Integral and isocapacitary inequalities

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To Victor Havin on his 75th birthday
with 45-year affection and admiration

Abstract. It is shown by a counterexample that isocapacitary and isoperimetric constants of a multi-dimensional Euclidean domain starshaped with respect to a ball are not equivalent. Sharp integral inequalities involving the harmonic capacity which imply Faber-Krahn property of the fundamental Dirichlet-Laplace eigenvalue are obtained. Necessary and sufficient conditions ensuring integral inequalities between a difference seminorm and the L_p-norm of the gradient are found.

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1 Introduction

Isocapacitary inequalities are intimately connected with various properties of Sobolev spaces, especially with norms of embedding operators [8], [12]–[16], [18], [19], [21], [26]. For instance, the best constants in some of these inequalities give two-sided estimates for eigenvalues of boundary value problems [15], [16], [18], [19]. Recently, in [7] and [3], isocapacitary inequalities were applied to the theory of quasi-linear second order elliptic equations.

The present paper deals with three topics related to isocapacitary inequalities. First we show by a counterexample in Sect. 2 that the fundamental eigenvalue of the Dirichlet Laplacian is not equivalent to an isoperimetric constant, called, as a rule, Cheeger's constant [6], in contrast with an isocapacitary constant introduced in [15] (see also [18]).[1] This equivalence, even uniform with respect to the dimension, holds for convex domains as proved recently by B. Klartag and E. Milman (oral communication) but, as I show, it fails even in the class of domains starshaped with respect to a ball.

Sect. 3 is devoted to certain integral capacitary inequalitites which are stronger than the classical Faber-Krahn property of the fundamental Dirichlet-Laplace eigenvalue (see [28]).

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[1] By the equivalence of the set functions a and b, defined on subsets of $\mathbb{R}^n$, I mean the existence of positive constants c_1 and c_2 depending only on n and such that $c_1 a \le b \le c_2 a$.

In Sect. 4 and 5 one can find necessary and sufficient conditions for the inequality

$$\Big(\int_\Omega\int_\Omega |u(x)-u(y)|^q \mu(dx,dy)\Big)^{1/q} \le C\,\|\nabla u\|_{L_p(\Omega)} \tag{1}$$

formulated in terms of the isoperimetric ($q \ge p = 1$) and isocapacitary ($q > p > 1$) inequalities of a new type. Here Ω is a subdomain of a Riemannian manifold, μ is a given measure of two subsets of Ω and u is an arbitrary smooth function.

No caracterization of (1) was known previously even for functions on the real line $\mathbb{R}$ (see Problem 3 in [11]). A particular case of a result obtained at the end of Sect. 5 is the criterion of the validity of (1) for all $u \in C_0^\infty(\mathbb{R})$:

$$\mu\big(([\alpha,\beta],\mathbb{R}\backslash(\alpha-r,\beta+r)\big) \le const.\, r^{-q(1-p)/p},$$

where $r > 0$, $\alpha < \beta$, and the constant factor does not depend on α, β, and r.

The marginal value $q = p > 1$ in (1) has special features and a sufficient condition for (1) is given in Sect. 6. The article is finished with a short discussion of the inequality

$$\Big(\int_\Omega |u|^q\,\nu(dx)\Big)^{1/q} \le \Big(\int_\Omega\int_\Omega |u(x)-u(y)|^p \mu(dx,dy)\Big)^{1/p}$$

with a nonnegative measure ν in Ω, μ as above, and $q \ge p \ge 1$.

It is worth mentioning that a Riemannian structure of Ω is not very important for most of the results presented in Sect. 3-5. It can be replaced by some natural requirements on the p-energy integral on a metric space (see [24], [10]).

In this article, I use a number of assertions from the book [22] which are not formulated in detail but supplied with references. It is therefore helpful to read the paper with [22] close at hand.

2 The first Dirichlet-Laplace eigenvalue and an isoperimetric constant

Let Ω be a subdomain of a n-dimensional Riemannian manifold $\mathfrak{R}_n$ and let $\Lambda(\Omega)$ be the first eigenvalue of the Dirichlet priblem for the Laplace operator $-\Delta$ in Ω or, more generally, the upper lower bound of the spectrum of this operator:

$$\Lambda(\Omega) = \inf_{u\in C_0^\infty(\Omega)} \frac{\|\nabla u\|^2_{L_2(\Omega)}}{\|u\|^2_{L_2(\Omega)}}. \tag{2}$$

By [15] (see also [18] and Corollary 2.3.3 [22]), $\Lambda(\Omega)$ admits the two-sided estimate

$$\frac{1}{4}\Gamma(\Omega) \le \Lambda(\Omega) \le \Gamma(\Omega) \tag{3}$$

with

$$\Gamma(\Omega) := \inf_{\{F\}} \frac{\mathrm{cap}(F;\Omega)}{m_n(F)}.$$

By m_n the n-dimensional Lebesgue measure on $\mathfrak{R}_n$ is meant, the infimum is taken over all compact subsets of Ω and $\mathrm{cap}(F;\Omega)$ stands for the relative harmonic capacity of F with respect to Ω:

$$\mathrm{cap}(F;\Omega) = \inf\Big\{\int_\Omega |\nabla u|^2 dx :\ u \in C_0^\infty(\Omega),\ u \ge 1 \text{ on } F\Big\}.$$

We write $\mathrm{cap}(F)$ instead of $\mathrm{cap}(F;\mathbb{R}^n)$.

By Theorem 2.2.1 [22], the set function

$$\gamma(\Omega) = \inf_{u\in C_0^\infty(\Omega)} \frac{\|\nabla u\|_{L_1(\Omega)}}{\|u\|_{L_1(\Omega)}} \tag{4}$$

admits the geometric representation

$$\gamma(\Omega) = \inf_{\{g\}} \frac{H_{n-1}(\partial g)}{m_n(g)}, \tag{5}$$

where g is an arbitrary open subset of $\mathfrak{R}_n$ with compact closure $\overline{g}$ in Ω and smooth boundary ∂g, and H_{n-1} is the $(n-1)$-dimensional Hausdorff measure. Obviously, for all $u \in C_0^\infty(\Omega)$,

$$\gamma(\Omega) \leq \frac{\displaystyle\int_\Omega |\nabla(u^2)|dx}{\displaystyle\int_\Omega u^2 dx} \leq 2\,\frac{\|\nabla u\|_{L_2(\Omega)}}{\|u\|_{L_2(\Omega)}}.$$

Hence

$$\gamma(\Omega)^2 \leq 4\,\Lambda(\Omega), \tag{6}$$

which shows, together with (3) and (4), that

$$\gamma(\Omega)^2 \leq 4\,\Gamma(\Omega) \tag{7}$$

(the square of the isoperimetric constant is dominated by the isocapacitary one).

One can ask whether an upper bound for $\Gamma(\Omega)$ formulated in terms of $\gamma(\Omega)$ exists. The negative answer is obtained easily if the class of sets Ω is not restricted. In fact, let F be a compact subset of the open n-dimensional unit cube Q in the Euclidean space $\mathbb{R}^n$, such that

$$H_{n-1}(F) = 0 \qquad \text{and} \qquad \text{cap}(F) > 0.$$

By Ω we shall mean the complement of the union of all integral shifts of F. Now, by Theorem 11.2 [22], $\gamma(\Omega) = 0$ and $\Gamma(\Omega) \geq \Lambda(\Omega) > 0$, which gives the negative answer to the question formulated above.

Let us put the same question for domains in $\mathbb{R}^n$ starshaped with respect to balls. We show that the answer stays negative in a certain sense.

Example. Let Ω be a subdomain of the n-dimensional unit ball B, starshaped with respect to a concentric ball $B(0;\rho) = \{x : |x| < \rho\}$. Here we show that the inequality opposite to (7):

$$\gamma(\Omega)^2 \geq C\,\Gamma(\Omega) \tag{8}$$

is imposssible with C independent of ρ. Moreover, we shall construct a sequence of domains $\{\Omega_N\}_{n\geq 1}$ situated in B and such that

(*i*) Ω_N is starshaped with respect to a ball $B(0,\rho_N)$, where $\rho_N \to 0$,

(*ii*) $\Gamma(\Omega_N) \to \infty$,

(*iii*) $\gamma(\Omega_N) \leq c$, where c depends only on n.

