Boundedness of the gradient of a solution to the Neumann-Laplace problem in a convex domain

Vladimir Maz'ya*

Department of Mathematical Sciences, University of Liverpool, Liverpool L69 7ZL
and
Department of Mathematics, Linköping University, Linköping, SE-581 83
e-mail: vlmaz@mai.liu.se

Abstract. It is shown that solutions of the Neumann problem for the Poisson equation in an arbitrary convex n-dimensional domain are uniformly Lipschitz. Applications of this result to some aspects of regularity of solutions to the Neumann problem on convex polyhedra are given.

Mathematics Subject Classification (2000): Primary 54C40, 14E20; Secondary 46E25, 20C20

Key words: Neumann problem, Laplace operator, Sobolev spaces, convex domain, boundedness of the gradient, eigenvalues of the Neumann-Beltrami operator

Résumé. On démontre que les solutions du problème de Neumann pour l'équation de Poisson dans un domaine convexe arbitraire de dimension n sont uniformément Lipschitz. Les applications de ce résultat à quelques aspects de régularité de solutions du problème de Neumann sur les polyèdres convexes sont données.

Mathematics Subject Classification (2000): Primary 54C40, 14E20; Secondary 46E25, 20C20

Mots clé: problème de Neumann, opérateur de Laplace, espaces de Sobolev, domaine convexe; bornitude du gradient, valeurs propres de l'opérateur de Neumann-Beltrami.

1 Introduction

Let Ω be a bounded convex domain in $\mathbb{R}^n$ and let $W^{l,p}(\Omega)$ stand for the Sobolev space of functions in $L^p(\Omega)$ with distributional derivatives of order l in $L^p(\Omega)$. By $L^p_\perp(\Omega)$ and $W^{l,p}_\perp(\Omega)$ we denote the subspaces of functions v in $L^p(\Omega)$ and $W^{l,p}(\Omega)$ subject to $\int_\Omega v dx = 0$.

Let $f \in L^2_\perp(\Omega)$ and let u be the unique function in $W^{1,2}(\Omega)$, also orthogonal to 1 in $L^2(\Omega)$, and satisfying the Neumann problem

$$-\Delta u = f \quad \text{in } \Omega, \tag{1}$$

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \tag{2}$$

where ν is the unit outward normal vector to $\partial\Omega$ and the problem (1), (2) is understood in the variational sense. It is well known that the inverse mapping

$$L^2_\perp(\Omega) \ni f \to u \in W^{2,2}_\perp(\Omega) \tag{3}$$

*The author was partially supported by the UK Engineering and Physical Sciences Research Council grant EP/F005563/1.

is continuous. (Since any attempt at reviewing the rich history of this fact and other ones, closely related to it, within frames of a short article is hopeless, I restrict myself to a number of relevant references [3], [4], [11], [13], [16]–[18], [20], [23], [28], [29].) As shown in [2] (see also [12] for a different proof, and [1], [8]–[10] for the Dirichlet problem), the operator

$$L^p_\perp(\Omega) \ni f \to u \in W^{2,p}_\perp(\Omega) \tag{4}$$

is also continuous if $1 < p < 2$. One cannot guarantee the continuity of (4) for any $p \in (2, \infty)$ without additional information about the domain. The situation is the same as in the case of the Dirichlet problem (see [4], [8]-[10]), which, moreover, possesses the following useful property: if Ω is convex, the gradient of the solution is uniformly bounded provided the right-hand side of the equation is good enough. This property can be easily checked by using a simple barrier. Other approaches to similar results were exploited in [21] and [14] for different equations and systems but only for the Dirichlet boundary conditions.

In this respect, other boundary value problems are in a nonsatisfactory state. For instance, it was unknown up to now whether solutions of the problem (1), (2) with a smooth f are uniformly Lipschitz under the only condition of convexity of Ω.

The main result of the present paper is *the boundedness of* $|\nabla u|$ *for the solution* u *of the Neumann problem* (*1*), (*2*) *in any convex domain* $\Omega \subset \mathbb{R}^n$, $n \geq 2$.

