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APPROXIMATION OF SOLUTIONS TO NONSTATIONARY STOKES SYSTEM

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We propose a fast method for high order approximations of the solution to the Cauchy problem for the linear nonstationary Stokes system in $\mathbb{R}^3$ *in the unknown velocity* $\mathbf{u}$ *and kinematic pressure* P*. The density* $\mathbf{f}(\mathbf{x},t)$ *and the divergence-free vector initial value* $\mathbf{g}(\mathbf{x})$ *are smooth and rapidly decreasing as* $|\mathbf{x}| \to \infty$*. We construct the vector* $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$ *where* $\mathbf{u}_1$ *solves a system of homogeneous heat equations and* $\mathbf{u}_2$ *solves a system of nonhomogeneous heat equations with right-hand side* $\mathbf{f} - \nabla P$*, where* $P = -\mathscr{L}(\nabla \cdot \mathbf{f})$ *and* $\mathscr{L}$ *denotes the harmonic potential. Fast semianalytic cubature formulas for computing the harmonic potential and the solution to the heat equation based on the approximation of the data by functions with analitically known potentials are considered. The gradient* ∇P *can be approximated by the gradient of the cubature of* P*, which is a semianalytic formula. We derive fast and accurate high order formulas for approximation of* $\mathbf{u}_1, \mathbf{u}_2$*,* P*, and* ∇P*. The accuracy of the method and the convergence order* $2, 4, 6, 8$ *are confirmed by numerical experiments. Bibliography: 18 titles.*

1 Introduction

In the present paper, we describe a fast method for numerical solution of the nonstationary Stokes system

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$$\frac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} + \nabla P = \mathbf{f}, \tag{1.1}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{1.2}$$

with the initial condition

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{g}(\mathbf{x}) \tag{1.3}$$

for $(\mathbf{x}, t) \in \mathbb{R}^3 \times \mathbb{R}_+$ with $\mathbb{R}_+ = [0, \infty)$. Here, $\nabla = \{\partial_{x_1}, \partial_{x_2}, \partial_{x_3}\}$ so that $\nabla P = \operatorname{grad} P$ and $\nabla \cdot \mathbf{u} = \operatorname{div} \mathbf{u}$. Equations (1.1)–(1.3) are solved for an unknown velocity vector field $\mathbf{u}(\mathbf{x}, t) = (u_1(\mathbf{x}, t), u_2(\mathbf{x}, t), u_3(\mathbf{x}, t))$ and kinematic pressure $P(\mathbf{x}, t)$ defined at all points $\mathbf{x} \in \mathbb{R}^3$ and time $t \geqslant 0$. Here, the viscosity ν is a positive coefficient, the data $\mathbf{f}(\mathbf{x}, t) = (f_1(\mathbf{x}, t), f_2(\mathbf{x}, t), f_3(\mathbf{x}, t))$ and $\mathbf{g}(\mathbf{x}) = (g_1(\mathbf{x}), g_2(\mathbf{x}), g_3(\mathbf{x}))$ are smooth and sufficiently rapidly decreasing as $|\mathbf{x}| \to \infty$ and $\mathbf{g}$ is a divergence-free vector field, i.e., $\nabla \cdot \mathbf{g} = 0$. The solution to the Cauchy problem (1.1)–(1.3) for smooth functions $\mathbf{f}$ and $\mathbf{g}$ decreasing at infinity is of the form

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{u}_1(\mathbf{x}, t) + \mathbf{u}_2(\mathbf{x}, t),$$

where the vector $\mathbf{u}_1$ is the solution to the homogeneous heat equations

$$\partial_t \mathbf{u}_1 - \nu \Delta \mathbf{u}_1 = 0, \quad \mathbf{u}_1(\mathbf{x}, 0) = \mathbf{g}(\mathbf{x}) \tag{1.4}$$

and the pair $(\mathbf{u}_2, P)$ is the solution to the Cauchy problem

$$\partial_t \mathbf{u}_2 - \nu \Delta \mathbf{u}_2 + \nabla P = \mathbf{f}, \quad \nabla \cdot \mathbf{u}_2 = 0, \quad \mathbf{u}_2(\mathbf{x}, 0) = 0\,. \tag{1.5}$$

