\rangle\, dx + \\ \int_\Omega \mathscr{R}e(\nabla^t(\mathbf{b}/p) - a)|v|^2 dx \geqslant -\omega \int_\Omega |v|^2 dx\end{gathered} \tag{5.14}$$

for any $v \in H_0^1(\Omega)$.

Proof. The result follows from Lemma 2. □

The next result permits to determine the best interval of p's for which the operator

$$Au = \nabla^t(\mathscr{A}\,\nabla u) \tag{5.15}$$

is L^p-dissipative. We set

$$\lambda = \inf_{(\xi,x)\in\mathcal{M}} \frac{\langle \mathscr{R}e\, \mathscr{A}(x)\xi, \xi\rangle}{|\langle \mathscr{I}m\, \mathscr{A}(x)\xi, \xi\rangle|}$$

where $\mathcal{M}$ is the set of (ξ, x) with $\xi \in \mathbb{R}^n$, $x \in \Omega$ such that $\langle \mathscr{I}m\, \mathscr{A}(x)\xi, \xi\rangle \neq 0$.

Corollary 6 *Let A be the operator* (5.15)*. Let us suppose that the matrix $\mathscr{I}m\,\mathscr{A}$ is symmetric and that*

$$\langle \mathscr{R}e\,\mathscr{A}(x)\xi,\xi\rangle \geqslant 0 \tag{5.16}$$

for any $x\in\Omega$, $\xi\in\mathbb{R}^n$. If $\mathscr{I}m\,\mathscr{A}(x)=0$ for any $x\in\Omega$, A is L^p-dissipative for any $p>1$. If $\mathscr{I}m\,\mathscr{A}$ does not vanish identically on Ω, A is L^p-dissipative if and only if

$$2+2\lambda(\lambda-\sqrt{\lambda^2+1})\leqslant p\leqslant 2+2\lambda(\lambda+\sqrt{\lambda^2+1}). \tag{5.17}$$

Proof.

When $\mathscr{I}m\,\mathscr{A}(x)=0$ for any $x\in\Omega$, the statement follows from Theorem 1. Let us assume that $\mathscr{I}m\,\mathscr{A}$ does not vanish identically; note that this implies $\mathcal{M}\neq\emptyset$.

Necessity. If the operator (5.15) is L^p-dissipative, Theorem 1 shows that

$$|p-2|\,|\langle \mathscr{I}m\,\mathscr{A}(x)\xi,\xi\rangle|\leqslant 2\sqrt{p-1}\,\langle \mathscr{R}e\,\mathscr{A}(x)\xi,\xi\rangle \tag{5.18}$$

for any $x\in\Omega$, $\xi\in\mathbb{R}^n$. In particular we have

$$\frac{|p-2|}{2\sqrt{p-1}}\leqslant\frac{\langle \mathscr{R}e\,\mathscr{A}(x)\xi,\xi\rangle}{|\langle \mathscr{I}m\,\mathscr{A}(x)\xi,\xi\rangle|}$$

for any $(\xi,x)\in\mathcal{M}$ and then

$$\frac{|p-2|}{2\sqrt{p-1}}\leqslant\lambda.$$

This inequality is equivalent to (5.17).

Sufficiency. If (5.17) holds, we have $(p-2)^2\leqslant 4(p-1)\lambda^2$. Note that $p>1$, because $2+2\lambda(\lambda-\sqrt{\lambda^2+1})>1$.

Since $\lambda\geqslant 0$ in view of (5.16), we find $|p-2|\leqslant 2\sqrt{p-1}\lambda$ and (5.18) is true for any $(\xi,x)\in\mathcal{M}$. On the other hand, if $x\in\Omega$ and $\xi\in\mathbb{R}^n$ with $(\xi,x)\notin\mathcal{M}$, (5.18) is trivially satisfied and then it holds for any $x\in\Omega$, $\xi\in\mathbb{R}^n$. Theorem 1 gives the result. □

The next Corollary provides a characterization of operators which are L^p-dissipative only for $p=2$.