A direct consequence of this fact is the sharp lower estimate $\Lambda \geq n-1$ for the first nonzero eigenvalue Λ of the Neumann problem for the Beltrami operator on a convex subdomain of a unit hemisphere. It was obtained by a different argument for manifolds of positive Ricci curvature by J. F. Escobar in [6], where the case of equality was settled as well. This estimate answered a question raised by M. Dauge [5], and it leads, in combination with known techniques of the theory of elliptic boundary value problems in domains with piecewise smooth boundaries (see [5], [25]–[27]), to estimates for solutions of the problem (1), (2) in various function spaces. Two examples are given at the end of this article.

2 Main result

In what follows, we need a constant C_Ω in the relative isoperimetric inequality

$$s(\Omega \cap \partial g) \geq C_\Omega \, |g|^{1-1/n}, \tag{5}$$

where g is an arbitrary open set in Ω such that $|g| \leq |\Omega|/2$ and $\Omega \cap \partial g$ is a smooth (not necessarily compact) submanifold of Ω. By s we denote the $(n-1)$-dimensional area and by $|g|$ the n-dimensional Lebesgue measure. The Poincaré-Gagliardo-Nirenberg inequality

$$\inf_{t \in \mathbb{R}} \|v - t\|_{L^{n/(n-1)}(\Omega)} \leq const. \, \|\nabla v\|_{L^1(\Omega)}, \qquad \forall v \in W^{1,1}(\Omega), \tag{6}$$

where $const. \leq C_\Omega^{-1}$ is a consequence of (5) (see Theorem 3.2.3 [24]).

Theorem. *Let* $f \in L^q_\perp(\Omega)$ *with a certain* $q > n$. *Then the solution* $u \in W^{1,2}_\perp(\Omega)$ *of the problem* (*1*), (*2*) *satisfies the estimate*

$$\|\nabla u\|_{L^\infty(\Omega)} \leq c(n,q) \, C_\Omega^{-1} \, |\Omega|^{(q-n)/qn} \|f\|_{L^q(\Omega)}, \tag{7}$$

where c *is a constant depending only on* n *and* q.

Proof. It suffices to prove (7) assuming that $f \in C_0^\infty(\Omega)$. Let us approximate Ω by a sequence $\{\Omega_m\}_{m \geq 1}$ of convex domains with smooth boundaries, $\Omega_m \supset \overline{\Omega}$. This can be done, for instance, by approximating Ω by a family of equidistant surfaces and by smoothing them with small perturbation of normal vectors. Then (6) implies

$$\inf_{t \in \mathbb{R}} \|w - t\|_{L^{\frac{n}{n-1}}(\Omega_m)} \leq (1+\varepsilon) \, C_\Omega^{-1} \, \|\nabla w\|_{L^1(\Omega_m)}, \tag{8}$$

for all $w \in W^{1,1}(\Omega_m)$, where ε is an arbitrary positive number and $m = m(\varepsilon)$.

By u_m we denote a solution of the problem (1), (2) in Ω_m with f extended by zero outside Ω. One can easily see that $\nabla u_m \to \nabla u$ in $L^2(\Omega)$. Hence, it is enough to obtain (7) assuming that $\partial\Omega$ is smooth.

Let $t > \tau > 0$ and let Ψ be a piecewise linear continuous function on $\mathbb{R}$ specified by $\Psi(\xi) = 0$ for $\xi < \tau$ and $\Psi(\xi) = 1$ for $\xi > t$. Note that

$$(\Delta u)^2 - |\nabla_2 u|^2 = (u_{x_j}\, \Delta u)_{x_j} - (u_{x_j} u_{x_i x_j})_{x_i}, \tag{9}$$

where

$$|\nabla_2 u| = \Big(\sum_{1\le i,j\le n} u^2_{x_i x_j} \Big)^{1/2}.$$

Hence

$$\int_\Omega \Psi(|\nabla u|)(f^2 - |\nabla_2 u|^2)\, dx = \int_{\partial\Omega} \Psi(|\nabla u|)\big(\nu_j\, u_{x_j} \Delta u - \nu_i\, u_{x_j}\, u_{x_i x_j}\big)\, ds_x$$
$$+ \int_\Omega \Psi'(|\nabla u|)\big(\, (|\nabla u|)_{x_j} u_{x_j} f + (|\nabla u|)_{x_i}\, u_{x_j}\, u_{x_i x_j}\big) dx, \tag{10}$$