By the unique solvability of the Cauchy problem for the heat equation, from the condition $\nabla \cdot \mathbf{g} = 0$ it follows that $\nabla \cdot \mathbf{u}_1 = 0$ for all $t > 0$. The pair $(\mathbf{u}_1, \mathbf{u}_2)$ and P have the form

$$\mathbf{u}_1(\mathbf{x}, t) = (\mathscr{P}\mathbf{g})(\mathbf{x}, t) := \int\limits_{\mathbb{R}^3} \Gamma(\mathbf{x} - \mathbf{y}, t) \mathbf{g}(\mathbf{y}) d\mathbf{y},$$

$$\mathbf{u}_2(\mathbf{x}, t) = (\mathbb{T}\,\mathbf{f})(\mathbf{x}, t) := \int\limits_0^t \int\limits_{\mathbb{R}^3} \mathbf{T}(\mathbf{x} - \mathbf{y}, t - \tau) \mathbf{f}(\mathbf{y}, \tau) d\mathbf{y} d\tau,$$

$$P(\mathbf{x}, t) = \int\limits_{\mathbb{R}^3} \nabla E(\mathbf{x} - \mathbf{y}) \cdot \mathbf{f}(\mathbf{y}, t) d\mathbf{y}$$

(cf. [1, p. 93] or [2, p. 343], and also [3]), where

$$\Gamma(\mathbf{x}, t) = \frac{\mathrm{e}^{-|\mathbf{x}|^2/(4\nu t)}}{(4\pi \nu t)^{3/2}}$$

is the fundamental solution to the heat equation and

$$E(\mathbf{x}) = -\frac{1}{4\pi |\mathbf{x}|}$$

is the fundamental solution to the Laplace equation. Hence

$$\nabla E(\mathbf{x}) = \frac{1}{4\pi} \frac{\mathbf{x}}{|\mathbf{x}|^3};$$

the matrix $\mathbf{T} = \{T_{ij}\}_{i,j=1,2,3}$ is the fundamental solution to the nonstationary Stokes system (the Oseen tensor)

$$T_{ij}(\mathbf{x}, t) = \Gamma(\mathbf{x}, t)\delta_{ij} + \frac{1}{4\pi} \frac{\partial^2}{\partial x_j \partial x_j} \int\limits_{\mathbb{R}^3} \frac{\Gamma(\mathbf{x}-\mathbf{y}, t)}{|\mathbf{y}|} d\mathbf{y} = \Gamma(\mathbf{x}, t)\delta_{ij} + \frac{1}{4\pi} \frac{\partial^2}{\partial x_j \partial x_j} \frac{1}{|\mathbf{x}|} \mathrm{erf}\left(\frac{|\mathbf{x}|}{2\sqrt{\nu t}}\right)$$

where

$$\mathrm{erf}\,(r) = \frac{2}{\sqrt{\pi}} \int\limits_0^r \mathrm{e}^{-x^2} dx.$$

The Stokes equations are the linearized form of the Navier–Stokes equations, and they model the simplest incompressible flow problems (cf. [4, 5]): the convection term is neglected, hence the arising model is linear. Thus, the difficulty is the coupling of velocity $\mathbf{u}$ and pressure P. A great deal of work has been done for the numerical solution of the Stokes equations and the Navier–Stokes equations, based on the use of finite element, finite-difference, or finite-volume methods (cf., for example, [6]–[10]). Here, we propose a fast method for approximation of $\mathbf{u}_1, \mathbf{u}_2$ and P within the framework of *approximate approximations* (cf. [11]) which provides very efficient high order approximations (cf. [12]).

The outline of the paper is the following. In Section 2, we consider the approximations of the solution $\mathbf{u}_1 = (u_{11}, u_{12}, u_{13})$ to Equations (1.4). Our method, proposed in [13] for n-dimensional parabolic problems, consists in approximating the functions $\mathbf{g} = (g_1, g_2, g_3)$ via the basis functions introduced by approximate approximations, which are product of Gaussians and special polynomials. The action of the Poisson integral $\mathscr{P}$ applied to the basis functions admits a separated representation (also a tensor product representation), i.e., it is represented as a product of functions depending on one of the variables. Then a separated representation of $\mathbf{g}$ provides a separated representation of the potential, and the resulting approximation formulas are very fast because only one-dimensional operations are used.

