Can one see the fundamental frequency of a drum?

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To the memory of Felix A. Berezin

Abstract

We establish two-sided estimates for the fundamental frequency (the lowest eigenvalue) of the Laplacian in an open set $\Omega \subset \mathbb{R}^n$ with the Dirichlet boundary condition. This is done in terms of the interior capacitary radius of Ω which is defined as the maximal possible radius of a ball B with a *negligible* intersection with the complement of Ω. Here negligibility of $F \subset B$ means that

$$\operatorname{cap}(F) \le \gamma \operatorname{cap}(B),$$

where cap means the Wiener (harmonic) capacity and γ is arbitrarily fixed with the sole restriction $0 < \gamma < 1$. We provide explicit values of constants in the two-sided estimates.

1 Main result

Let us consider an open set $\Omega \subset \mathbb{R}^n$ and denote the bottom of the spectrum of its minus Dirichlet Laplacian $(-\Delta)_{\mathrm{Dir}}$ by $\lambda(\Omega)$. (We understand the minus Dirichlet Laplacian as the self-adjoint operator which is the Friedrichs extension of the operator $-\Delta$ defined on $C_0^\infty(\Omega)$.) In case when Ω is a bounded domain with a sufficiently regular boundary, $\lambda(\Omega)$ is the lowest eigenvalue of $-\Delta$ with the Dirichlet boundary condition on $\partial\Omega$. In the general case we can write

$$\lambda(\Omega) = \inf_{u \in C_0^\infty(\Omega)} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega |u|^2 dx}. \tag{1.1}$$

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It follows that $\Omega' \subset \Omega$ implies $\lambda(\Omega) \le \lambda(\Omega'))$. In particular, if B_r is an open ball of radius r, such that $B_r \subset \Omega$, then $\lambda(\Omega) \le \lambda(B_r) = C_n r^{-2}$ where $C_n = \lambda(B_1)$. It follows that for the interior radius of Ω, which is defined as

$$r_\Omega = \sup\{r |\, \exists B_r \subset \Omega\},$$

we have

$$\lambda(\Omega) \le C_n r_\Omega^{-2}.$$

But this estimate is not good for unbounded domains or domains with complicated boundaries. In particular, a similar estimate from below does not hold in general.

The way to improve this estimate is to relax the requirement for B_r to be completely inside Ω by allowing some part of B_r, which has a "small" Wiener capacity, to stick out of Ω. Namely, let us take an *arbitrary* $\gamma \in (0,1)$ and call a compact set $F \subset \bar{B}_r$ negligible (or, more precisely, γ-negligible) if

$$\operatorname{cap}(F) \le \gamma \operatorname{cap}(\bar{B}_r). \tag{1.2}$$

(Here $\operatorname{cap}(F)$ denotes the Wiener capacity of F, $\bar{B}_r$ is the closure of B_r.)

Now denote

$$r_{\Omega,\gamma} = \inf\{r |\, \exists B_r,\ \bar{B}_r \setminus \Omega \text{ is } \gamma\text{-negligible}\}.$$

This is the *interior capacitary radius.*

Theorem 1.1 *Let us fix $\gamma \in (0,1)$. Then there exist $c = c(\gamma, n) > 0$ and $C = C(\gamma, n) > 0$, such that for every open set $\Omega \subset \mathbb{R}^n$*

$$c r_{\Omega,\gamma}^{-2} \le \lambda(\Omega) \le C r_{\Omega,\gamma}^{-2}. \tag{1.3}$$

Explicit values of constants $c = c(\gamma, n)$ and $C(\gamma, n)$ are provided below in (3.19) *and* (4.16) *respectively.*

Let us formulate some corollaries of this theorem.

Corollary 1.2 $\lambda(\Omega) > 0$ *if and only if* $r_{\Omega,\gamma} < \infty$.

This corollary gives a necessary and sufficient condition of strict positivity of the operator $(-\Delta)_{\mathrm{Dir}}$ in Ω.

Since the condition $\lambda(\Omega) > 0$ does not contain γ, we immediately obtain

Corollary 1.3 *Conditions $r_{\Omega,\gamma} < \infty$, taken for different γ's, are equivalent.*

Denoting $F = \mathbb{R}^n \setminus \Omega$ (which can be an arbitrary closed subset in $\mathbb{R}^n$), we obtain from the previous corollary (comparing $\gamma = 0.01$ and $\gamma = 0.99$):

Corollary 1.4 *Let F be a closed subset in $\mathbb{R}^n$, which has the following property: there exists $r > 0$ such that*

$$\operatorname{cap}(F \cap \bar{B}_r) \geq 0.01 \operatorname{cap}(\bar{B}_r)$$

for all B_r. Then there exists $r_1 > 0$ such that

$$\operatorname{cap}(F \cap \bar{B}_{r_1}) \geq 0.99 \operatorname{cap}(\bar{B}_{r_1})$$

for all B_{r_1}.

This is a new property of capacity which is proved by spectral theory arguments.

Once upon a time Marc Kac [7] formulated a fascinating question: **"Can one hear the shape of a drum?"** The precise meaning of this question is as follows: is it possible to reconstruct the drum (a bounded domain in $\mathbb{R}^2$) up to an isometry by the spectrum of its Dirichlet Laplacian?

Theorem 1.1 suggests formulation of a question, which is roughly inverse to the question above: **"Can one see the fundamental frequency of a drum?"** More precisely, can one find a simple visual image related to a domain in $\mathbb{R}^2$ (or $\mathbb{R}^n$), such that it allows to recover the lowest eigenvalue of the Dirichlet Laplacian in this domain, or at least give reasonably good estimates of this eigenvalue? Assuming that our eye (possibly armed by a visual aid device) can filter out the sets of "small" capacity, a partial answer to this question is given by Theorem 1.1.