Let N stand for a sufficiently large integer number. By $\{\omega_j\}_{j=1}^{N^{n-1}}$ we denote a collection of points on the unit sphere S^{n-1} placed uniformly in the sense that the distance from every point ω_j to the set of other points of the collection lies between $c_1\,N^{-1}$ and $c_2\,N^{-1}$, where c_1 and c_2 are positive constants, depending only on n. Consider a closed rotational cone C_j with the axis $O\omega_j$ and the vertex at the distance $c_0 N^{-1}$ from O, where c_0 is an absolute constant large enough. The opening of C_j will be independent of j and denoted by ε_N. Let $\varepsilon_N = o(N^{\frac{1-n}{n-2}})$. Clearly, the complement of C_j is visible from a sufficiently small ball $B(0;\rho_N)$. Therefore, the domain

$$\Omega_N := B \backslash \cup_j C_j$$

is starshaped with respect to $B(0,\rho_N)$.

We shall find the limit of $\gamma(\Omega_N)$ as $N \to \infty$ as well as a lower estimate for $\Gamma(\Omega_N)$. Clearly, $\gamma(\Omega_N) \geq \gamma(B) = n$. Furthermore, by (5),

$$\begin{aligned}
\gamma(\Omega_N) \quad &\leq \quad \frac{H_{n-1}(\partial\Omega_N)}{m_n(\Omega_N)} = \frac{H_{n-1}(\partial B) + H_{n-1}\big(\cup_j(B\cap\partial C_j)\big)}{m_n(B) - m_n\big(\cup_j(B\cap C_j)\big)} \\
&\leq \quad \frac{|S^{n-1}| + c_1\,\varepsilon_N^{n-1}\,N^{n-1}}{|S^{n-1}|/n - c_2\,\varepsilon_N^{n-2}\,N^{n-1}}
\end{aligned}$$

and therefore,

$$\lim_{N\to\infty} \gamma(\Omega_N) = n.$$

In order to estimate $\Gamma(\Omega_N)$ from below, we construct a covering of B by the balls $\mathcal{B}_k := B(x_k, 4c_0 N^{-1})$, whose multiplicity does not exceed a constant depending only on n. Let $|x_k| \geq c_0\, N^{-1}$. Theorem 10.1.2 [22] implies

$$c\, N^n \mathrm{cap}(\mathcal{B}_k \backslash \Omega_N) \int_{\mathcal{B}_k} u^2\, dx \leq \int_{\mathcal{B}_k} |\nabla u|^2 dx \tag{9}$$

for all $u \in C_0^\infty(\Omega_N)$, and the result will stem from a proper lower bound for $\mathrm{cap}(\mathcal{B}_k\backslash\Omega_N)$.

First, let us consider $n = 3$. Clearly, $\mathcal{B}_k\backslash\Omega_N$ contains a right rotational cylinder T_k with height $c_0\, N^{-1}$ and diameter of the base $\varepsilon_N\, N^{-1}$. Now, by Proposition 9.1.3/1 [22],

$$\mathrm{cap}(T_k) \geq c\, N^{-1} |\log \varepsilon_N|^{-1}.$$

This estimate in combination with (9) gives

$$c\, N^2 |\log \varepsilon_N|^{-1} \int_{\mathcal{B}_k} u^2\, dx \leq \int_{\mathcal{B}_k} |\nabla u|^2 dx. \tag{10}$$

Choosing $\varepsilon_N = \exp(-N)$ and summing (10) over all balls $\mathcal{B}_k$, we obtain $\lambda(\Omega_N) \geq c\, N$. Hence $\lambda(\Omega_N) \to \infty$ where as $\gamma(\Omega_N) \leq c$. Thus, in particular, there is no inequality

$$\Big(\inf_{\{g\}} \frac{s(\partial g)}{m_3(g)}\Big)^2 \geq C \inf_{\{F\}} \frac{\mathrm{cap}(F;\Omega)}{m_3(F)}$$

and, equivalently,

$$\Big(\inf_{\{g\}} \frac{s(\partial g)}{m_3(g)}\Big)^2 \geq C\Lambda(\Omega)$$

with constant factors C independent of the radius ρ.

For dimensions greater than 3, the very end of the argument remains intact but the estimation of $\mathrm{cap}(\mathcal{B}_k\backslash\Omega_N)$ becomes a bit more complicated and the choice of ε_N will be different.

Let $\alpha\, \mathcal{B}_k$ stand for the ball concentric with $\mathcal{B}_k$ and dilated with coefficient α. We introduce the set $s_k = \{j : C_j \cap \frac{1}{2}\mathcal{B}_k \neq \emptyset\}$. With every j in s_k we associate a right rotational cylinder T_j coaxial with the cone C_j and situated in $C_j \cap \frac{1}{2}\mathcal{B}_k$. The height of T_j will be equal to $c_0\, N^{-1}$ and the diameter of the base equal to $\varepsilon_N\, |x_k|$. We define a cut-off function η_j, equal to 1 on the $\varepsilon_N\, |x_k|$-neighbourhood of T_j, zero outside the $2\varepsilon_N\, |x_k|$-neighbourhood of T_j and satisfying the estimate

$$|\nabla \eta_j(x)| \leq c\, \delta(x)^{-1},$$

where $\delta(x)$ is the distance from x to the intersection of T_j with the axis of C_j.

By $\mathcal{P}_k$ we denote the equilibrium potential of $\mathcal{B}_k\backslash\Omega_N$. We have

$$\sum_{j\in s_k} \mathrm{cap}(C_j \cap \mathcal{B}_k) \leq \sum_{j\in s_k} \int_{\mathbb{R}^n} |\nabla(\mathcal{P}_k \eta_j)|^2 dx$$

$$\leq c\Big(\int_{\mathbb{R}^n} |\nabla \mathcal{P}_k|^2 dx + \int_{\mathbb{R}^n} \mathcal{P}_k^2 \delta^{-2} dx\Big).$$

Changing the constant c, one can majorize the last integral by the previous one due to Hardy's inequality. Hence,

$$\mathrm{cap}(\mathcal{B}_k\backslash\Omega_N) \geq c \sum_{j\in s_k} \mathrm{cap}(T_j). \tag{11}$$

By Proposition 9.1.3/1 [22],

$$\mathrm{cap}(T_j) \geq c\, \big(\varepsilon_N |x_k|\big)^{n-3} N^{-1}.$$

Furthermore, it is visible that the number of integers in s_k is between two multiples of $|x_k|^{1-n}$. Now, by (11)

$$\operatorname{cap}(\mathcal{B}_k\backslash\Omega_N) \geq c\,|x_k|^{1-n}\left(\varepsilon_N|x_k|\right)^{n-3}N^{-1}$$

and by (9)

$$c\,N^{n-1}|x_k|^{-2}\varepsilon_N^{n-3}\int_{\mathcal{B}_k}u^2dx \leq \int_{\mathcal{B}_k}|\nabla u|^2dx. \tag{12}$$

Since $|x_k| \leq 1$, it follows by summation of (12) over k that

$$\lambda(\Omega_N) \geq c\,\varepsilon_N^{n-3}N^{n-1}.$$

Putting, for instance,

$$\varepsilon_N = N^{(1-n)/(n-5/2)},$$

we see that $\Gamma(\Omega_N) \to \infty$, and the desired counterexample is constructed for $n > 3$.

3 Capacitary improvement of the Faber-Krahn inequality

We state and prove the main result of this section. Here Ω is an open subset of an arbitrary n-dimensional Riemannian manifold.

Theorem 1. *Let $\mathcal{R} > 0$, $u \in C_0^\infty(\Omega)$, and $N_t = \{x \in \Omega : |u(x)| \geq t\}$. If $n > 2$, then*

$$\begin{aligned}&\Big(\frac{j_{(n-2)/2}}{\mathcal{R}}\Big)^2 m_n(B_{\mathcal{R}})\int_0^\infty\Big(\frac{\operatorname{cap}(N_t;\Omega)}{\operatorname{cap}(B_{\mathcal{R}})+\operatorname{cap}(N_t;\Omega)}\Big)^{\frac{n}{n-2}}d(t^2)\\ &\leq \|\nabla u\|^2_{L_2(\Omega)},\end{aligned} \tag{13}$$

where j_ν is the first positive root of the Bessel function J_ν. If $n = 2$, then

$$\pi j_0^2\int_0^\infty \exp\Big(\frac{-4\pi}{\operatorname{cap}(N_t;\Omega)}\Big)d(t^2) \leq \|\nabla u\|^2_{L_2(\Omega)}. \tag{14}$$

Proof. Let w be an arbitrary absolutely continuous function on $(0, \mathcal{R}]$, such that $w(\mathcal{R}) = 0$. The inequality

$$\Big(\frac{j_{(n-2)/2}}{\mathcal{R}}\Big)^2\int_0^{\mathcal{R}} w(\rho)^2\rho^{n-1}d\rho \leq \int_0^{\mathcal{R}} w'(\rho)^2\rho^{n-1}d\rho, \tag{15}$$

where $n > 2$, is equivalent to the fact that the first eigenvalue of the Dirichlet-Laplace operator in the unit ball B equals $j^2_{(n-2)/2}$. Similarly, with $n = 2$ the inequality

$$\Big(\frac{j_0}{\mathcal{R}}\Big)^2\int_0^{\mathcal{R}} w(\rho)^2\rho\,d\rho \leq \int_0^{\mathcal{R}} w'(\rho)^2\rho\,d\rho \tag{16}$$

is associated.