where $(\nu_1, \dots, \nu_n)$ are components of the outward unit normal. By the Bernshtein-type identity (see, for instance, [11] or [17]), the first integral on the right-hand side of (10) equals

$$-2 \int_{\partial\Omega} Q(\nabla_{tan} u,\, \nabla_{tan} u)\, ds_x,$$

where Q is the second fundamental quadratic form on $\partial\Omega$ and ∇_{tan} is the tangential gradient. The form Q is nonpositive by convexity of Ω, which leads, together with (10), to the inequality

$$\int_\Omega \Psi'(|\nabla u|)\big(\, (|\nabla u|)_{x_j} u_{x_j} f + (|\nabla u|)_{x_i}\, u_{x_j}\, u_{x_i x_j}\big) dx \le \int_\Omega \Psi(|\nabla u|)\, f^2\, dx. \tag{11}$$

By the co-area formula [7], the left-hand side of (11) is identical to

$$(t-\tau)^{-1} \int_\tau^t \int_{|\nabla u| = \sigma} \big(\, (|\nabla u|)_{x_j} u_{x_j} f + (|\nabla u|)_{x_i}\, u_{x_j}\, u_{x_i x_j}\big) \frac{ds_x}{|\nabla |\nabla u|\,|} d\sigma,$$

which is equal to

$$(t-\tau)^{-1} \int_\tau^t \int_{|\nabla u| = \sigma} \Big(\frac{\partial u}{\partial \nu} f - |\nabla u| \frac{\partial}{\partial \nu} |\nabla u| \Big) ds_x\, d\sigma \tag{12}$$

because $\nabla |\nabla u| = -\nu\, |\nabla |\nabla u|\,|$ on the level surface $|\nabla u| = \sigma$, where ν is the unit normal, outward with respect to the set $\{x : |\nabla u| > \sigma\}$. The expression (12) can be written as

$$(\tau - t)^{-1} \int_\tau^t \int_{|\nabla u| = \sigma} \Big(\frac{\partial u}{\partial \nu} f - |\nabla u|\, |\nabla |\nabla u|\,| \Big) ds_x\, d\sigma.$$

Passing here to the limit as $\tau \uparrow t$ and using (11), we arrive at the estimate

$$\int_{|\nabla u| = t} \Big(|\nabla u|\, |\nabla |\nabla u|\,| - f \frac{\partial u}{\partial \nu} \Big) ds_x \le \int_{|\nabla u| > t} f^2 dx,$$

which implies

$$t \int_{|\,\nabla u| = t} |\nabla |\nabla u|\,| ds_x \le t \int_{|\nabla u| = t} |f|\, ds_x + \int_{|\nabla u| > t} f^2 dx. \tag{13}$$

We define the median of $|\nabla u|$ as

$$med\, |\nabla u| = \sup\{ t \in \mathbb{R} : |\{ |\nabla u| > t\}| \ge |\Omega|/2 \},$$

and we note that

$$\big| \{ |\nabla u| > med\, |\nabla u| \} \big| \le |\Omega|/2$$

and

$$\big|\{|\nabla u| \geq med\,|\nabla u|\}\big| \geq |\Omega|/2.$$

Clearly,

$$med\,|\nabla u| \leq \Big(\frac{2}{|\Omega|}\Big)^{1/2} \|\nabla u\|_{L^2(\Omega)} \tag{14}$$

and, by Hölder's inequality and (8),

$$\begin{aligned} \|\nabla u\|_{L^2(\Omega)} &\leq \inf_{\gamma\in\mathbb{R}} \|u-\gamma\|^{1/2}_{L^{\frac{n}{n-1}}(\Omega)} \|f\|^{1/2}_{L^n(\Omega)} \\ &\leq (1+\varepsilon)\, C_\Omega^{-1/2} \|\nabla u\|^{1/2}_{L^1(\Omega)} \|f\|^{1/2}_{L^n(\Omega)}. \end{aligned}$$