The inequalities (1.3) for sufficiently small $\gamma > 0$ (comparable with $(4n)^{-4n}$) were established in [13] (see also Chapters 10 and 11 in [14]). Theorem 1.1 provides a substantial improvement, in particular allowing corollaries 1.3 and 1.4 and providing explicit values of the constants.

The proof of Theorem 1.1 is based on the ideas of our paper [17]. Necessary definitions and results about the Wiener capacity can be found e.g. in [14, 25].

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2 Preliminaries on capacity

In this section we will recall some definitions and introduce necessary notations. For simplicity we will always assume that $n \geq 3$.

For every subset $\mathcal{D} \subset \mathbb{R}^n$ denote by $\operatorname{Lip}(\mathcal{D})$ the space of (real-valued) functions satisfying the uniform Lipschitz condition in $\mathcal{D}$, and by $\operatorname{Lip}_c(\mathcal{D})$ the subspace in $\operatorname{Lip}(\mathcal{D})$ of all functions with compact support in $\mathcal{D}$ (this will be only used when $\mathcal{D}$ is open). By $\operatorname{Lip}_{loc}(\mathcal{D})$ we will denote the set of functions on (an open set) $\mathcal{D}$ which are Lipschitz on any compact subset $K \subset \mathcal{D}$. Note that $\operatorname{Lip}(\mathcal{D}) = \operatorname{Lip}(\bar{\mathcal{D}})$ for any bounded $\mathcal{D}$.

If F is a compact subset in $\mathbb{R}^n$, then the Wiener capacity of F is defined as

$$\operatorname{cap}(F) = \inf\left\{\int_{\mathbb{R}^n} |\nabla u(x)|^2 dx \,\middle|\, u \in \operatorname{Lip}_c(\mathbb{R}^n), u|_F = 1\right\}. \tag{2.1}$$

Note that the infimum does not change if we restrict ourselves to the functions $u \in \operatorname{Lip}_c(\mathbb{R}^n)$ such that $0 \le u \le 1$ everywhere (see e.g. [14], Sect. 2.2.1).

We will also need another (equivalent) definition of the Wiener capacity $\operatorname{cap}(F)$ for a compact set $F \subset \bar{B}_r$. For $n \ge 3$ it is as follows:

$$\operatorname{cap}(F) = \sup\{\mu(F) \,\Big|\, \int_F \mathcal{E}(x-y)d\mu(y) \le 1 \quad \text{on } \mathbb{R}^n \setminus F\}, \tag{2.2}$$

where the supremum is taken over all positive finite Radon measures μ on F and $\mathcal{E} = \mathcal{E}_n$ is the standard fundamental solution of $-\Delta$ in $\mathbb{R}^n$ i.e.

$$\mathcal{E}(x) = \frac{1}{(n-2)\omega_n}|x|^{2-n}\ ,$$

with ω_n being the area of the unit sphere $S^{n-1} \subset \mathbb{R}^n$. The maximizing measure in (2.2) exists and is unique. We will denote it μ_F and call it the *equilibrium measure*. Note that

$$\operatorname{cap}(F) = \mu_F(F) = \mu_F(\mathbb{R}^n) = \langle \mu_F, 1\rangle = \int_F d\mu_F. \tag{2.3}$$

The corresponding potential will be denoted P_F, so

$$P_F(x) = \int_F \mathcal{E}(x-y)d\mu_F(y), \quad x \in \mathbb{R}^n \setminus F.$$

We will call P_F the *equilibrium potential* or *capacitary potential*. We will extend it to F by setting $P_F(x) = 1$ for all $x \in F$.

It follows from the maximum principle that $0 \le P_F \le 1$ everywhere in $\mathbb{R}^n$.

In case when F is the closure of an open subset with a smooth boundary, $u = P_F$ is the unique minimizer for the Dirichlet integral in (2.1). In particular,

$$\int |\nabla P_F|^2 dx = \operatorname{cap}(F).$$

where the integration is taken over $\mathbb{R}^n$ (or $\mathbb{R}^n \setminus F$).

The capacity of the ball $\bar{B}_r$ is easily calculated and is given by

$$\operatorname{cap}(\bar{B}_r) = (n-2)\omega_n r^{n-2}. \tag{2.4}$$

3 Lower bound

In this section we will establish the lower bound for $\lambda(\Omega)$ from Theorem 1.1 which is an easier part of this theorem. The key part of the lower bound proof is presented in the following lemma, which was first proved in [11] (see also [14], where it is present as a particular case of a much more general Theorem 10.1.2, part 1), though without an explicit constant, which we provide to specify explicit constants in Theorem 1.1.

Lemma 3.1 *The following inequality holds for every complex-valued function $u \in \mathrm{Lip}(\bar{B}_r)$ which vanishes on a co.mpact set $F \subset \bar{B}_r$ (but is not identically 0 on $\bar{B}_r$):*

$$\mathrm{cap}\,(F) \le \frac{C_n \int_{B_r} |\nabla u(x)|^2 dx}{r^{-n} \int_{B_r} |u(x)|^2 dx}\ , \tag{3.1}$$

where

$$C_n = 4\omega_n \left(1 - \frac{2}{n^2}\right) \tag{3.2}$$

Beginning of Proof. A. Clearly, it is sufficient to consider the ball B_r centered at 0, and real-valued functions $u \in \mathrm{Lip}(\bar{B}_r)$. By scaling we see that it suffices to consider the case $r = 1$. (The corresponding estimate for an arbitrary $r > 0$ follows from the one with $r = 1$ with the same constant C_n.) So we need to prove the estimate

$$\int_{B_1} |u|^2 dx \le \frac{C_n}{\mathrm{cap}\,(F)} \int_{B_1} |\nabla u|^2 dx, \tag{3.3}$$

where F is a compact subset of $\bar{B}_1$, $u \in \mathrm{Lip}(\bar{B}_1)$ and $u|_F = 0$.