In the case $n > 2$, we introduce the new variables

$$\psi = \frac{\rho^{2-n}-\mathcal{R}^{2-n}}{(n-2)|S^{n-1}|}, \qquad t(\psi) = w(\rho(\psi)),$$

and write (15) in the form

$$\begin{aligned}&\big(|S^{n-1}|j_{(n-2)/2}\mathcal{R}^{-1}\big)^2\int_0^\infty\frac{t(\psi)^2d\psi}{\big((n-2)|S^{n-1}|\,\psi+\mathcal{R}^{2-n}\big)^{2(n-1)/(n-2)}}\\ &\leq \int_0^\infty t'(\psi)^2d\psi.\end{aligned} \tag{17}$$

Similarly, for $n = 2$, putting

$$\psi = (2\pi)^{-1} \log \frac{\mathcal{R}}{\rho}, \qquad t(\psi) = w(\rho(\psi)),$$

we write (16) as

$$(2\pi j_0)^2 \int_0^\infty t(\psi)^2 \exp(-4\pi\psi)\, d\psi \le \int_0^\infty t'(\psi)^2 d\psi. \tag{18}$$

Note that the function t in (17) and (18) is subject to the boundary condition $t(0) = 0$. We write (17) and (18) as

$$n^{-1}|S^{n-1}|\Big(\frac{j_{(n-2)/2}}{\mathcal{R}}\Big)^2 \int_0^\infty \frac{dt(\psi)^2}{\big((n-2)|S^{n-1}|\,\psi + \mathcal{R}^{2-n}\big)^{n/(n-2)}}$$
$$\le \int_0^\infty t'(\psi)^2 d\psi \tag{19}$$

and

$$\pi j_0^2 \int_0^\infty \exp(-4\pi\psi)\, dt(\psi)^2 \le \int_0^\infty t'(\psi)^2 d\psi. \tag{20}$$

Now, as in Sect. 2.2.1 [22], we introduce the function

$$\psi(t) = \int_0^t \frac{d\tau}{\displaystyle\int_{|u|=\tau} |\nabla u|\, dH_{n-1}}, \tag{21}$$

as well as its inverse $\psi \to t(\psi)$, and replace the integral in the right-hand side of (19) and (20) by $\|\nabla u\|^2_{L_2(\Omega)}$. It remains to note that

$$\psi \le \big(\mathrm{cap}(N_{t(\psi)}; \Omega)\big)^{-1} \tag{22}$$

by Lemma 2.2.2/1 [22].

Let us use the area minimizing function of Ω:

$$\lambda(v) = \inf H_{n-1}(\partial g), \tag{23}$$

where the infimum is extended over all sets g with smooth boudaries and compact closures $\overline{g} \subset \Omega$, subject to the inequality $m_n(g) \ge v$. This and related geometrical characterizations of Ω proved to be useful in the theory of Sobolev spaces and elliptic equations, see [12], [14], [5], [20]. The function λ appears in the lower estimate of the capacity

$$\mathrm{cap}(F; \Omega) \ge \Big(\int_{m_n(F)}^{m_n(\Omega)} \frac{dv}{\lambda(v)^2}\Big)^{-1}$$

(see Corollary 2.2.3/2 [22]). Therefore, (13), (14), and the identity

$$\mathrm{cap}(B_{\mathcal{R}}) = (n-2)|S^{n-1}|\mathcal{R}^{n-2}$$

lead to the following Lorentz-type estimates.

Corollary 1. *If $n > 2$ and $\mathcal{R} > 0$, then, for all $u \in C_0^\infty(\Omega)$,*

$$\Big(\frac{j_{(n-2)/2}}{\mathcal{R}}\Big)^2 m_n(B_{\mathcal{R}}) \int_0^\infty \Big(\mathrm{cap}(B_{\mathcal{R}}) \int_{m_n(N_t)}^{m_n(\Omega)} \frac{dv}{\lambda(v)^2} + 1\Big)^{\frac{n}{2-n}} d(t^2)$$
$$\le \|\nabla u\|^2_{L_2(\Omega)}. \tag{24}$$

If $n = 2$, then, for all $u \in C_0^\infty(\Omega)$,

$$\pi j_0^2 \int_0^\infty \exp\Big(-4\pi \int_{m_n(N_t)}^{m_n(\Omega)} \frac{dv}{\lambda(v)^2}\Big) d(t^2) \le \|\nabla u\|^2_{L_2(\Omega)}. \tag{25}$$

Remark 1. Since

$$\lambda(v) \geq n^{\frac{n-1}{n}} |S^{n-1}|^{\frac{1}{n}}\, v^{\frac{n-1}{n}} \tag{26}$$

by the classical isoperimetric inequality for $\mathbb{R}^n$, the estimates (24) and (25) imply the Faber-Krahn property

$$\Lambda(\Omega) \geq \Big(\frac{j_{(n-2)/2}}{\mathcal{R}}\Big)^2$$

for any n-dimensional Euclidean domain Ω with $m_n(\Omega) = n^{-1}|S^{n-1}|\,\mathcal{R}^n$. $\square$

Theorem 1 is a very special case of the following general assertion.

Theorem 2. *Let* $\mathbf{M}$ *be a decreasing nonnegative function on* $[0,\infty)$ *and let* $q > 0$ *and* $p \geq 1$. *Suppose that for all absolutely continuous functions* $\psi \to t(\psi)$ *on* $[0,\infty)$, *the inequality*

$$\Big(-\int_0^\infty |t(\psi)|^q d\mathbf{M}(\psi)\Big)^{1/q} \leq \Big(\int_0^\infty |t'(\psi)|^p d\psi\Big)^{1/p} \tag{27}$$

holds. Then, for all $u \in C_0^\infty(\Omega)$,

$$\Big(\int_0^\infty \mathbf{M}\Big(\big(\mathrm{cap}_p(N_t;\Omega)\big)^{1/(1-p)}\Big) d(t^q)\Big)^{1/q} \leq \|\nabla u\|_{L_p(\Omega)}, \tag{28}$$

where cap_p *is the p-capacity defined by*

$$\mathrm{cap}_p(F;\Omega) = \inf\Big\{\int_\Omega |\nabla u|^p dx :\ u \in C_0^\infty(\Omega),\ u \geq 1 \text{ on } F\Big\}. \tag{29}$$

Proof. The role of the function ψ given by (21) is played in the present proof by

$$\psi(t) = \int_0^t \frac{d\tau}{\Big(\int_{|u|=\tau} |\nabla u|^{p-1}\, dH_{n-1}\Big)^{1/(p-1)}}. \tag{30}$$

We write the left-hand side of (27) in the form

$$\Big(\int_0^\infty \mathbf{M}(\psi)\, d(t(\psi))^q\Big)^{1/q}$$

and use the monotonicity of $\mathbf{M}$ and the inequality

$$\psi \leq \big(\mathrm{cap}_p(N_{t(\psi)};\Omega)\big)^{1/(1-p)} \tag{31}$$

proved in Lemma 2.2.2/1 [22]. It remains to apply (27) and the identity

$$\int_0^\infty |f'(\psi)|^p d\psi = \int_\Omega |\nabla u|^p dx \tag{32}$$

found in Lemma 2.3.1 [22]. $\square$

Using the area minimizing function λ defined by (23) and the estimate

$$\mathrm{cap}_p(F;\Omega) \geq \Big(\int_{m_n(F)}^{m_n(\Omega)} \frac{dv}{\lambda(v)^{p/(p-1)}}\Big)^{1-p} \tag{33}$$

(see Corollary 2.2.3/2 [22]), we obtain from Theorem 2

Corollary 2. *Let* μ, p, *and* q *be the same as in Theorem* 2 *and let* (27) *hold. Then*

$$\Big(\int_0^\infty \mathbf{M}\Big(\int_{m_n(N_t)}^{m_n(\Omega)} \frac{dv}{\lambda(v)^{p/(p-1)}}\Big) d(t^q)\Big)^{1/q} \leq \|\nabla u\|_{L_p(\Omega)} \tag{34}$$

for all $u \in C_0^\infty(\Omega)$.