Hence,

$$\|\nabla u\|_{L^2(\Omega)} \leq (1+\varepsilon)^2 C_\Omega^{-1} |\Omega|^{1/2}\, \|f\|_{L^n(\Omega)},$$

which, in combination with (14) and Hölder's inequality, implies

$$med\,|\nabla u| \leq 2^{1/2}(1+\varepsilon)^2 C_\Omega^{-1} |\Omega|^{(q-n)/qn}\, \|f\|_{L^q(\Omega)}. \tag{15}$$

We introduce the function

$$\psi : \big[med\,|\nabla u|, \max|\nabla u|\,\big] \to [0,\infty)$$

by the equality

$$\psi(t) = \int_{med\,|\nabla u|}^{t} \Big(\int_{|\nabla u|=\sigma} |\nabla|\nabla u|\,|ds_x\Big)^{-1} d\sigma. \tag{16}$$

Let $E_\psi = \big\{x : |\nabla u(x)| = t(\psi)\big\}$ and $M_\psi = \big\{x : |\nabla u(x)| > t(\psi)\big\}$. Putting $t = t(\psi)$ in (13) and integrating in ψ over $\mathbb{R}_+$, we arrive at

$$\max|\nabla u|^2 \leq (med\,|\nabla u|)^2 + 2\max|\nabla u| \int_0^\infty \int_{E_\psi} |f|\, ds_x\, d\psi + 2\int_0^\infty \int_{M_\psi} f^2\, dx\, d\psi.$$

Recalling (15), we see that in order to obtain (7), it suffices to prove the inequalities

$$\int_0^\infty \int_{E_\psi} |f|\, ds_x\, d\psi \leq c\, C_\Omega^{-1} |\Omega|^{(q-n)/qn} \|f\|_{L^q(\Omega)} \tag{17}$$

and

$$\Big(\int_0^\infty \int_{M_\psi} f^2\, dx\, d\psi\Big)^{1/2} \leq c\, C_\Omega^{-1} |\Omega|^{(q-n)/qn} \|f\|_{L^q(\Omega)}, \tag{18}$$

where $c = c(n,q)$.

The following argument leading to (17) is an obvious modification of the proof of Lemma 4 [22]. We start with the estimate for the area of the set E_ψ

$$s(E_\psi)^2 \leq -\frac{d}{d\psi}|M_\psi| \tag{19}$$

obtained in Lemma 2 [22]. By the triple Hölder inequality and (19)

$$\int_{E_\psi} |f|\, ds_x \leq \Big(\int_{E_\psi} |f|^q \frac{ds_x}{|\nabla|\nabla u|\,|}\Big)^{1/q} \Big(\int_{E_\psi} |\nabla|\nabla u|\,|\, ds_x\Big)^{1/q} s(E_\psi)^{1-2/q}$$

$$\leq s(E_\psi)^{-1} \Big(-\frac{d}{d\psi}|M_\psi|/t'(\psi)\Big)^{1-1/q} \Big(\int_{E_\psi} |f|^q \frac{ds_x}{|\nabla|\nabla u|\,|}\Big)^{1/q} \int_{E_\psi} |\nabla|\nabla u|\,|\, ds_x.$$

This, in combination with the inequality

$$s(E_\psi) \geq (1+\varepsilon)^{-1} C_\Omega\, |M_\psi|^{1-1/n}$$

(see (5)), implies the estimate for the integral on the left-hand side of (17)

$$\int_0^\infty \int_{E_\psi} |f|\, ds_x\, d\psi \le (1+\varepsilon) C_\Omega^{-1} \Big(\int_0^{|\Omega|/2} \mu^{\frac{(1-n)q}{n(q-1)}} d\mu \Big)^{1-1/q} \Big(\int_0^\infty \int_{E_\psi} |f|^q \frac{ds_x}{|\nabla |\nabla u|\,|}\, t'(\psi)\, d\psi \Big)^{1/q}$$

$$\le (1+\varepsilon) C_\Omega^{-1} \Big(\frac{n(q-1)}{q-n} \Big(\frac{|\Omega|}{2} \Big)^{\frac{q-n}{n(q-1)}} \Big)^{1-1/q} \|f\|_{L^q(\Omega)}$$

and the proof of (17) is complete.