To be able to use (2.1), consider the following function $U \in \mathrm{Lip}(\mathbb{R}^n)$:

$$U(x) = \begin{cases} 1 - |u(x)|, & \text{if } |x| \le 1, \\ |x|^{2-n}(1 - |u(|x|^{-2}x)|), & \text{if } |x| \ge 1, \end{cases}$$

i.e. U extends $1 - |u|$ to $\{x : |x| \ge 1\}$ as the Kelvin transform of $1 - |u|$. Clearly, $U|_F = 1$, $|\nabla U| = |\nabla u|$ almost everywhere in B_1, $U(x) = O(|x|^{n-2})$ and $|\nabla U(x)| = O(|x|^{1-n})$ as $|x| \to \infty$. It follows that U can serve as a test function in (2.1), i.e.

$$\mathrm{cap}\,(F) \le \int_{\mathbb{R}^n} |\nabla U|^2 dx. \tag{3.4}$$

Using the harmonicity of $|x|^{2-n}$ and the Green-Stokes formula, we obtain by a straightforward calculation

$$\int_{\mathbb{R}^n} |\nabla U|^2 dx = 2 \int_{B_1} |\nabla u|^2 dx + (n-2) \int_{\partial B_1} (1 - |u(\omega)|)^2 d\omega, \tag{3.5}$$

where $\partial B_1 = \{\omega \in \mathbb{R}^n,\ |\omega| = 1\}$ is the boundary of B_1 (the unit sphere in $\mathbb{R}^n$), $d\omega$ means the standard volume element on ∂B_1.

B. For a function v on ∂B_1 define its average

$$\bar{v} = \fint_{\partial B_1} v d\omega = \frac{1}{\omega_n} \int_{\partial B_1} v d\omega \,.$$

To continue the proof of Lemma 3.1, we will need the following elementary lemma.

Lemma 3.2 *For any* $v \in \mathrm{Lip}(B_1)$,

$$\int_{\partial B_1} |v - \bar{v}|^2 d\omega \leq \int_{B_1} |\nabla v|^2 dx. \tag{3.6}$$

Proof of Lemma 3.2. It suffices to prove it for real-valued functions v. Let us expand v in spherical functions. Let

$$\{Y_{k,l} |\, l = 0, 1, \ldots, n_k,\ k = 0, 1, \ldots\}$$

be an orthonormal basis in $L^2(\partial B_1)$ which consists of eigenfunctions of the (negative) Laplace-Beltrami operator Δ_ω on ∂B_1, so that the eigenfunctions $Y_{k,l} = Y_{k,l}(\omega)$ with a fixed k have the same eigenvalue $-k(k+n-2)$ (which has multiplicity $n_k + 1$). Note that the zero eigenvalue (corresponding to $k = 0$) has multiplicity 1 and $Y_{0,0} = const = \omega_n^{-1/2}$ for the corresponding eigenfunction.

Writing $x = r\omega$, where $r = |x|$, $\omega = x/|x|$, we can present v in the form

$$v(x) = v(r, \omega) = \sum_{k,l} v_{k,l}(r) Y_{k,l}(\omega). \tag{3.7}$$

Then

$$\int_{B_1} |v(x)|^2 dx = \sum_{k,l} \int_0^1 |v_{k,l}(r)|^2 r^{n-1} dr, \tag{3.8}$$

and

$$\int_{\partial B_1} |v(\omega)|^2 d\omega = \sum_{k,l} |v_{k,l}(1)|^2. \tag{3.9}$$

It follows that

$$\int_{\partial B_1} |v(\omega) - \bar{v}|^2 d\omega = \sum_{\{k,l: k \geq 1\}} |v_{k,l}(1)|^2. \tag{3.10}$$

Taking into account that

$$|\nabla v|^2 = \left|\frac{\partial v}{\partial r}\right|^2 + r^{-2} |\nabla_\omega v|^2,$$

where ∇_ω means the gradient along the unit sphere with variable ω and fixed r, we also get

$$\int_{B_1} |\nabla v|^2 dx = \sum_{k,l} \int_0^1 \left(|v'_{k,l}(r)|^2 + \frac{k(k+n-2)}{r^2} |v_{k,l}(r)|^2 \right) r^{n-1} dr. \tag{3.11}$$

Comparing (3.10) and (3.11), and taking into account that $k(k+n-2)$ increases with k, we see that it suffices to establish that the inequality

$$|g(1)|^2 \le \int_0^1 \left(|g'(r)|^2 + \frac{n-1}{r^2} |g(r)|^2 \right) r^{n-1} dr,$$

holds for any real-valued function $g \in \mathrm{Lip}([0,1]))$. To this end write

$$\begin{aligned} g(1)^2 &= \int_0^1 (r^{n-2} g^2)' dr = \int_0^1 [2r^{n-2} g' g + (n-2) r^{n-3} g^2] dr \\ &\le \int_0^1 [r^{n-1} g'^2 + (n-1) r^{n-3} g^2] dr = \int_0^1 \left(g'^2 + \frac{n-1}{r^2} g^2 \right) r^{n-1} dr, \end{aligned}$$

which proves Lemma 3.2. □

C. Proof of Lemma 3.1 (continuation). Let us normalize u by requiring $\overline{|u|} = 1$, i.e. average of $|u|$ over ∂B_1 equals 1. This can be done if $u \not\equiv 0$ on ∂B_1. Then by Lemma 3.2

$$\int_{\partial B_1} (1 - |u|)^2 d\omega \le \int_{B_1} |\nabla u|^2 dx.$$

Combining this with (3.4) and (3.5), we obtain

$$\mathrm{cap}\,(F) \le n \int_{B_1} |\nabla u|^2 dx.$$

Removing the restriction $\overline{|u|} = 1$, we can conclude that for any $u \in \mathrm{Lip}(B_1)$

$$\left(\fint_{\partial B_1} |u| d\omega \right)^2 \le \frac{n}{\mathrm{cap}\,(F)} \int_{B_1} |\nabla u|^2 dx. \tag{3.12}$$

(This obviously also holds in case when $u \equiv 0$ on ∂B_1.)