Clearly, (34) is a generalization of the estimates (24) and (25) which were obtained for $p = 2$ with a particular choice of μ. Another obvious remark is that (27), where $\mathbf{M}$ is defined on the interval $0 < t < m_n(\Omega)$ by

$$\mathbf{M}\Big(\int_t^{m_n(\Omega)} \frac{dv}{\lambda(v)^{p/(p-1)}}\Big) = \Lambda_{p,q} t$$

with a constant $\Lambda_{p,q}$ depending on $m_n(\Omega)$, implies the inequality

$$\Lambda_{p,q}^{1/q} \|u\|_{L_q(\Omega)} \le \|\nabla u\|_{L_p(\Omega)} \tag{35}$$

for all $u \in C_0^\infty(\Omega)$.

4 Criterion for an upper estimate of a difference seminorm (the case $p = 1$)

Let us consider the seminorm

$$\langle u\rangle_{q,\mu} = \Big(\int_\Omega \int_\Omega |u(x) - u(y)|^q \mu(dx, dy)\Big)^{1/q}, \tag{36}$$

where Ω is an open subset of a Riemannian manifold and μ is a non-negative measure on $\Omega \times \Omega$, locally finite outside the diagonal $\{(x, y) : \ x = y\}$. By definition, the product $0 \cdot \infty$ equals zero.

In this section, first, we characterize both μ and Ω subject to the inequality

$$\langle u\rangle_{q,\mu} \le C \,\|\nabla u\|_{L_1(\Omega)}, \tag{37}$$

where $q \ge 1$ and u is an arbitrary function in $C^\infty(\Omega)$. We show that (37) is equivalent to a somewhat unusual relative isoperimetric inequality.

Theorem 3. *Inequality (37) holds for all $u \in C^\infty(\Omega)$ with $q \ge 1$ if and only if for any open subset g of Ω, such that $\Omega \cap \partial g$ is smooth, the inequality*

$$\big(\mu(g, \Omega\backslash\overline{g}) + \mu(\Omega\backslash\overline{g}, g)\big)^{1/q} \le C H_{n-1}(\Omega \cap \partial g) \tag{38}$$

holds with the same value of C as in (37). In particular, a constant C in (37) exists if and only if

$$\sup_{\{g\}} \frac{\mu(g, \Omega\backslash\overline{g})^{1/q}}{H_{n-1}(\Omega \cap \partial g)} < \infty. \tag{39}$$

Proof. *Sufficiency.* Denote by u_+ and u_- the positive and negative parts of u, so that $u = u_+ - u_-$. We notice that

$$\langle u\rangle_{q,\mu} \le \langle u_+\rangle_{q,\mu} + \langle u_-\rangle_{q,\mu} \tag{40}$$

and

$$\int_\Omega |\nabla u| dx = \int_\Omega |\nabla u_+| dx + \int_\Omega |\nabla u_-| dx. \tag{41}$$

First, we obtain (37) separately for for $u = u_+$ and $u = u_-$. Let $a > b$ and let $\chi_t(a, b) = 1$ if $a > t > b$ and $\chi_t(a, b) = 0$ otherwise.

Clearly,

$$\begin{aligned} \langle u\rangle_{q,\mu} &= \Big(\int_\Omega \int_\Omega \Big|\int_{u(x)}^{u(y)} dt\Big|^q \mu(dx, dy)\Big)^{1/q} \\ &= \Big(\int_\Omega \int_\Omega \Big|\int_0^\infty \big(\chi_t(u(x), u(y)) + \chi_t(u(y), u(x))\big) dt\Big|^q \mu(dx, dy)\Big)^{1/q}. \end{aligned}$$

By Minkowski's inequalitiy,

$$\begin{aligned}
\langle u\rangle_{q,\mu} &\leq \int_0^\infty \left(\int_\Omega\int_\Omega \big(\chi_t(u(x),u(y))+\chi_t(u(y),u(x))\big)^q \mu(dx,dy)\right)^{1/q} dt \\
&= \int_0^\infty \left(\int_\Omega\int_\Omega \big(\chi_t(u(x),u(y))+\chi_t(u(y),u(x))\big) \mu(dx,dy)\right)^{1/q} dt \\
&= \int_0^\infty \big(\mu(M_t,\Omega\backslash N_t)+\mu(\Omega\backslash N_t,M_t)\big)^{1/q} dt,
\end{aligned}$$

where $M_t=\{x\in\Omega : u(x)>t\}$ and $N_t=\{x\in\Omega : u(x)\geq t\}$.

By (38) and the co-area formula, the last integral does not exceed

$$C\int_0^\infty H_{n-1}\big(\{x\in\Omega : u(x)>t\}\big)dt = C\int_\Omega |\nabla u(x)|dx.$$

Therefore,

$$\langle u_\pm\rangle_{q,\mu} \leq C\int_\Omega |\nabla u_\pm(x)|dx$$

and the reference to (40) and (41) completes the proof of sufficiency.

Necessity. Let $\{w_m\}$ be the sequence of locally Lipschitz functions in Ω constructed in Lemma 3.2.2 [22] with the following properties:

1. $w_m(x)=0$ in $\Omega\backslash g$,
2. $w_m(x)\in[0,1]$ in Ω,
3. for any compactum $K\subset g$ there exists an integer $N(e)$ such that $w_m(x)=1$ for $x\in K$ and $m\geq N(e)$,
4. the limit relation holds

$$\limsup_{m\to\infty}\int_\Omega |\nabla w_m(x)|dx = H_{n-1}(\Omega\cap\partial g).$$

By Theorem 1.1.5/1 [22], the inequality (37) holds for all locally Lipschitz functions. Therefore,

$$\langle w_m\rangle_{q,\mu} \leq C\,\|\nabla w_m\|_{L_1(\Omega)} \tag{42}$$

and due to property 4,

$$\limsup_{m\to\infty}\langle w_m\rangle_{q,\mu} \leq CH_{n-1}(\Omega\cap\partial g). \tag{43}$$

On the other hand,

$$\begin{aligned}
\langle w_m\rangle_{q,\mu}^q &= \int_{x\in g}\int_{y\in\Omega\backslash g} w_m(x)^q\mu(dx,dy) \\
&+ \int_{x\in\Omega\backslash g}\int_{y\in g} w_m(y)^q\mu(dx,dy) + \int_g\int_g |w_m(x)-w_m(y)|^q\mu(dx,dy)
\end{aligned}$$

which implies

$$\langle w_m\rangle_{q,\mu}^q \geq \int_g w_m(x)^q\mu(dx,\Omega\backslash\overline{g}) + \int_g w_m(y)^q\mu(\Omega\backslash\overline{g},dy).$$

This, along with property 3, leads to

$$\liminf_{m\to\infty}\langle w_m\rangle_{q,\mu}^q \geq \mu(g,\Omega\backslash\overline{g})+\mu(\Omega\backslash g,\overline{g}).$$

Combining this relation with (42) and (43), we arrive at (38). □

Corollary 3 (One-dimensional case). *Let*

$$\Omega=(\alpha,\beta), \quad \textit{where} \quad -\infty\leq\alpha<\beta\leq\infty.$$

The inequality
$$\Big(\int_\Omega\int_\Omega |u(x)-u(y)|^q \mu(dx,dy)\Big)^{1/q} \le C\int_\Omega |u'(x)|dx \tag{44}$$
with $q\ge 1$ holds for all $u\in C^\infty(\Omega)$ if and only if
$$\big(\mu(I,\Omega\backslash\overline{I})+\mu(\Omega\backslash\overline{I},I)\big)^{1/q}\le 2C \tag{45}$$
for all open intervals I such that $\overline{I}\subset\Omega$, and
$$\big(\mu(I,\Omega\backslash\overline{I})+\mu(\Omega\backslash\overline{I},I)\big)^{1/q}\le C \tag{46}$$
for all intervals $I\subset\Omega$, such that $\overline{I}$ contains one of the end points of Ω.