Corollary 7 *Let A be as in Corollary 6. The operator A is L^p-dissipative only for $p=2$ if and only if $\mathscr{I}m\,\mathscr{A}$ does not vanish identically and $\lambda=0$.*

Proof. Inequalities (5.17) are satisfied only for $p=2$ if and only if $\lambda(\lambda-\sqrt{\lambda^2-1})=\lambda(\lambda+\sqrt{\lambda^2+1})$ and this happens if and only if $\lambda=0$. Thus the result is a consequence of Corollary 6. □

From now on we suppose that the operator is strongly elliptic in Ω in the sense that

$$\langle \mathscr{R}e\,\mathscr{A}(x)\xi,\xi\rangle > 0$$

for any $x\in\overline{\Omega}$, $\xi\in\mathbb{R}^n\setminus\{0\}$.

We have proved that, if $\mathscr{I}m\,\mathscr{A}$ is symmetric, the algebraic condition (3.2) is necessary and sufficient for the L^p-dissipativity of the operator (5.15). We have shown that this is not true for the more general operator (5.1). The next result shows that condition (3.2) is necessary and sufficient for the L^p-quasi-dissipativity of (5.1). We emphasize that here we do not require the symmetry of $\mathscr{I}m\,\mathscr{A}$.

Theorem 4 *The strongly elliptic operator* (5.1) *is L^p-quasi-dissipative if and only if*

$$|p-2|\,|\langle \mathscr{I}m\,\mathscr{A}(x)\xi,\xi\rangle| \leqslant 2\sqrt{p-1}\,\langle \mathscr{R}e\,\mathscr{A}(x)\xi,\xi\rangle \tag{5.19}$$

for any $x\in\Omega$, $\xi\in\mathbb{R}^n$.

Proof.

Necessity. By using the functions X, Y introduced in Corollary 1, we write condition (5.14) in the form

$$\int_\Omega \Big\{\frac{4}{p\,p'}\langle \mathscr{R}e\,\mathscr{A}\,X,X\rangle + \langle \mathscr{R}e\,\mathscr{A}\,Y,Y\rangle +$$
$$2\langle (p^{-1}\,\mathscr{I}m\,\mathscr{A} + p'^{-1}\,\mathscr{I}m\,\mathscr{A}^*)X,Y\rangle + \langle \mathscr{I}m\,\mathbf{b},Y\rangle |v| +$$
$$\mathscr{R}e\left[\nabla^t(\mathbf{b}/p) - a + \omega\right]|v|^2\Big\}dx \geqslant 0\,.$$

As in the proof of Corollary 1, this inequality implies

$$\frac{4}{p\,p'}\int_\Omega \langle \mathscr{R}e\,\mathscr{A}\,\nabla\varrho,\nabla\varrho\rangle dx + \mu^2\int_\Omega \langle \mathscr{R}e\,\mathscr{A}\,\nabla\varrho,\nabla\varrho\rangle dx +$$
$$2\mu\int_\Omega \langle (p^{-1}\,\mathscr{I}m\,\mathscr{A} + p'^{-1}\,\mathscr{I}m\,\mathscr{A}^*)\nabla\varrho,\nabla\varrho\rangle dx +$$
$$\mu\int_\Omega \varrho\langle \mathscr{I}m\,\mathbf{b},\nabla\varrho\rangle dx + \int_\Omega \mathscr{R}e\left[\nabla^t(\mathbf{b}/p) - a + \omega\right]\varrho^2 dx \geqslant 0$$

for any $\varrho \in C_0^1(\Omega)$, $\mu \in \mathbb{R}$. Since

$$\langle \mathscr{I}m\, \mathscr{A}^* \nabla\varrho, \nabla\varrho\rangle = -\langle \mathscr{I}m\, \mathscr{A}^t \nabla\varrho, \nabla\varrho\rangle = -\langle \mathscr{I}m\, \mathscr{A} \nabla\varrho, \nabla\varrho\rangle$$

we have

$$\frac{4}{p\,p'} \int_\Omega \langle \mathscr{R}e\, \mathscr{A} \nabla\varrho, \nabla\varrho\rangle dx + \mu^2 \int_\Omega \langle \mathscr{R}e\, \mathscr{A} \nabla\varrho, \nabla\varrho\rangle dx -$$
$$2(1-2/p)\mu \int_\Omega \langle \mathscr{I}m\, \mathscr{A} \nabla\varrho, \nabla\varrho\rangle dx +$$
$$\mu \int_\Omega \varrho \langle \mathscr{I}m\, \mathbf{b}, \nabla\varrho\rangle dx + \int_\Omega \mathscr{R}e \left[\nabla^t (\mathbf{b}/p) - a + \omega\right] \varrho^2 dx \geqslant 0$$

for any $\varrho \in C_0^1(\Omega)$, $\mu \in \mathbb{R}$.