Note that for any real-valued function $v \in \mathrm{Lip}(B_1)$

$$\fint_{\partial B_1} |v - \bar{v}|^2 d\omega = \fint_{\partial B_1} |v|^2 d\omega - \bar{v}^2,$$

hence, using (3.6), we get

$$\fint_{\partial B_1} |v|^2 d\omega = \bar{v}^2 + \fint_{\partial B_1} |v - \bar{v}|^2 d\omega \le \bar{v}^2 + \frac{1}{\omega_n} \int_{B_1} |\nabla v|^2 dx.$$

Applying this to $v = |u|$ and using (3.12), we obtain

$$\int_{\partial B_1} |u|^2 d\omega \leq \left(1 + \frac{n\omega_n}{\operatorname{cap}(F)}\right) \int_{B_1} |\nabla u|^2 dx. \tag{3.13}$$

D. Note that out goal is an estimate which is similar to (3.13) but with the integral over ∂B_1 in the left hand side replaced by the integral over B_1. To this end we will again use the expansion (3.7) of $v = |u|$ over spherical functions, and the identities (3.8), (3.9) and (3.11). Let us take a real-valued function $g \in \mathrm{Lip}([0,1])$ and denote

$$Q = \int_0^1 g^2(r) r^{n-1} dr.$$

Integrating by parts, we obtain

$$Q = -\frac{2}{n} \int_0^1 g g' r^n dr + \frac{1}{n} g^2(1).$$

Using an elementary inequality $2ab \leq \varepsilon a^2 + \varepsilon^{-1} b^2$, where $a, b \in \mathbb{R}$, $\varepsilon > 0$, and taking into account that $r \leq 1$, we obtain

$$\begin{aligned} Q &\leq \frac{1}{n} \int_0^1 \left(\varepsilon g^2(r) + \frac{1}{\varepsilon} g'^2(r)\right) r^{n-1} dr + \frac{1}{n} g^2(1) \\ &= \frac{\varepsilon}{n} Q + \frac{1}{n\varepsilon} \int_0^1 g'^2(r) r^{n-1} dr + \frac{1}{n} g^2(1), \end{aligned}$$

hence for any $\varepsilon \in (0, n)$

$$Q \leq \frac{1}{(n-\varepsilon)\varepsilon} \int_0^1 g'^2(r) r^{n-1} dr + \frac{1}{n-\varepsilon} g^2(1).$$

Taking $\varepsilon = n/2$, we obtain

$$Q \leq \frac{4}{n^2} \int_0^1 g'^2(r) r^{n-1} dr + \frac{2}{n} g^2(1). \tag{3.14}$$

Now we can argue as in the proof of Lemma 3.2, expanding $v = |u|$ over spherical harmonics $Y_{k,l}$. Then the desired inequality follows from the inequalities for the coefficients $v_{k,l} = v_{k,l}(r)$, with the strongest one corresponding to the case $k = 0$ (unlike $k = 1$ in Lemma 3.2). Then using the inequality (3.14) for $g = v_{0,0}$ we obtain

$$\int_{B_1} |u|^2 dx \leq \frac{4}{n^2} \int_{B_1} |\nabla u|^2 dx + \frac{2}{n} \int_{\partial B_1} |u|^2 d\omega. \tag{3.15}$$

Using (3.13), we deduce from (3.15):

$$\int_{B_1} |u|^2 dx \leq \left[\frac{4}{n^2} + \frac{2}{n}\left(1 + \frac{n\omega_n}{\operatorname{cap}(F)}\right)\right] \int_{B_1} |\nabla u|^2 dx. \tag{3.16}$$

Taking into account the inequality

$$\operatorname{cap}(F) \le \operatorname{cap}(\bar{B}_1) = (n-2)\omega_n,$$

we can estimate the constant in front of the integral in the right hand side of (3.16) as follows:

$$\frac{4}{n^2} + \frac{2}{n}\left(1 + \frac{n\omega_n}{\operatorname{cap}(F)}\right) \le 4\omega_n\left(1 - \frac{2}{n^2}\right),$$

which ends the proof of Lemma 3.1. □

The lower bound in (1.3) is given by

Lemma 3.3 *There exists $c = c(\gamma, n) > 0$ such that for all open sets $\Omega \subset \mathbb{R}^n$*

$$\lambda(\Omega) \ge c r_{\Omega,\gamma}^{-2}. \tag{3.17}$$

Proof. Let us fix $\gamma \in (0,1)$ and choose any $r > r_{\Omega,\gamma}$. Then any ball $\bar{B}_r$ has a non-negligible intersection with $\mathbb{R}^n \setminus \Omega$, i.e.

$$\operatorname{cap}(\bar{B}_r \setminus \Omega) \ge \gamma \operatorname{cap}(\bar{B}_r).$$