In particular, a constant in (44) exists if and only if
$$\sup_{\{I\}}\mu(I,\Omega\backslash\overline{I})<\infty.$$

Proof. *Necessity* follows directly from (38) by setting $g=I$. Let us check the sufficiency of (45). Represent an arbitrary open set g of Ω as the union of non-overlapping open intervals I_k. Then by (45) and (46)
$$\big(\mu(g,\Omega\backslash\overline{g})+\mu(\Omega\backslash\overline{g},g)\big)^{1/q}=\Bigg(\sum_k\big(\mu(I_k,\Omega\backslash\overline{g})+\mu(\Omega\backslash\overline{g},I_k)\big)\Bigg)^{1/q}$$
$$\le\sum_k\big(\mu(I_k,\Omega\backslash\overline{g})+\mu(\Omega\backslash\overline{g},I_k)\big)^{1/q}\le C\sum_k H_0(\Omega\cap\partial I_k)$$
which is the same as (38). The result follows from Theorem 3.□

Remark 2. Suppose that the class of admissible functions in Theorem 3 is diminished by the requirement $u=0$ *in a neighbourhood of a closed subset F of $\overline{\Omega}$*. Then the same proof leads to the same criterion (38) with the only difference that the admissible sets g should be at a positive distance from F. For the example $F=\partial\Omega$, i.e. for the inequality (37) with any $u\in C_0^\infty(\Omega)$, the necessary and sufficient condition (38) becomes the isoperimetric inequality
$$\big(\mu(g,\Omega\backslash\overline{g})+\mu(\Omega\backslash\overline{g},g)\big)^{1/q}\le CH_{n-1}(\partial g) \tag{47}$$
for all open sets g with smooth boundary and compact closure $\overline{g}\subset\Omega$. If, in particular, in Corollary 3, the criterion of the validity of (44) for all $u\in C_0^\infty(\Omega)$ is the inequality (45) for every interval I, $\overline{I}\subset\Omega$. In the case $u=0$ near one of the end points $\Omega=(\alpha,\beta)$, one should require both (45) and (46) but the intervals I should be at a positive distance from that end point.

Needless to say, the condition (38) is simplified as follows for a symmetric measure μ, i.e. under the assumption $\mu(\mathcal{E},\mathcal{F})=\mu(\mathcal{F},\mathcal{E})$:
$$\mu(g,\Omega\backslash\overline{g})^{1/q}\le 2^{-1/q}CH_{n-1}(\Omega\cap\partial g)$$
for the same open sets g as in Theorem 3.

Remark 3. The integration domain $\Omega\times\Omega$ in (36) excludes inequalities for integrals taken over $\partial\Omega$. This can be easily avoided assuming additionally that μ is defined on compact subsets of $\overline{\Omega}\times\overline{\Omega}$ and that $u\in C(\overline{\Omega})\cap C^\infty(\Omega)$. Then, with the same proof, one obtains the corresponding criterion, similar to (38):
$$\big(\mu(\overline{g},\,\overline{\Omega}\backslash\overline{g})+\mu(\overline{\Omega}\backslash\overline{g},\,\overline{g})\big)^{1/q}\le C\,H_{n-1}(\Omega\cap\partial g).$$
As an application, consider the inequality
$$\int_{\partial\Omega}\int_{\partial\Omega}|u(x)-u(y)|\,H_{n-1}(dx)\,H_{n-1}(dy)\le C\int_\Omega|\nabla u|\,dx \tag{48}$$

which holds if and only if

$$H_{n-1}(\partial\Omega \cap \partial g)\, H_{n-1}(\partial\Omega \backslash \partial g) \le 2^{-1} C\, H_{n-1}(\Omega \cap \partial g) \tag{49}$$

for the same sets g as in Theorem 3.

By Corollary 6.4.4/3 [22], which appeared first in [5],

(i) If Ω is the unit ball in $\mathbb{R}^3$, then

$$4\pi H_2(\Omega \cap \partial g) \ge H_2(\partial\Omega \cap \partial g)\, H_2(\partial\Omega \backslash \partial g)$$

and

(ii) If Ω is the unit disk on the plane, then

$$H_1(\Omega \cap \partial g) \ge 2 \sin\bigl(\frac{1}{2} H_1(\partial\Omega \cap \partial g)\bigr).$$

Moreover, the last two inequalities are sharp. Hence, the inequality (48) holds with the best constant $C = 8\pi$ if $\Omega = B$. In the case (ii),

$$H_1(\Omega \cap \partial g) \ge 2^{-1} \min_{0 \le \varphi \le \pi} \frac{\sin\varphi}{\varphi(\pi - \varphi)} H_1(\partial\Omega \cap \partial g) H_1(\partial\Omega \backslash \partial g).$$

Since the last minimum equals π^{-1}, it follows that the best value of C in the inequality (48) for the unit disk is 4π. $\Box$

We can simplify the criterion (38) for $\Omega = \mathbb{R}^n$, replacing arbitrary sets g by arbitrary balls $B(x, \rho)$ similarly to Theorem 1.4.2/2 [22], where the norm

$$\|u\|_{L_q(\mu)} = \Bigl(\int_{\mathbb{R}^n} |u|^q d\mu\Bigr)^{1/q}$$

is treated in place of $\langle u \rangle_{q,\mu}$. Unfortunately, the best constant in the sufficiency part will be lost.

Corollary 4. (i) *If $q \ge 1$ and*

$$\sup_{x \in \mathbb{R}^n, \rho > 0} \rho^{(1-n)q} \bigl(\mu(B(x,\rho), \mathbb{R}^n \backslash B(x,\rho)) + \mu(\mathbb{R}^n \backslash B(x,\rho), B(x,\rho)) \bigr) < \infty, \tag{50}$$

then the inequality

$$\Bigl(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |u(x) - u(y)|^q \mu(dx, dy)\Bigr)^{1/q} \le C\, \|\nabla u\|_{L_1(\mathbb{R}^n)} \tag{51}$$

holds for all $u \in C^\infty(\mathbb{R}^n)$ and

$$C^q \le c^q \sup_{x \in \mathbb{R}^n, \rho > 0} \rho^{(1-n)q} \bigl(\mu(B(x,\rho), \mathbb{R}^n \backslash B(x,\rho)) + \mu(\mathbb{R}^n \backslash B(x,\rho), B(x,\rho)) \bigr), \tag{52}$$

where c depends only on n.

(ii) *If (51) holds for all $u \in C^\infty(\mathbb{R}^n)$, then*

$$C^q \ge |S^{n-1}|^{-q} \sup_{x \in \mathbb{R}^n, \rho > 0} \rho^{(1-n)q} \bigl(\mu(B(x,\rho), \mathbb{R}^n \backslash B(x,\rho)) + \mu(\mathbb{R}^n \backslash B(x,\rho), B(x,\rho)) \bigr).$$

Proof. Let g be an arbitrary open set in $\mathbb{R}^n$ with smooth boundary and let $\{B(x_j, \rho_j)\}$ be the Gustin covering of g subject to

$$\sum_j \rho_j^{n-1} \le c\, H_{n-1}(\partial g), \tag{53}$$

where c depends only on n (see Theorem 1.2.2/2 [22]). Then

$$\begin{aligned}
\mu(g, \mathbb{R}^n \backslash g) &\leq \sum_j \mu(B(x_j,\rho_j), \mathbb{R}^n\backslash g) \\
&\leq \Big(\sum_j \mu(B(x_j,\rho_j), \mathbb{R}^n\backslash g)^{1/q}\Big)^q \\
&\leq \Big(\sum_j \mu(B(x_j,\rho_j), \mathbb{R}^n\backslash B(x_j,\rho_j))^{1/q}\Big)^q \\
&\leq (c\,B)^q \big(\sum_j \rho_j^{n-1}\big)^q,
\end{aligned}$$

where B is the value of the supremum in (50). This and (53) imply

$$\mu(g, B(x_j,\rho_j) \leq (c\,B\,H_{n-1}(\partial g))^q.$$

Similarly,

$$\mu(\mathbb{R}^n\backslash g, g) \leq (c\,B\,H_{n-1}(\partial g))^q$$

and the result follows from Theorem 3.

The assertion (ii) stems from (38) by setting $g = B(x,\rho)$. $\square$

5 Criterion for an upper estimate of a difference norm (the case $p > 1$)

Now we deal with the inequality

$$\langle u\rangle_{q,\mu} \leq C\,\|\nabla u\|_{L_p(\Omega)}, \tag{54}$$

where $q > p > 1$, and show that it is equivalent to a certain isocapacitary inequality.

The capacity to appear in the present context is defined as follows. Let F_1 and F_2 be non-overlapping subsets of Ω, closed in Ω. The p-capacity of the pair (F_1, F_2) with respect to Ω is given by

$$\text{cap}_p(F_1, F_2; \Omega) = \inf_{\{u\}} \int_\Omega |\nabla u(x)|^p dx,$$

where $\{u\}$ is the set of all $u \in C^\infty(\Omega)$, such that $u \geq 1$ on F_1 and $u \leq 0$ on F_2.

Obviously, this capacity does not change if F_1 and F_2 change places. Furthermore, if F is a closed set in $\mathbb{R}^n$ and $F \subset G$, where G is an open set, such that $\overline{G} \subset \Omega$, then $\text{cap}_p(F, \Omega\backslash G; \Omega)$ coincides with the p-capacity $\text{cap}_p(F; G)$ defined in (29).