Taking $\varrho(x) = \psi(x)\cos\langle\xi, x\rangle$ and $\varrho(x) = \psi(x)\sin\langle\xi, x\rangle$ with $\psi \in C_0^1(\Omega)$ and arguing as in the proof of Corollary 1, we find

$$\int_\Omega \langle \mathscr{B} \nabla\psi, \nabla\psi\rangle dx + \int_\Omega \langle \mathscr{B}\,\xi, \xi\rangle \psi^2 dx +$$
$$\mu \int_\Omega \langle \mathscr{I}m\, \mathbf{b}, \nabla\psi\rangle \psi\, dx + \int_\Omega \mathscr{R}e \left[\nabla^t (\mathbf{b}/p) - a + \omega\right] \psi^2 dx \geqslant 0\,,$$

where $\mu \in \mathbb{R}$ and

$$\mathscr{B} = \frac{4}{p\,p'} \mathscr{R}e\, \mathscr{A} + \mu^2\, \mathscr{R}e\, \mathscr{A} - 2(1-2/p)\mu\, \mathscr{I}m\, \mathscr{A}\ .$$

Because of the arbitrariness of ξ we see that

$$\int_\Omega \langle \mathscr{B}\,\xi, \xi\rangle \psi^2 dx \geqslant 0$$

for any $\psi \in C_0^1(\Omega)$. Hence $\langle \mathscr{B}\,\xi, \xi\rangle \geqslant 0$, i.e.

$$\frac{4}{p\,p'} \langle \mathscr{R}e\, \mathscr{A}\, \xi, \xi\rangle + \mu^2 \langle \mathscr{R}e\, \mathscr{A}\, \xi, \xi\rangle - 2(1-2/p)\mu \langle \mathscr{I}m\, \mathscr{A}\, \xi, \xi\rangle \geqslant 0$$

for any $x \in \Omega$, $\xi \in \mathbb{R}^n$, $\mu \in \mathbb{R}$. Inequality (5.19) follows from the arbitrariness of μ.

Sufficiency. Assume first that $\mathscr{I}m\, \mathscr{A}$ is symmetric. By repeating the first part of the proof of sufficiency of Theorem 1, we find that (5.19) implies

$$\frac{4}{p\,p'} \langle \mathscr{R}e\, \mathscr{A}\, \xi, \xi\rangle + \langle \mathscr{R}e\, \mathscr{A}\, \eta, \eta\rangle - 2(1-p/2) \langle \mathscr{I}m\, \mathscr{A}\, \xi, \eta\rangle \geqslant 0 \qquad (5.20)$$

for any $x \in \Omega$, $\xi, \eta \in \mathbb{R}^n$.

In order to prove (5.14), it is not restrictive to suppose

$$\mathscr{R}e(\nabla^t(\mathbf{b}/p) - a) = 0.$$

Since A is strongly elliptic, there exists a non singular real matrix $\mathscr{C} \in C^1(\overline{\Omega})$ such that

$$\langle \mathscr{R}e\, \mathscr{A}\, \eta, \eta\rangle = \langle \mathscr{C}\, \eta, \mathscr{C}\, \eta\rangle$$

for any $\eta \in \mathbb{R}^n$. Setting

$$\mathscr{S} = (1-2/p)(\mathscr{C}^t)^{-1}\, \mathscr{I}m\, \mathscr{A},$$

we have

$$|\,\mathscr{C}\, \eta - \mathscr{S}\, \xi|^2 = \langle \mathscr{R}e\, \mathscr{A}\, \eta, \eta\rangle - 2(1-p/2)\langle \mathscr{I}m\, \mathscr{A}\, \xi, \eta\rangle + |\, \mathscr{S}\, \xi|^2.$$