Since any $u \in C_0^\infty(\Omega)$ vanishes on $\bar{B}_r \setminus \Omega$, it follows from Lemma 3.1 that for any such u

$$\int_{\bar{B}_r} |u|^2 dx \le \frac{C_n}{r^{-n}\operatorname{cap}(\bar{B}_r \setminus \Omega)} \int_{\bar{B}_r} |\nabla u|^2 dx \le \frac{C_n}{r^{-n}\gamma \operatorname{cap}(\bar{B}_r)} \int_{\bar{B}_r} |\nabla u|^2 dx.$$

Taking into account that $\operatorname{cap}(\bar{B}_r) = \operatorname{cap}(\bar{B}_1) r^{n-2}$, we obtain

$$\int_{\bar{B}_r} |u|^2 dx \le \frac{C_n r^2}{\gamma \operatorname{cap}(\bar{B}_1)} \int_{\bar{B}_r} |\nabla u|^2 dx.$$

Now let us choose a covering of $\mathbb{R}^n$ by balls $\bar{B}_r = \bar{B}_r^{(k)}$, $k = 1, 2, \ldots$, so that the multiplicity of this covering is at most $N = N(n)$. For example, we can make

$$N(n) \le n \log n + n \log(\log n) + 5n, \quad n \ge 2, \tag{3.18}$$

which holds also for the smallest multiplicity of coverings of $\mathbb{R}^n$ by translations of any convex body (see Theorem 3.2 in [23]).

Then summing up the estimates above over all balls in this covering, we see that

$$\begin{aligned}\int_{\mathbb{R}^n} |u|^2 dx \le \sum_k \int_{\bar{B}_r^{(k)}} |u|^2 dx &\le \frac{C_n r^2}{\gamma \operatorname{cap}(\bar{B}_1)} \sum_k \int_{\bar{B}_r^{(k)}} |\nabla u|^2 dx \\ &\le \frac{C_n N r^2}{\gamma \operatorname{cap}(\bar{B}_1)} \int_{\mathbb{R}^n} |\nabla u|^2 dx.\end{aligned}$$

Recalling (1.1), we see that

$$\lambda(\Omega) \geq cr^{-2}$$

with

$$c = c(\gamma, n) = \frac{\gamma \operatorname{cap}(\bar{B}_1)}{C_n N} = \frac{\gamma n^2(n-2)}{4(n^2-2)N}. \tag{3.19}$$

Taking limit as $r \downarrow r_{\Omega,\gamma}$, we obtain (3.17) with the same c. $\square$

4 Upper bound

4.1

According to (1.1), to get an upper bound for $\lambda(\Omega)$ it is enough to take any test function $u \in C_0^\infty(\Omega)$ and write

$$\lambda(\Omega) \leq \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega |u|^2 dx}. \tag{4.1}$$

For simplicity of notations we will write λ instead of $\lambda(\Omega)$ everywhere in this section. The inequality (4.1) can be rewritten as follows:

$$\int_\Omega |u|^2 dx \leq \lambda^{-1} \int_\Omega |\nabla u|^2 dx. \tag{4.2}$$

By approximation, it suffices to take $u \in \operatorname{Lip}_c(\Omega)$ or even $u \in H_0^1(\Omega)$, where $H_0^1(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the standard Sobolev space $H^1(\Omega)$ (which consists of all $u \in L^2(\Omega)$ with the distributional derivatives $\partial u/\partial x_j \in L^2(\Omega)$, $j = 1, \ldots, n$).

In particular, choosing a ball B_r, we can take

$$u \in \operatorname{Lip}_c(\Omega \cap B_r) = \operatorname{Lip}_c(\Omega) \cap \operatorname{Lip}_c(B_r). \tag{4.3}$$

Let us take a compact set $F \subset \bar{B}_{3r/2}$, such that F is the closure of an open set with a smooth boundary. (In this section we will call such sets *regular* subsets of $\bar{B}_{3r/2}$.) Denote by P_F its equilibrium potential (see Sect. 2). Regularity of F implies that $P_F \in \operatorname{Lip}(\mathbb{R}^n)$. By definition $P_F = 1$ on F, so $1 - P_F = 0$ on F. Let us also assume that

$$\operatorname{Int} F \supset \bar{B}_r \setminus \Omega,$$

where $\operatorname{Int} F$ means the set of all interior points of F (so $\operatorname{Int} F$ is an open subset in $\mathbb{R}^n$). Then $1 - P_F = 0$ in a neighborhood of $\bar{B}_r \setminus \Omega$. Therefore, multiplying $1 - P_F$ by a cut-off function $\eta \in C_0^\infty(B_r)$, we will get a function $u = \eta(1 - P_F)$, satisfying the requirenment (4.3), hence fit to be a test function in (4.1).

In the future we will also assume that the cut-off function $\eta \in C_0^\infty(B_r)$ has the following properties:

$$0 \leq \eta \leq 1 \text{ on } B_r, \quad \eta = 1 \text{ on } B_{(1-\kappa)r}, \quad |\nabla \eta| \leq 2(\kappa r)^{-1} \text{ on } B_r,$$

where $0 < \kappa < 1$ and the balls B_r and $B_{(1-\kappa)r}$ are supposed to have the same center. Using integration by parts and and the equation $\Delta P_F = 0$ on $B_r \setminus F$, we obtain for the test function $u = \eta(1 - P_F)$:

$$\begin{aligned}\int_{B_r} |\nabla u|^2 dx &= \int_{B_r} \left(|\nabla \eta|^2 (1-P_F)^2 - \nabla(\eta^2)\cdot(1-P_F)\nabla P_F + \eta^2 |\nabla P_F|^2\right) dx \\ &= \int_{B_r} |\nabla \eta|^2 (1-P_F)^2 dx \le 4(\kappa r)^{-2} \int_{B_r} (1-P_F)^2 dx.\end{aligned}$$