We turn to inequality (18). By (19), its left-hand side does not exceed

$$\Big(-\int_0^\infty \int_{M_\psi} |f|^2 dx \frac{d\,|M_\psi|}{s(E_\psi)^2} \Big)^{1/2}$$

which is dominated by

$$(1+\varepsilon)\, C_\Omega^{-1} \Big(-\int_0^\infty \int_{M_\psi} |f|^2 dx \frac{d\,|M_\psi|}{|M_\psi|^{2\frac{n-1}{n}}} \Big)^{1/2}$$

$$\le (1+\varepsilon)\, C_\Omega^{-1} \Big(\int_0^{|\Omega|/2} \int_0^\sigma f_*(\tau)^2 d\tau \frac{d\sigma}{\sigma^{2\frac{n-1}{n}}} \Big)^{1/2},$$

where f_* is the nonincreasing rearrangement of $|f|$. Now, (18) follows by integration by parts and Hölder's inequality. The proof of Theorem is complete.

3 Regularity of solutions to the Neumann problem in a convex polyhedron

The following assertion essentially stemming from the above theorem is a particular case of Escobar's result in [6] mentioned in Introduction.

Corollary. *Let ω be a convex subdomain of the upper unit hemisphere S_+^{n-1}. The first positive eigenvalue Λ of the Beltrami operator on ω with zero Neumann data on $\partial\omega$ is not less than $n-1$.*

Proof. Let $\lambda(\lambda+n-2)=\Lambda$ and $\lambda>0$. In the convex domain

$$\Omega = \big\{ x \in \mathbb{R}^n : 0 < |x| < 1,\ \frac{x}{|x|} \in \omega \big\},$$

we define the function

$$u(x) = |x|^\lambda \Phi\Big(\frac{x}{|x|}\Big) \eta(|x|), \tag{20}$$

where Φ is an eigenfunction corresponding to Λ and η is a smooth cut-off function on $[0,\infty)$, equal to one on $[0,1/2]$ and vanishing outside $[0,\,1]$.

Let N be an integer satisfying $4N > n-1 \ge 4(N-1)$ and let $j=0,1,\dots,N$,

$$q_j = \begin{cases} \dfrac{2(n-1)}{n-1-4j}, & \text{if } 0 \le j < (n-1)/4, \\ \text{arbitrary} & \text{if } j = (n-1)/4, \end{cases}$$

and $q_N = \infty$. Iterating the estimate

$$\|\Phi\|_{L^{q_j+1}(\omega)} \le c\,\Lambda\, \|\Phi\|_{L^{q_j}(\omega)}$$

obtained in Theorems 5 and 6 [22], we see that $\Phi \in L^\infty(\omega)$.

The function u, defined by (20), satisfies the Neumann problem (1), (2) with

$$f(x) = -\Phi\Big(\frac{x}{|x|}\Big) \big[\Delta, \eta(|x|)\big]\, |x|^\lambda.$$

Since $\Phi \in L^\infty(\omega)$, it follows that $f \in L^\infty(\Omega)$ and by Theorem, $|\nabla u| \in L^\infty(\Omega)$, which is possible only if $\lambda \geq 1$, i.e. $\Lambda \geq n-1$. The proof is complete.

Two applications of the above estimate for Λ will be formulated.

Let Ω be a convex bounded 3-dimensional polyhedron. By the techniques, well-known nowadays (see [5], [25]–[27]), one can show the unique solvability of the variational Neumann problem in $W^{1,p}_\perp(\Omega)$ for every $p \in (1, \infty)$. By definition of this problem, its solution is subject to the integral identity

$$\int_\Omega \nabla u \cdot \nabla \eta \, dx = f(\eta),$$

where f is a given distribution in the space $(W^{1,p'}(\Omega))^*$, $f(1) = 0$ and η is an arbitrary function in $W^{1,p'}(\Omega)$, $p + p' = pp'$.

Let us turn to the second application of Corollary. We continue to deal with the polyhedron Ω in $\mathbb{R}^3$. Let $\{\mathcal{O}\}$ be the collection of all vertices and let $\{U_\mathcal{O}\}$ be an open finite covering of $\overline{\Omega}$ such that $\mathcal{O}$ is the only vertex in $U_\mathcal{O}$. Let also $\{E\}$ be the collection of all edges and let α_E denote the opening of the dihedral angle with edge E, $0 < \alpha_E < \pi$. The notation $r_E(x)$ stands for the distance between $x \in U_\mathcal{O}$ and the edge E such that $\mathcal{O} \in \overline{E}$.