Theorem 4. *Inequality (54) with $p \in (1, q)$ holds for all $u \in C^\infty(\Omega)$ if and only if for any pair (F_1, F_2) of non-overlapping sets, closed in Ω,*

$$\mu(F_1, F_2)^{p/q} \leq B\,\text{cap}_p(F_1, F_2; \Omega), \tag{55}$$

where B depends only on p and q. In the sufficiency part we may assume that F_1 and F_2 are sets with smooth $\Omega \cap \partial F_i$.

In the proof of this theorem, we use the inequality

$$\Big(\int_{\mathbb{R}_+} |f(\psi)|^q \psi^{-1-q/p'} d\psi\Big)^{1/q} \leq c\,\|f'\|_{L_p(\mathbb{R}_+)} \tag{56}$$

due to Bliss [4] and the inequality

$$\Big(\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \frac{|f(\psi) - f(\phi)|^q}{|\psi - \phi|^{2+q/p'}} d\phi d\psi\Big)^{1/q} \leq c\,\|f'\|_{L_p(\mathbb{R}_+)}, \tag{57}$$

where $q > p > 1, p' = p/(p-1)$ and f is an arbitrary absolutely continuous function on $\overline{\mathbb{R}}_+$.

A short argument leading to (57) is as follows. Clearly, (57) results from the same inequality with $\mathbb{R}$ in place of $\mathbb{R}_+$, which follows, in its turn, from the estimate

$$\|f\|_{B_q^{1-(q-p)/pq}(\mathbb{R})} \leq c\,\|f\|_{W_p^1(\mathbb{R})} \tag{58}$$

by dilation with a coefficient λ and the limit passage as $\lambda \to 0_+$. (The standard notations B and W for Besov and Sobolev spaces with non-homogeneous norms is used in (57).) In order to obtain (58), we recall the well-known Sobolev type inequality

$$\|h\|_{L_{p'}(\mathbb{R}_+)} \leq c\,\|h\|_{B_{q'}^{(q-p)/pq}(\mathbb{R})}$$

(see Theorem 4′, Sect. 5.1 [29]) and put $h = (-\Delta + 1)^{-1/2} f$, which shows that

$$\|f\|_{W_{p'}^{-1}(\mathbb{R})} \leq c\,\|f\|_{B_{q'}^{-1+(q-p)/pq}(\mathbb{R})}. \tag{59}$$

By duality, (59) is equivalent to (58).

With (57) at hand, we return to Theorem 4.

Proof. *Sufficiency.* Arguing as at the beginning of the the proof of Theorem 2, we see that it sufficies to prove (54) for a non-negative u. By the definition of the Lebesque integral

$$\int_\Omega u d\nu = \int_{\mathbb{R}_+} \nu(N_\tau) d\tau = \int_{\mathbb{R}_+} \nu(M_\tau) d\tau,$$

where ν is a measure, and therefore

$$\int_\Omega P(u) d\nu = \int_{\mathbb{R}_+} \nu(N_\tau) dP(\tau), \tag{60}$$

where P is a non-decreasing function on $\mathbb{R}_+$. Putting here $u = 1/v$ and $Q(\tau) = P(\tau^{-1})$, we deduce

$$\int_\Omega Q(u) d\nu = -\int_{\mathbb{R}_+} \nu(\Omega\backslash M_\tau) dQ(\tau), \tag{61}$$

where Q is non-increasing. We obtain

$$\begin{aligned}\int_\Omega\int_\Omega |u(x)-u(y)|^q \mu(dx,dy) &= \int_\Omega\int_\Omega (u(x)-u(y))_+^q \mu(dx,dy)\\ &+ \int_\Omega\int_\Omega (u(y)-u(x))_+^q \mu(dx,dy)\\ &= \int_\Omega\int_\Omega (u(x)-u(y))_+^q (\mu(dx,dy)+\mu(dy,dx)).\end{aligned}$$

By (60) and (61), the last double integral is equal to

$$\begin{aligned}&q\int_{\mathbb{R}_+}\int_\Omega (t-u(y))_+^{q-1}(\mu(N_\tau,dy)+\mu(dy,N_\tau))d\tau\\ &= q(q-1)\int_{\mathbb{R}_+}\int_{\mathbb{R}_+}(\tau-\sigma)_+^{q-2}\big(\mu(N_\tau,\Omega\backslash M_\sigma)+\mu(\Omega\backslash M_\sigma,N_\tau)\big)d\tau d\sigma.\end{aligned}$$

Now, (55) implies

$$\langle u\rangle_{q,\mu}^q \leq 2q(q-1)B\int_{\mathbb{R}_+}\int_{\mathbb{R}_+}(\tau-\sigma)_+^{q-2}\mathrm{cap}_p(N_\tau,\Omega\backslash M_\sigma;\Omega)d\tau d\sigma$$

and using the function $\psi \to t(\psi)$, inverse of (30), we arrive at the inequality

$$\begin{aligned}&\langle u\rangle_{q,\mu}^q \leq 2q(q-1)B^{q/p}\\ &\quad\times\int_{\mathbb{R}_+}\int_0^\psi (t(\psi)-t(\phi))^{q-2}\big(\mathrm{cap}(N_{t(\psi)},\Omega\backslash M_{t(\phi)};\Omega)^{q/p} t'(\phi)t'(\psi)d\phi d\psi.\end{aligned}$$

By Lemma 2.2.2/1 [22], for $\psi > \phi$

$$\operatorname{cap}(N_{t(\psi)}, \Omega\backslash M_{t(\phi)}; \Omega) \le (\psi - \phi)^{1-p}$$

and therefore,

$$\langle u\rangle_{q,\mu}^q \le 2q(q-1)B^{q/p}\int_{\mathbb{R}_+}\int_0^\psi (\psi-\phi)^{-q/p'}(t(\psi)-t(\phi))^{q-2}t'(\phi)t'(\psi)d\phi d\psi. \qquad (62)$$

Integrating by parts twice on the right-hand side of (62), we obtain

$$\langle u\rangle_{q,\mu}^q \le 2B^{q/p}\frac{q}{p'}\Big(\Big(\frac{q}{p'}+1\Big)\int_{\mathbb{R}_+}\int_0^\psi \frac{(t(\psi)-t(\phi))^q}{(\psi-\phi)^{2+q/p'}}d\phi d\psi + \int_{\mathbb{R}_+}\psi^{-q/p'}t(\psi)^{q-1}t'(\psi)d\psi\Big)$$

$$= B^{q/p}\frac{q}{p'}\Big(\Big(\frac{q}{p'}+1\Big)\int_{\mathbb{R}_+}\int_{\mathbb{R}_+}\frac{|t(\psi)-t(\phi)|^q}{|\psi-\phi|^{2+q/p'}}d\phi d\psi + \frac{1}{p'}\int_{\mathbb{R}_+}t(\psi)^q\psi^{-1-q/p'}d\psi\Big).$$

Hence, we deduce from (56) and (57) that

$$\langle u\rangle_{q,\mu} \le c\,B^{1/p}\|t'\|_{L_p(\mathbb{R}_+)}, \qquad (63)$$

where c depends only on p and q. It remains to refer to (32).

Necessity. Let F_1 and F_2 be subsets of Ω, closed in Ω. We take an arbitrary function $u \in C^\infty(\Omega)$, such that $u \ge 1$ on F_1 and $u \le 0$ on F_2, and put it into (54)

$$\mu(F_1, F_2; \Omega)^{p/q} \le \Big(\int_{F_1}\int_{F_2}|u(x)-u(y)|^q\mu(dx,dy)\Big)^{1/q} \le C\int_\Omega |\nabla u|^p dx.$$

It remains to minimize the right-hand side, in order to obtain

$$\mu(F_1, F_2; \Omega)^{p/q} \le C\operatorname{cap}_p(F_1, F_2; \Omega).$$

The result follows.□

A direct consequence of Theorem 4 and the isocapacitary inequality for $\operatorname{cap}_p(F; G)$ (see (5) and (6) in Sect. 2.2.3 [22]) is the following sufficient condition for (54) formulated in terms of the n-dimensional Lebesgue measure:

$$\mu(F, \Omega\backslash G) \le c\Big(\log\frac{m_n(G)}{m_n(F)}\Big)^{q(1-n)/n}, \quad \text{if } p = n \qquad (64)$$

and

$$\mu(F, \Omega\backslash G) \le c\big|m_n(G)^{(p-n)/n(p-1)} - m_n(F)^{(p-n)/n(p-1)}\big|^{1-p}, \quad \text{if } p \ne n. \qquad (65)$$

Choosing two concentric balls situated in Ω as the sets F_1 and $\Omega\backslash F_2$ in (55) and using the explicit fofmulae for the p-capacity of spherical condensers (see (1) and (2) in Sect. 2.2.4 [22]) we see that the inequalities (64) and (65), with concentric balls F and G placed in Ω, is a necessary condition for (54).