Therefore, from (4.2) we obtain

$$\int_{B_r} |u|^2 dx \le \lambda^{-1} 4(\kappa r)^{-2} \int_{B_r} (1-P_F)^2 dx.$$

Since $0 \le P_F \le 1$, the last integral in the right hand side is estimated by

$$\mathrm{mes}\,(B_r) = n^{-1}\omega_n r^n.$$

where mes means the usual Lebesgue measure on $\mathbb{R}^n$. Therefore,

$$\int_{B_r} |u|^2 dx \le 4 n^{-1}\omega_n \lambda^{-1}\kappa^{-2} r^{n-2}.$$

Restricting the intergal in the left hand side to $B_{(1-\kappa)r}$, we obtain

$$\int_{B_{(1-\kappa)r}} (1-P_F)^2 dx \le 4 n^{-1}\omega_n \lambda^{-1}\kappa^{-2} r^{n-2}. \tag{4.4}$$

4.2

Now we need to provide an appropriate lower bound for the left hand side of (4.4). To this end we first restrict the integration to the spherical layer

$$S_{r_1,r_2} = B_{r_2} \setminus B_{r_1},$$

where $0 < r_1 < r_2 < r$. In the future we will take

$$r_1 = (1-2\kappa)r, \quad r_2 = (1-\kappa)r, \tag{4.5}$$

where $0 < \kappa < 1/2$, though it is convenient to write some formulas in a bigger generality. Let us denote the volume of the layer S_{r_1,r_2} by $|S_{r_1,r_2}|$, i.e.

$$|S_{r_1,r_2}| = \mathrm{mes}\, S_{r_1,r_2} = n^{-1}\omega_n (r_2^n - r_1^n].$$

We will also need the notation

$$⨍_{S_{r_1,r_2}} f(x) dx = \frac{1}{|S_{r_1,r_2}|} \int_{S_{r_1,r_2}} f(x) dx$$

for the average of a positive function f over S_{r_1,r_2}. In particular, restricting the integration in (4.4) to S_{r_1,r_2} (with $r_2 \le (1-\kappa)r$) and dividing by $|S_{r_1,r_2}|$, we obtain

$$\fint_{S_{r_1,r_2}} (1-P_F)^2 dx \le \frac{4\lambda^{-1}\kappa^{-2}r^{n-2}}{r_2^n - r_1^n}.$$

Hence, by the Cauchy-Schwarz inequality,

$$\left[1 - \fint_{S_{r_1,r_2}} P_F dx\right]^2 = \left[\fint_{S_{r_1,r_2}} (1-P_F)dx\right]^2 \le \frac{4\lambda^{-1}\kappa^{-2}r^{n-2}}{r_2^n - r_1^n}. \tag{4.6}$$

To simplify the right hand side, let us estimate $(r_2^n - r_1^n)^{-1}$ from above. Applying the Bernoulli inequality, we see that

$$\begin{aligned} r_2^n - r_1^n &= (r_2 - r_1)(r_2^{n-1} + r_2^{n-2}r_1 + \cdots + r_1^{n-1}) \\ &\ge n\kappa r r_1^{n-1} = n\kappa r^n (1-2\kappa)^{n-1} \ge n\kappa r^n [1 - 2(n-1)\kappa]. \end{aligned}$$

Now note that

$$\frac{1}{1-2(n-1)\kappa} \le 1 + 4(n-1)\kappa,$$

provided

$$0 < \kappa \le \frac{1}{4(n-1)}. \tag{4.7}$$

Under this condition it follows that

$$\frac{1}{r_2^n - r_1^n} \le n^{-1}\kappa^{-1}r^{-n}\left[1 + 4(n-1)\kappa\right], \tag{4.8}$$

and (4.6) takes the form

$$\left[1 - \fint_{S_{r_1,r_2}} P_F dx\right]^2 \le 4n^{-1}\kappa^{-3}\left[1 + 4(n-1)\kappa\right]\lambda^{-1}r^{-2}. \tag{4.9}$$

4.3

For simplicity of notations and without loss of generality we may assume that the ball B_r is centered at $0 \in \mathbb{R}^n$ (and so are smaller balls and spherical layers).

To provide a lower bound for the left hand side of (4.9), we will give an upper bound for the average of P_F. According to the definition of P_F and notations from Section 2, we can write

$$\begin{aligned} \fint_{S_{r_1,r_2}} P_F dx &= \fint_{S_{r_1,r_2}} \left(\int_F \mathcal{E}(x-y)d\mu_F(y)\right) dx \\ &= \int_F \left(\fint_{S_{r_1,r_2}} \mathcal{E}(x-y)dx\right) d\mu_F(y). \end{aligned} \tag{4.10}$$

The inside integral in the right hand side can be explicitly calculated (due to Newton) as the potential of a uniformly charged spherical layer with total charge 1. The result of this calculation is $|S_{r_1,r_2}|^{-1}V_{r_1,r_2}(y)$, where

$$
(4.11)\qquad V_{r_1,r_2}(y)=\begin{cases}\frac{r_2^2-r_1^2}{2(n-2)}\,, & \text{if } |y|\le r_1,\\ -\frac{|y|^2}{2n}+\frac{r_2^2}{2(n-2)}-\frac{r_1^n}{n(n-2)|y|^{n-2}}\,, & \text{if } r_1\le|y|\le r_2,\\ \frac{r_2^n-r_1^n}{n(n-2)|y|^{n-2}}\,, & \text{if } |y|\ge r_2.\end{cases}
$$