This leads to the identity

$$\begin{gathered}\frac{4}{p\,p'}\langle \mathscr{R}e\, \mathscr{A}\, \xi, \xi\rangle + \langle \mathscr{R}e\, \mathscr{A}\, \eta, \eta\rangle - 2(1-p/2)\langle \mathscr{I}m\, \mathscr{A}\, \xi, \eta\rangle = \\ |\,\mathscr{C}\, \eta - \mathscr{S}\, \xi|^2 + \frac{4}{p\,p'}\langle \mathscr{R}e\, \mathscr{A}\, \xi, \xi\rangle - |\, \mathscr{S}\, \xi|^2\end{gathered} \tag{5.21}$$

for any $\xi, \eta \in \mathbb{R}^n$. In view of (5.20), putting $\eta = \mathscr{C}^{-1}\, \mathscr{S}\, \xi$ in (5.21), we obtain

$$\frac{4}{p\,p'}\langle \mathscr{R}e\, \mathscr{A}\, \xi, \xi\rangle - |\, \mathscr{S}\, \xi|^2 \geqslant 0 \tag{5.22}$$

for any $\xi \in \mathbb{R}^n$.

On the other hand, we may write

$$\begin{gathered}\langle \mathscr{I}m\, \mathbf{b}, Y\rangle = \langle (\mathscr{C}^{-1})^t\, \mathscr{I}m\, \mathbf{b}, \mathscr{C}\, Y\rangle = \\ \langle (\mathscr{C}^{-1})^t\, \mathscr{I}m\, \mathbf{b}, \mathscr{C}\, Y - \mathscr{S}\, X\rangle + \langle (\mathscr{C}^{-1})^t\, \mathscr{I}m\, \mathbf{b}, \mathscr{S}\, X\rangle\, .\end{gathered}$$

By the Cauchy inequality

$$\begin{gathered}\int_\Omega \langle (\mathscr{C}^{-1})^t\, \mathscr{I}m\, \mathbf{b}, \mathscr{C}\, Y - \mathscr{S}\, X\rangle |v|\, dx \geqslant \\ -\int_\Omega |\,\mathscr{C}\, Y - \mathscr{S}\, X|^2 dx - \frac{1}{4}\int_\Omega |(\mathscr{C}^{-1})^t\, \mathscr{I}m\, \mathbf{b}|^2 |v|^2 dx\end{gathered}$$

and, integrating by parts,

$$\int_\Omega \langle (\mathscr{C}^{-1})^t\, \mathscr{I}m\, \mathbf{b}, \mathscr{S}\, X\rangle |v|\, dx = \frac{1}{2}\int_\Omega \langle (\mathscr{C}^{-1}\,\mathscr{S})^t\, \mathscr{I}m\, \mathbf{b}, \nabla(|v|^2)\rangle\, dx = \\ -\frac{1}{2}\int_\Omega \nabla^t((\mathscr{C}^{-1}\,\mathscr{S})^t\, \mathscr{I}m\, \mathbf{b})\, |v|^2\, dx\,.$$

This implies that there exists $\omega \geqslant 0$ such that

$$\int_\Omega \langle \mathscr{I}m\, \mathbf{b}, Y\rangle\, |v|\, dx \geqslant -\int_\Omega |\,\mathscr{C}\, Y - \mathscr{S}\, X|^2 dx - \omega \int_\Omega |v|^2 dx$$

and then, in view of (5.21),

$$\int_\Omega \Big\{ \frac{4}{p\,p'} \langle \mathscr{R}e\, \mathscr{A}\, X, X\rangle + \langle \mathscr{R}e\, \mathscr{A}\, Y, Y\rangle + \\ 2(1-p/2)\langle \mathscr{I}m\, \mathscr{A}\, X, Y\rangle + \langle \mathscr{I}m\, \mathbf{b}, Y\rangle |v| \Big\} dx \geqslant \\ \int_\Omega \left(\frac{4}{p\,p'} \langle \mathscr{R}e\, \mathscr{A}\, X, X\rangle - |\,\mathscr{S}\, X|^2 \right) dx - \omega \int_\Omega |v|^2 dx\ .$$

Inequality (5.22) gives the result.