In the one-dimensional case Theorem 4 can be written in a much simpler form.

Corollary 5. *Let*

$$\Omega = (\alpha, \beta), \qquad -\infty \le \alpha < \beta \le \infty.$$

The inequality

$$\Big(\int_\Omega\int_\Omega |u(x)-u(y)|^q\mu(dx,dy)\Big)^{1/q} \le C\Big(\int_\Omega |u'(x)|^p dx\Big)^{1/p} \qquad (66)$$

holds for every $u \in C^\infty(\Omega)$ if and only if, for all pair of intervals I and J of the three types:

$$I = [x-d, x+d] \quad \text{and} \quad J = (x-d-r,\, x+d+r),$$

$$I = (\alpha, x] \quad \text{and} \quad J = (\alpha, x + r), \tag{67}$$

$$I = [x, \beta) \quad \text{and} \quad J = (x - r, \beta), \tag{68}$$

where d and r are positive and $J \subset \Omega$, we have

$$r^{(p-1)/p} \big(\mu(I, \Omega \backslash J)\big)^{1/q} \le B, \tag{69}$$

where B does not depend on I and J.

Proof. The necessity of (69) follows directly from that in Theorem 4 and the inequality

$$\mathrm{cap}_p(I, \Omega \backslash J; \Omega) \le 2\, r^{1-p}$$

(see Lemma 2.2.2/2 [22]).

Let us prove the sufficiency. By G_1 we mean an open subset of Ω such that $F_1 \subset G_1$ and $\overline{G}_1 \subset \Omega \backslash F_2$. Connected components of $\Omega \backslash F_2$ will be denoted by J_k. Let J_k contain the closed convex hull $\tilde{J}_k$ of those connected components of G_1 which are situated in J_k.

Then

$$\mu(F_1, F_2)^{p/q} \le \mu(G_1, F_2)^{p/q} \le \Big(\sum_k \mu\big(\tilde{J}_k, \Omega \backslash J_k\big)\Big)^{p/q} \le \sum_k \mu\big(\tilde{J}_k, \Omega \backslash J_k\big)^{p/q}$$

and since by (69)

$$\mu\left(\tilde{J}_k, \Omega \backslash J_k\right)^{p/q} \le B^p \big(\mathrm{dist}\{I_k, \mathbb{R} \backslash J_k\}\big)^{1-p}$$

we obtain

$$\mu(F_1, F_2)^{p/q} \le B^p \sum_k \big(\mathrm{dist}\{I_k, \mathbb{R} \backslash J_k\}\big)^{1-p}. \tag{70}$$

Consider an arbitrary function $u \in C^\infty(\Omega)$, such that $u = 1$ on G_1 and $u = 0$ on F_2. Clearly, $u = 0$ on ∂J_k. We have

$$\int_\Omega |u'|^p dx \ge \sum_k \int_{J_k} |u'|^p dx \ge \sum_k \int_{J_k} |\tilde{u}_k'|^p dx, \tag{71}$$

where $\tilde{u} = u$ on $J_k \backslash \tilde{I}_k$, $\tilde{u}_k = 1$ on $\tilde{I}_k$, and $\tilde{u}_k = 0$ on ∂J_k. Hence

$$\int_\Omega |u'|^p dx \ge \sum_k \big(\mathrm{dist}\{\tilde{I}_k, \mathbb{R} \backslash J_k\}\big)^{1-p}.$$

Comparing this estimate with (70), we arrive at

$$\int_\Omega |u'|^p dx \ge \mu(F_1, F_2)^{p/q}$$

and minimizing the integral in the left-hand side over all functions u, we obtain (69).□

Remark 4. It is straightforward but somewhat cumbersome to obtain a more general criterion by replacing the seminorm on the right-hand side of (66) with

$$\Big(\int_\Omega |u'(x)|^p \sigma(dx)\Big)^{1/p}, \tag{72}$$

where σ is a measure in Ω. In fact, one can replace σ by its absolutely continuous part $(d\sigma^*/dx)dx$ and further, roughly speaking, the criterion will follow from Corollary 5 by the change of variable $x \to \xi$, where

$$d\xi = (d\sigma^*/dx)^{1/(1-p)} dx.$$

Restricting myself to this hint, I leave details to the interested reader.

6 Capacitary sufficient condition in the case $q = p$

In the marginal case $q = p$ the condition (55) in Theorem 4, being necessary, is not generally sufficient. In fact, let $n = 1, \Omega = \mathbb{R}$, and

$$\mu(dx, dy) = \frac{dxdy}{|x-y|^{p+1}}.$$

Then as shown in the proof of Corollary 4, (55) is equivalent to (69), and (69) holds, since

$$\begin{aligned}
\mu(I, \mathbb{R}\backslash J) &= \int_{|t-x|<d} dt \int_{|\tau-x|>d+r} \frac{d\tau}{|t-\tau|^{p+1}} \\
&= \int_{|t|<d} dt \int_{|\tau|>d+r} \frac{d\tau}{|t-\tau|^{p+1}} \leq c\, r^{1-p}
\end{aligned}$$

and the same estimate holds for I and J defined by (67) and (68).

On the other hand, (47) fails, because

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|u(x)-u(y)|^p}{|x-y|^{p+1}} dxdy = \infty$$

for every non-constant function u.

In the next theorem we give a sufficient condition for (54) with $q = p > 1$ formulated in terms of an isocapacitary inequality.

Theorem 5. *Given $p \in (1, \infty)$ and a positive, vanishing at infinity, non-increasing absolutely continuous function ν on $\mathbb{R}_+$, such that*

$$S := \sup_{\tau>0} \Big(\int_0^\tau |\nu'(\sigma)|^{1/(1-p)} \frac{d\sigma}{\sigma} \Big)^{p-1} \int_\tau^\infty |\nu'(\sigma)| \frac{d\sigma}{\sigma} < \infty.$$

Suppose that

$$\mu(F_1, F_2) \leq \nu\big((\mathrm{cap}_p(F_1, F_2; \Omega))^{1-p} \big) \tag{73}$$

for all non-overlapping sets F_1 and F_2 closed in Ω. Assume also that

$$\mathcal{K} := \int_0^\infty |\nu'(\sigma)|\, \sigma^{p-1} d\sigma < \infty. \tag{74}$$

Then

$$\|u\|_{p,\mu} \leq 2^{1/p} p \Big(\frac{S}{(p-1)^{p-1}} \Big)^{1/pp'} \mathcal{K}^{1/p} \|\nabla u\|_{L_p(\Omega)} \tag{75}$$

for all $u \in C^\infty(\Omega)$.

Proof. We assume that $\nabla u \in L_p(\Omega)$ and the integral in (75) involving derivatives of ν is convergent. Arguing as in the proof of Theorem 4 and using (73) instead of (55), we obtain

$$\langle u \rangle_{p,\mu}^p \leq 2p(p-1) \int_0^\infty \int_\phi^\infty \nu(\psi - \phi) \big(t(\psi) - t(\phi) \big)^{p-2} t'(\psi) d\psi\, t'(\phi) d\phi. \tag{76}$$

Owing to (74), we can integrate by parts in the inner integral in (76) and obtain

$$\begin{aligned}
\langle u \rangle_{p,\mu}^p &\leq 2p \int_0^\infty \int_\phi^\infty |\nu'(\psi - \phi)| \big(t(\psi) - t(\phi) \big)^{p-1} d\psi\, t'(\phi) d\phi \\
&= 2p \int_0^\infty \int_0^\psi |\nu'(\psi - \phi)| \big(t(\psi) - t(\phi) \big)^{p-1} t'(\phi) d\phi\, d\psi.
\end{aligned}$$

By Hölder's inequality

$$\langle u \rangle_{p,\mu}^p \leq 2p \int_0^\infty \mathcal{A}(\phi)^{1/p'} \mathcal{B}(\phi)^{1/p} d\phi, \tag{77}$$

where

$$\mathcal{A} = \int_0^{\psi} \frac{|\nu'(\psi-\phi)|}{\psi-\phi} \big(t(\psi)-t(\phi)\big)^p d\phi$$

and

$$\mathcal{B} = \int_0^{\psi} |\nu'(\psi-\phi)|\,(\psi-\phi)^{p-1}\,|t'(\psi)|^p d\phi.$$