The function $y\mapsto V_{r_1,r_2}(y)$ belongs to $C^1(\mathbb{R}^n)$ and is spherically symmetric; it tends to 0 as $|y|\to\infty$; it is harmonic in $\mathbb{R}^n\setminus S_{r_1,r_2}$ and satisfies the equation $\Delta V_{r_1,r_2}=-1$ in S_{r_1,r_2}. These properties uniquely define the function V_{r_1,r_2}. Differentiating it with respect to $|y|$, we easily see that it is decreasing with respect to $|y|$, hence its maximum is at $y=0$ (hence given by the first row in (4.11)). So we obtain, using (4.8):

$$
\begin{aligned}
&⨍_{S_{r_1,r_2}}\mathcal{E}(x-y)dx\le|S_{r_1,r_2}|^{-1}V_{r_1,r_2}(0)=\frac{n(r_2^2-r_1^2)}{2(n-2)\omega_n(r_2^n-r_1^n)}\\
&=\frac{n\kappa r(r_1+r_2)}{2(n-2)\omega_n(r_2^n-r_1^n)}\le\frac{nr^2\kappa(1-\kappa)}{(n-2)\omega_n(r_2^n-r_1^n)}\\
&\le\frac{(1-\kappa)[1+4(n-1)\kappa]}{(n-2)\omega_n r^{n-2}}\le\frac{1+(4n-5)\kappa}{(n-2)\omega_n r^{n-2}}
\end{aligned}
$$

Finally, using (2.4), we obtain

$$
(4.12)\qquad ⨍_{S_{r_1,r_2}}\mathcal{E}(x-y)dx\le\frac{1+(4n-5)\kappa}{\operatorname{cap}(\bar B_r)}\,.
$$

provided r_1,r_2 choosen as in (4.5) and (4.7) is satisfied.

4.4

Using (4.12) in (4.10) and taking into account (2.3), we obtain

$$
\begin{aligned}
(4.13)\qquad &⨍_{S_{r_1,r_2}}P_F(x)dx\le\frac{1+(4n-5)\kappa}{\operatorname{cap}(\bar B_r)}\int_F d\mu_F(y)\\
&=[(1+(4n-5)\kappa]\frac{\operatorname{cap}(F)}{\operatorname{cap}(\bar B_r)}\le[(1+(4n-5)\kappa]\gamma,
\end{aligned}
$$

provided F is γ-negligible. (i.e. satisfies (1.2)). Note that we do not assume that $F\subset\bar B_r$ but do assume that $0<\gamma<1$. In this case, taking into account (4.7), we can take

$$
(4.14)\qquad \kappa=\min\left\{\frac{1}{4(n-1)},\ \frac{1-\gamma}{2(4n-5)\gamma}\right\},
$$

so that (4.7) is satisfied, and, besides,

$$[(1+(4n-5)\kappa]\gamma \leq \frac{1+\gamma}{2} = 1 - \frac{1-\gamma}{2},$$

so that (4.13) becomes

$$\fint_{S_{r_1,r_2}} P_F(x)dx \leq 1 - \frac{1-\gamma}{2}.$$

Taking this into account in (4.9) and using (4.7), we obtain

$$\frac{(1-\gamma)^2}{4} \leq 4n^{-1}\kappa^{-3}[1+4(n-1)\kappa]\lambda^{-1}r^{-2} \leq 8n^{-1}\kappa^{-3}\lambda^{-1}r^{-2},$$

hence

$$\lambda \leq 32(1-\gamma)^{-2}\kappa^{-3}r^{-2}. \tag{4.15}$$

4.5

We are now ready for

Proof of Theorem 1.1.
The lower bound for λ was established in Lemma 3.3.

We proved the estimate (4.15) above under the condition that there exist $\gamma \in (0,1)$, a ball B_r and a regular compact set $F \subset \bar{B}_{3/2}$ (here the balls B_r and $B_{3/2}$ have the same center), such that F is γ-negligible and its interior includes $\bar{B}_r \setminus \Omega$. (The estimate then holds with $\kappa = \kappa(\gamma, n)$ given by (4.14).) It follows in particular that $\bar{B}_r \setminus \Omega$ is γ-negligible.

Vice versa, if $\bar{B}_r \setminus \Omega$ is γ-negligible, then we can approximate it by regular compact sets F_k, $k = 1, 2, \ldots$, such that $\operatorname{Int} F_k \supset \bar{B}_r \setminus \Omega$, $\operatorname{Int} F_k \supset F_{k+1}$, and $\bar{B}_r \setminus \Omega$ is the intersection of all F_k's. In this case

$$\lim_{k\to\infty} \operatorname{cap}(F_k) = \operatorname{cap}(\bar{B}_r \setminus \Omega),$$

due to the continuity property of the capacity. (See e.g. [14], Sect. 2.2.1.) In this case, for any $\varepsilon > 0$ the sets F_k will be $(\gamma+\varepsilon)$-negligible for sufficiently large k. It follows that the estimate (4.15) will hold if we only know that there exists a ball B_r such that $\bar{B}_r \setminus \Omega$ is γ-negligible. Then the estimate still holds if we replace r by the least upper bound of the radii of such balls which is exactly the interior capacitary radius $r_{\Omega,\gamma}$. This proves the upper bound in (1.3) with

$$C(\gamma, n) = 32(1-\gamma)^{-2}\kappa^{-3}, \tag{4.16}$$

where κ is defined by (4.14). $\Box$

5 Further remarks

5.1 Measure instead of capacity

E. Lieb [9] used geometric arguments to establish a lower bound for $\lambda(\Omega)$ which is similar to (3.17) but with capacity replaced by the Lebesgue measure. Such lower bounds can be also deduced from Theorem 1.1 if we use isoperimetric inequalities between the capacity and Lebesgue measure:

$$\operatorname{mes} F \le A_n(\operatorname{cap}(F))^{n/(n-2)}, \tag{5.1}$$

with the equality for balls (see e.g. [22] or Sect. 2.2.3, 2.2.4 in [14]), so

$$A_n = (\operatorname{mes} B_1)\,[\operatorname{cap}(B_1)]^{-n/(n-2)} = n^{-1}(n-2)^{-n/(n-2)}\omega_n^{-2/(n-2)}.$$