We have proved the sufficiency under the assumption $\mathscr{I}m\, \mathscr{A}^t = \mathscr{I}m\, \mathscr{A}$. In the general case, the operator A can be written in the form

$$Au = \nabla^t((\mathscr{A} + \mathscr{A}^t)\nabla u)/2 + \mathbf{c}\nabla u + au$$

where

$$\mathbf{c} = \nabla^t(\mathscr{A} - \mathscr{A}^t)/2 + \mathbf{b}.$$

Since $(\mathscr{A} + \mathscr{A}^t)$ is symmetric, we know that A is L^p-quasi-dissipative if and only if

$$|p-2|\, |\langle \mathscr{I}m(\mathscr{A} + \mathscr{A}^t)\xi, \xi\rangle| \leqslant 2\sqrt{p-1}\, \langle \mathscr{R}e(\mathscr{A} + \mathscr{A}^t)\xi, \xi\rangle$$

for any $\xi \in \mathbb{R}^n$, which is exactly condition (5.19). □

Corollary 8 *Let A be the strongly elliptic operator* (5.1)*. If $\mathscr{I}m\, \mathscr{A}(x) = 0$ for any $x \in \Omega$, A is L^p-quasi-dissipative for any $p > 1$. If $\mathscr{I}m\, \mathscr{A}$ does not vanish identically on Ω, A is L^p-quasi-dissipative if and only if* (5.17) *holds.*

Proof. The proof is similar to that of Corollary 6, the role of Theorem 1 being played by Theorem 4. □

We give a criterion for the L^p-contractivity of the semigroup generated by A.

Theorem 5 *Let A be the strongly elliptic operator* (5.15) *with $\mathscr{I}m\,\mathscr{A} = \mathscr{I}m\,\mathscr{A}^t$. The operator A generates a contraction semigroup on L^p if and only if*

$$|p-2|\,|\langle \mathscr{I}m\,\mathscr{A}(x)\xi,\xi\rangle| \leqslant 2\sqrt{p-1}\,\langle \mathscr{R}e\,\mathscr{A}(x)\xi,\xi\rangle \tag{5.23}$$

for any $x\in\Omega$, $\xi\in\mathbb{R}^n$.

Proof.

Sufficiency. It is a classical result that the operator A defined on (5.2) and acting in $L^p(\Omega)$ is a densely defined closed operator (see [1], [17, Theorem 1, p.302]).

From Theorem 1 we know that the form $\mathscr{L}$ is L^p-dissipative and Theorem 3 shows that A is L^p-dissipative. Finally the formal adjoint operator

$$A^*u = \nabla^t(\mathscr{A}^*\,\nabla u)$$

with $\mathscr{D}(A^*) = W^{2,p'}(\Omega)\cap W_0^{1,p'}(\Omega)$, is the adjoint operator of A and since $\mathscr{I}m\,\mathscr{A}^* = \mathscr{I}m(\mathscr{A}^*)^t$ and (5.23) can be written as

$$|p'-2|\,|\langle \mathscr{I}m\,\mathscr{A}^*(x)\xi,\xi\rangle| \leqslant 2\sqrt{p'-1}\,\langle \mathscr{R}e\,\mathscr{A}^*(x)\xi,\xi\rangle, \tag{5.24}$$

we have also the $L^{p'}$-dissipativity of A^*.

The result is a consequence of the following well known result: *if A is a densely defined closed operator and if both A and A^* are dissipative, then A is the infinitesimal generator of a C_0 contraction semigroup* (see, e.g., [20, p.15]).

Necessity. If A generates a contraction semigroup on L^p, it is L^p-dissipative. Therefore (5.23) holds because of Theorem 1. □

Let us assume that either A has lower order terms or they are absent and $\mathscr{I}m\,\mathscr{A}$ is not symmetric. The next Theorem gives a criterion for the L^p-quasi-contractivity of the semigroup generated by A (i.e. the L^p-contractivity of the semigroup generated by $A-\omega I$).

Theorem 6 *Let A be the strongly elliptic operator* (5.1)*. The operator A generates a quasi-contraction semigroup on L^p if and only if* (5.23) *holds for any $x \in \Omega$, $\xi \in \mathbb{R}^n$.*

Proof.

Sufficiency. Let us consider A as an operator defined on (5.2) and acting in $L^p(\Omega)$. As in the proof of Theorem 5, one can see that A is a densely defined closed operator and that the formal adjoint coincides with the adjoint A^*. Theorem 4 shows that A is L^p-quasi-dissipative. On the other hand, condition (5.24) holds and then A^* is $L^{p'}$-quasi-dissipative. As in Theorem 5, this implies that A generates a quasi-contraction semigroup on L^p.

Necessity. If A generates a quasi-contraction semigroup on L^p, A is L^p-quasi-dissipative and (5.23) holds. □

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