With every vertex $\mathcal{O}$ and edge E we associate real numbers $\beta_\mathcal{O}$ and δ_E, and we introduce the weighted L^p-norm

$$\|v\|_{L^p(\Omega;\{\beta_\mathcal{O}\},\{\delta_E\})} := \Big(\sum_{\{\mathcal{O}\}} \int_{U_\mathcal{O}} |x-\mathcal{O}|^{p\beta_\mathcal{O}} \prod_{\{E:\mathcal{O}\in\overline{E}\}} \Big(\frac{r_E(x)}{|x-\mathcal{O}|}\Big)^{p\delta_E} |v(x)|^p dx\Big)^{1/p},$$

where $1 < p < \infty$. Under the conditions

$$\begin{aligned} 3/p' &> \beta_\mathcal{O} > -2 + 3/p', \\ 2/p' &> \delta_E > -\min\{2, \pi/\alpha_E\} + 2/p', \end{aligned}$$

the inclusion of the function f in $L^p(\Omega; \{\beta_\mathcal{O}\}, \{\delta_E\})$ implies the unique solvability of the problem (1), (2) in the class of functions with all derivatives of the second order belonging to $L^p(\Omega; \{\beta_\mathcal{O}\}, \{\delta_E\})$. This fact follows from Corollary and a result essentially established in Sect. 7.5 [27].

An important particular case when all $\beta_\mathcal{O}$ and δ_E vanish, i.e. when we deal with a standard Sobolev space $W^{2,p}(\Omega)$, is also included here. To be more precise, if

$$1 < p < \min\Big\{3, \frac{2\alpha_E}{(2\alpha_E - \pi)_+}\Big\} \tag{21}$$

for all edges E, then the inverse operator of the problem (1), (2):

$$L^p_\perp(\Omega) \ni f \to u \in W^{2,p}_\perp(\Omega)$$

is continuous whatever the convex polyhedron $\Omega \subset \mathbb{R}^3$ may be. The bounds for p in (21) are sharp for the class of all convex polyhedra.

References

[1] Adolfsson, V. *L^p-integrability of the second order derivatives of Green potentials in convex domains*. Pacific J. Math. **159** (1993), no. 2, 201–225.

[2] Adolfsson, V., Jerison, D. *L^p-integrability of the second order derivatives for the Neumann problem in convex domains*. Indiana Univ. Math. J. **43** (1994), no. 4, 1123–1138.

[3] Bernshtein, S. N. *Sur la nature analytique des solutions des équations aux dérivées partielles du second ordre*. Math. Ann. **59** (1904), no. 1-2, 20–76.

[4] Birman, M. S., Skvortsov, G. E. *On square summability of highest derivatives of the solution of the Dirichlet problem in a domain with piecewise smooth boundary.* (Russian) Izv. Vysš. Učebn. Zaved. Matematika, 1962 no. 5 (30), 11–21.

[5] Dauge, M. *Neumann and mixed problems on curvilinear polyhedra*. Integral Equations Operator Theory **15** (1992), no. 2, 227–261.

[6] Escobar, J. F. *Uniqueness Theorems on Conformal Deformation of Metrics, Sobolev inequalities and an Eigenvalue Estimate*. Communications on Pure and Applied Mathematics **XLIII** (1990), 857–883.

[7] Federer, H., *Curvature measures*, Trans. Amer. Math. Soc. **93** (1959), no. 3, 418–491.

[8] Fromm, S. J. *Potential space estimates for Green potentials in convex domains.* Proc. Amer. Math. Soc. **119** (1993), no. 1, 225–233.

[9] Fromm, S. J. *Regularity of the Dirichlet problem in convex domains in the plane.* Michigan Math. J. **41** (1994), no. 3, 491–507.

[10] Fromm, S. J., Jerison, D. *Third derivative estimates for Dirichlet's problem in convex domains.* Duke Math. J. **73** (1994), no. 2, 257–268.