Namely, let us denote for any $\alpha \in (0,1)$

$$r^{(mes)}_{\Omega,\alpha} = \sup\{r \mid \exists B_r,\ \operatorname{mes}(B_r \setminus \Omega) \le \alpha \operatorname{mes} B_r\}.$$

Then (5.1) implies that

$$r^{(mes)}_{\Omega,\alpha} \ge r_{\Omega,\gamma} \quad \text{provided} \quad \alpha = \gamma^{n/(n-2)}.$$

Therefore, we obtain

Proposition 5.1 *For every $\alpha \in (0,1)$*

$$\lambda(\Omega) \ge c(\gamma, n)\left(r^{(mes)}_{\Omega,\alpha}\right)^{-2}, \quad \textit{where} \quad \gamma = \alpha^{(n-2)/n}.$$

Here $c(\gamma, n)$ is given by (3.19).

This is exactly Lieb's inequality (1.2) in [9], though with a different constant.

There are numerous results which give lower bounds for $\lambda(\Omega)$. We will mention only a few. The famous Faber-Krahn inequality ([4, 8, 22]) gives a lower bound of $\lambda(\Omega)$ in terms of the area of $\Omega \subset \mathbb{R}^2$. Under miscellaneous topological and geometric restrictions on Ω the interior radius was shown to provide a lower bound (hence a two-sided estimate) for $\lambda(\Omega)$ in case $n = 2$ by Hayman [6], Osserman [18, 19, 20], Taylor [24], Croke [3], Bañuelos and Carroll [1], and also in case $n \ge 3$ ([6], [20]).

Let $\operatorname{cap}_\Omega(F)$ denote the capacity of a compact set $F \subset \Omega$ with respect to an open set $\Omega \subset \mathbb{R}^n$. It is defined similarly to $\operatorname{cap}(F)$ in (2.1), except the allowed test functions u should be supported in Ω. The following 2-sided estimate for $\lambda(\Omega)$ was established in [10, 12]:

$$\frac{1}{4}\inf_F \frac{\operatorname{cap}_\Omega(F)}{\operatorname{mes} F} \le \lambda(\Omega) \le \inf_F \frac{\operatorname{cap}_\Omega(F)}{\operatorname{mes} F}, \tag{5.2}$$

where the infimum is taken over all compact sets $F \subset \Omega$. The constant $1/4$ in the lower bound is precise. Both inequalities in (5.2) hold on Riemannian manifolds as well.

The lower bound (the first inequality) in (5.2) implies the Cheeger inequality [2] (see Sect. 3 in [15] for this implication), which also provides a geometric lower bound for $\lambda(\Omega)$ on manifolds. (See also Grigor'yan [5] for a review and related results.)

5.2 Bounds for essential spectrum

Let $\lambda_\infty(\Omega)$ denote the bottom of the essential spectrum of $-\Delta$ with the Dirichlet boundary conditions in Ω. Then Persson's arguments [21] give

$$\lambda_\infty(\Omega) = \lim_{R\to+\infty} \lambda(\Omega \setminus B_R(0)),$$

where $B_R(0)$ is the ball with the radius R and the center at the origin. Applying two-sided estimates from Theorem 1.1 to $\lambda(\Omega \setminus B_R(0))$, we obtain

Theorem 5.2 *For any $\gamma \in (0,1)$ and any open set $\Omega \subset \mathbb{R}^n$,*

$$c r_{\Omega,\gamma,\infty}^{-2} \le \lambda_\infty(\Omega) \le C r_{\Omega,\gamma,\infty}^{-2},$$

where

$$r_{\Omega,\gamma,\infty} = \lim_{R\to\infty} r_{\Omega\setminus B_R(0),\gamma},$$

and the constants $c = c(\gamma, n)$, $C = C(\gamma, n)$ are the same as in Theorem 1.1.

For small γ this theorem is due to Maz'ya and Otelbaev (see [16] and also Theorem 12.3.1 in [14]).

Theorem 5.2 implies that for any $\gamma \in (0,1)$ the condition $r_{\Omega,\gamma,\infty} = 0$ is necessary and sufficient for the discreteness of spectrum of the operator $-\Delta$ with the Dirichlet boundary conditions in Ω. (This is also a particular case of the main results of [17]).

5.3 Bounds for spectra of Schrödinger operators

Theorem 1.1, Proposition 5.1 and Theorem 5.2 can be extended to Schrödinger operators with positive potentials (which are even allowed to be positive measures, which are absolutely continuous with respect to the Wiener capacity). For small γ these results can be found in Chapters 10 – 12 of [14] with appropriate references.

For simplicity of formulation we will consider operators $H_V = -\Delta + V$, $V \ge 0$, where V is locally integrable. Then 2-sided estimates of the type (1.3) can be obtained for the bottom of the spectrum (and essential spectrum) of H_V, if $r_{\Omega,\gamma}$ is replaced by the quantity

$$r_{V,\gamma} = \sup\left\{ r \,|\, \exists B_r, \quad \text{such that} \quad r^{n-2} \ge \inf_F \int_{B_r\setminus F} V\,dx \right\},$$

where the infimum is taken over all negligible subsets in $F \subset \bar{B}_r$, i.e. sets satisfying (1.2).

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