Using Theorem 1.3.1/1 [22] concerning a two-weight Hardy inequality, we obtain

$$\mathcal{A} \le \frac{p^p}{(p-1)^{p-1}}\,S\,\mathcal{B}$$

which together with (77) gives

$$\langle u\rangle_{p,\mu}^p \le 2p^p(p-1)^{(1-p)/p'} S^{1/p'} \int_0^{\infty}\int_0^{\psi} |\nu'(\psi-\phi)|\,(\psi-\phi)^{p-1}\,|t'(\psi)|^p d\phi\, d\psi.$$

Changing the order of integration, we arrive at

$$\langle u\rangle_{p,\mu} \le 2^{1/p} p\,\big((p-1)^{1-p}S\big)^{1/pp'}\mathcal{K}^{1/p}\|\,t'\,\|_{L_p(\mathbb{R}_+)}.$$

It remains to apply (32). $\square$

Remark 5. If the requirement

$$u=0 \quad \textit{on a neighbourhood of a closed subset } E \textit{ of } \overline{\Omega}$$

is added in the formulation of Theorems 4 and 5, the same proofs give conditions for the validity of (54), similar to (55) and (69). The only new feature is the a restriction

$$\Omega\cap\partial(\Omega\backslash F_2) \quad \textit{is at a positive distance from } E.$$

In the important particular case $E=\partial\Omega$, which corresponds to zero Dirichlet data on $\partial\Omega$, the conditions (55) and (73) become

$$\mu(F,\Omega\backslash G)^{p/q} \le B\,\mathrm{cap}_p(F;G) \tag{78}$$

and

$$\mu(F,\Omega\backslash G) \le \nu\big((\mathrm{cap}_p(F;G))^{1-p}\big), \tag{79}$$

respectively, where F is closed and G is open, $G\supset F$, and the closure of G is compact and situated in Ω. The capacity $\mathrm{cap}_p(F;G)$ is defined by (29) with $\Omega=G$.

Using lower estimates for the p-capacity in terms of area minimizing functions, one obtains sufficient conditions from (55), (69) (78) and (79) formulated in geometrical terms in the spirit of Corollary 2. For example, by (78) and (79), inequalities (55) and (73) hold for all $u\in C_0^\infty(\Omega)$ if, respectively

$$\mu(F,\Omega\backslash G)^{p/q} \le B\Big(\int_{m_n(F)}^{m_n(\Omega\backslash G)} \frac{dv}{\lambda(v)^{p/(p-1)}}\Big)$$

and

$$\mu(F,\Omega\backslash G) \le \nu\Big(\int_{m_n(F)}^{m_n(\Omega\backslash G)} \frac{dv}{\lambda(v)^{p/(p-1)}}\Big),$$

where F and G are the same as in (78) and (79). $\square$

By obvious modifications of the proof of sufficiency in Corollary 4 one deduces the following assertion from Theorem 5.

Corollary 6. (One-dimensional case) *With the notation used in Corollary 5, suppose that*

$$\mu(I,\Omega\backslash J) \le \nu(r).$$

Then there exists a positive constant c depending only on p and such that

$$\langle u\rangle_{p,\mu} \le c\, S^{1/pp'} K^{1/p}\, \|u'\|_{L_p(\Omega)}$$

for all $u \in C^\infty(\Omega)$.

Remark 6. Let us show that the condition $K < \infty$, which appeared in Theorem 5, is sharp. Suppose that there exists a positive constant C independent of u and such that

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |u(t)-u(\tau)|^p\, \nu''(t-\tau)dt\, d\tau \le C \int_{\mathbb{R}} |u'(t)|^p dt, \tag{80}$$

where ν is a convex function in $C^2(\mathbb{R})$. We take an arbitrary $N > 0$ and put $u(t) = \min\{|t|, N\}$ into (80). Then

$$\int_0^{N/2}\int_\tau^N (t-\tau)^p\, \nu''(t-\tau)dt\, d\tau \le 2CN$$

and setting here $t = \tau + s$, we obtain

$$\frac{1}{2}pN\int_0^{N/2} s^{p-1}\,|\nu'(s)|\, ds \le p\int_0^{N/2}\int_0^{N-\tau} s^{p-1}|\nu'(s)|ds\, d\tau \le 2CN.$$

Hence $K \le 4p^{-1}C$.

Remark 7. It seems appropriate, in conclusion, to say a few words about the lower estimate for the difference seminorm $\langle u\rangle_{p,\mu}$, similar to the classical Sobolev inequality:

$$\Big(\int_\Omega |u|^q\, \nu(dx)\Big)^{1/q} \le C\,\langle u\rangle_{p,\mu}, \tag{81}$$

where Ω is a subdomain of a Riemannian manifold, μ and ν are measures in $\Omega\times\Omega$ and Ω, respectively, and u is an arbitrary function in $C_0^\infty(\Omega)$. Suppose that $q \ge p \ge 1$. Then a condition, necessary and sufficient for (81), is the isocapacitary inequality

$$\sup_{\{F\}} \frac{\nu(F)^{p/q}}{\mathrm{cap}_{p,\mu}(F;\Omega)} < \infty, \tag{82}$$

where F is an arbitrary compact set in Ω and the capacity is defined by

$$\mathrm{cap}_{p,\mu}(F;\Omega) = \inf\big\{\langle u\rangle_{p,\mu}^p :\ u \in C_0^\infty(\Omega),\ u \ge 1 \text{ on } F\big\}.$$

The necessity of (82) is obvious and the sufficiency results directly from the inequality

$$\int_0^\infty \mathrm{cap}_{p,\mu}(N_t;\Omega)\, d(t^p) \le c(p)\,\langle u\rangle_{p,\mu}^p \tag{83}$$

(see [24] for the proof and history of (83)).

Although providing a universal characterization of (81), the condition (82) does not seem satisfactory when dealing with concrete measures and domains. This is related even to one-dimensional case (cfr. Problem 2 [11]). As an example of a more visible criterion, consider the measure μ on $\mathbb{R}^n\times\mathbb{R}^n$ given by

$$\mu(dx,dy) = |x-y|^{-n-p\alpha}dx\, dy \tag{84}$$

with $0 < \alpha < 1$ and $\alpha p < n$. This measure generates a seminorm in the homogeneous Besov space $b_p^\alpha(\mathbb{R}^n)$. With this particular choice of μ, we have by Theorem 8.7.1 and Remark 8.6/3 [22] that (81) holds with $q > p > 1$ and $q \ge p = 1$ if and only if

$$\sup_{x\in\mathbb{R}^n,\, \rho>0} \frac{\nu(B(x,\rho))^{p/q}}{\rho^{n-p\alpha}} < \infty. \tag{85}$$

The inequality (85) is the same as (82) with $F = B(x, \rho)$. It is unknown whether the replacement of arbitrary sets by balls in (82) is possible for the general μ and $\Omega = \mathbb{R}^n$ in (82). If not, what are sharp requirements allowing this replacement?

Let $q = p > 1$, $\Omega = \mathbb{R}^n$ and let μ be given by (84). Then (81) holds simultaneously with the inequality

$$\int_{\mathbb{R}^n} |u|^p \, \nu(dx) \leq c \, \|(-\Delta)^{\alpha/2} u\|^p_{L_p(\mathbb{R}^n)} \tag{86}$$

because both (81) and (86) are equivalent to isocapacitary inequalities of the type (82) with equivalent capacities in the right-hand side (see [2], Sect.4.4).

Note that (86) is the so called trace inequality for the Riesz potential operator $I_\alpha := (-\Delta)^{-\alpha/2}$. This inequality has been studied intensively (see [30] for a survey of this area). First of all, the simplest estimate

$$\nu(B) \leq c \, m_n(B)^{1-p\alpha/n} \quad \text{for all balls } B,$$

being necessary for (86), is not sufficient for it (see [1] and [2]). However, there exist other conditions involving no capacity, which are necessary and sufficient for (86). They are as follows:

(i) For every ball B,

$$\int_B (I_\alpha \nu_B)^p dx \leq c \, \nu(B),$$

where ν_B be the restriction of ν on B, see [9].

(ii) Almost everywhere in $\mathbb{R}^n$,

$$I_\alpha (I_\alpha \nu)^{p'} \leq c \, I_\alpha \nu,$$

see [27].

(iii) For every dyadic cube P of side length $\ell(P)$,

$$\sum_{Q \subset P} \big(\nu(Q) \, \ell(Q)^{\alpha - n/p}\big)^{p'} \leq c \, \nu(P),$$

where the sum is taken over all dyadic cubes Q contained in P, see [30], Sect. 3.

In accordance with the equivalence of (81) and (86) mentioned previously, the criteria (i)-(iii) characterize not only (86) but also (81) with $q = p > 1$ and μ defined by (84). It is unclear how these criteria could be modified to characterize (81) with an arbitrary μ.

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