[11] Grisvard, P. *Elliptic Problems in Nonsmooth Domains*, Pitman, 1985.

[12] Jakab, T., Mitrea, I., Mitrea, M. *Sobolev estimates for the Green potential associated with the Robin-Laplacian in Lipschitz domains satisfying a uniform exteriour ball condition*, Sobolev Spaces in mathematics II, Applications in Analysis and Partial Differential Equations, International Mathematical Series, Vol. 9, Springer, 2008.

[13] Kadlec, J. *The regularity of the solution of the Poisson problem in a domain whose boundary is similar to that of a convex domain.* (Russian) Czechoslovak Math. J. **14** (**89**), (1964), 386–393.

[14] Kozlov, V., Maz'ya, V. *Asymptotic formula for solutions to elliptic equations near the Lipschitz boundary*, Ann. Mat. Pura ed Appl. **184** (2005), , no. 2, 185–213.

[15] Kozlov, V. A.; Maz'ya, V. G.; Rossmann, J., *Spectral Problems Associated with Corner Singularities of Solutions to Elliptic Equations*, Mathematical Surveys and Monographs, vol. 85, 436 pp., American Mathematical Society, 2001.

[16] Ladyzhenskaya, O. A. *Closure of an elliptic operator*, (Russian) Dokl. Akad. Nauk SSSR, **79** (1951), 723–725.

[17] Ladyzhenskaya, O. A. *Smeshannaya Zadacha dlya Giperbolicheskogo Uravneniya.* (Russian) [*The Mixed Problem for a Hyperbolic Equation.*] Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1953.

[18] Ladyzhenskaya, O. A., Uraltseva, N. N. *Linear and Quasilinear Elliptic Equations.* Academic Press, New York-London, 1968.

[19] Mayboroda, S., Maz'ya, V. *Boundedness of the Hessian of a biharmonic function in a convex domain*, Comm. Partial Diff. Eq. **33** (2008), no 8.

[20] Maz'ya, V. G. *Solvability in* $\overset{\circ}{W}{}^2_2$ *of the Dirichlet problem in a region with smooth irregular boundary.* (Russian) Vestnik Leningrad. Univ. **22** (1967), no. 7, 87–95.

[21] Maz'ya, V. G. *The boundedness of the first derivatives of the solution of the Dirichlet problem in a region with smooth nonregular boundary.* (Russian) Vestnik Leningrad. Univ. **24** (1969), no.1, 72–79.

[22] Maz'ya, V. G. *On weak solutions of the Dirichlet and Neumann problems*. Trans. Moscow Math. Soc. **20** (1969), 135–172.

[23] Maz'ya, V. G. *The coercivity of the Dirichlet problem in a domain with irregular boundary.* (Russian) Izv. Vysš. Učebn. Zaved. Matematika, no. 4(131) (1973), 64–76.

[24] Maz'ya, V. G. *Sobolev Spaces*, Springer, 1985.

[25] Maz'ya, V.G., Plamenevskii, B.A. *L_p-estimates of solutions of elliptic boundary value problems in domains with ribs*. (Russian) Trudy Moskov. Mat. Obshch. **37** (1978), 49–93. English translation: Trans. Moscow Math. Soc. **1** (1980), 49–97.

[26] Maz'ya, V. G., Plamenevskii, B. A. *Estimates in L_p and in Hölder classes, and the Miranda-Agmon maximum principle for the solutions of elliptic boundary value problems in domains with singular points on the boundary*. Amer. Math. Soc. Transl. (2) **123** (1984), 1–56.

[27] Maz'ya, V., Rossmann, J. *Weighted L_p estimates of solutions to boundary value problems for second order elliptic systems in polyhedral domains*, Z. Angew. Math. Mech. **83** (2003), no. 7, 435-467.

[28] Schauder, J. *Sur les équations linéaires du type élliptique à coefficients continus*, C. R. Acad. Sci. Paris, **199** (1934), 1366–1368.

[29] Sobolev, S. L. *Sur la presque périodicité des solutions de l'équations des ondes.* I. Comptes Rendus (Doklady) de l'Acad. Sci. de l'URSS, **48** (1945), 542–545; II, ibid. **48** (1945), 618–620.