In Section 3, we consider approximation of the pressure P given in (1.6) and its gradient ∇P. By (1.2), for fixed $t > 0$ the divergence operator applied to the system (1.1) gives

$$-\Delta P = F$$

with

$$F(\mathbf{x}, t) = -\nabla \cdot \mathbf{f}(\mathbf{x}, t) = -\sum_{j=1}^{3} \partial_{x_j} f_j(\mathbf{x}, t).$$

The solution $P(\mathbf{x}, t)$ with $P(\mathbf{x}, t) \to 0$ as $|\mathbf{x}| \to \infty$ is given by

$$P(\mathbf{x}, t) = \mathscr{L}F(\mathbf{x}, t) = -\frac{1}{4\pi} \int\limits_{\mathbb{R}^3} \frac{F(\mathbf{y}, t)}{|\mathbf{x}-\mathbf{y}|} d\mathbf{y} = \frac{1}{4\pi} \sum_{j=1}^{3} \int\limits_{\mathbb{R}^3} \frac{\partial_{y_j} f_j(\mathbf{y}, t)}{|\mathbf{x}-\mathbf{y}|} d\mathbf{y}, \tag{1.6}$$

where $\mathscr{L}$ denotes the harmonic potential. Fast semianalytic cubature formulas for computing $\mathscr{L}F$ were constructed in [14]. These formulas are based on approximation of the density $F(\cdot, t)$

by functions with analytically known harmonic potentials. The gradient $\nabla P = \nabla \mathscr{L} F$ can be approximated by the gradient of the cubature of $\mathscr{L} F$, which is also a semianalytic formula. If $F(\cdot, t)$ admits a separated representation, then we derive the tensor product representations of P and ∇P, which admits efficient one-dimensional operations.

In Section 4, we describe an approximation of $\mathbf{u}_2 = (u_{21}, u_{22}, u_{23})$ satisfying the nonhomogeneous heat equations

$$\frac{\partial \mathbf{u}_2}{\partial t} - \nu \Delta \mathbf{u}_2 = \Phi, \quad \mathbf{u}_2(\mathbf{x}, 0) = 0, \quad \mathbf{x} \in \mathbb{R}^3, \quad t > 0, \tag{1.7}$$

where $\Phi = \mathbf{f} - \nabla P$ and $\nabla P = \nabla \mathscr{L} F$ is computed in Section 3. Since $\nabla \cdot \Phi = 0$, the solution to Equations (1.7) satisfies the condition $\nabla \cdot \mathbf{u}_2 = 0$. The solution to Equations (1.7) admits the representation

$$\mathbf{u}_2(\mathbf{x}, t) = \mathscr{H} \Phi(\mathbf{x}, t) = (\mathscr{H} \varphi_1, \mathscr{H} \varphi_2, \mathscr{H} \varphi_3), \quad \Phi = (\varphi_1, \varphi_2, \varphi_3)$$

with

$$\begin{aligned} \mathscr{H} \varphi_j(\mathbf{x}, t) &= \int_0^t \frac{ds}{(4\pi \nu s)^{3/2}} \int_{\mathbb{R}^3} \mathrm{e}^{-|\mathbf{x} - \mathbf{y}|^2/(4\nu s)} \varphi_j(\mathbf{y}, t - s) d\mathbf{y} \\ &= \int_0^t (\mathscr{P} \varphi_j(\cdot, s))(\mathbf{x}, t - s) ds, \quad j = 1, 2, 3 \end{aligned} \tag{1.8}$$

(cf. [15]). The cubature formula of (1.8) proposed in [13] is based on replacing the density φ_j by approximate quasiinterpolants on rectangular grids. The action of $\mathscr{H}$ on the basis functions yields one-dimensional integral representations with separated integrands. This construction, combined with an accurate quadrature rule as suggested in [16] and a separated representation of the density φ_j provides a separated representation of the integral operator (1.8).

We derive fast and accurate formulas for the approximation of $(\mathbf{u}_1, \mathbf{u}_2, P)$ of an arbitrary high order. In Section 5, the accuracy of the method and convergence orders 2,4, 6, 8 are confirmed by numerical experiments.

2 Approximation of $\mathbf{u}_1$

In this section, we consider approximation of $\mathbf{u}_1 = (u_{11}, u_{12}, u_{13})$. Each component u_{1j}, $j = 1, 2, 3$, is a solution to the homogeneous heat equation

$$\frac{\partial u_{1j}}{\partial t} - \nu \Delta u_j = 0, \quad u_{1j}(\mathbf{x}, 0) = g_j(\mathbf{x}) \quad \mathbf{x} \in \mathbb{R}^3, \quad t > 0.$$

The solution is given by

$$u_{1j}(\mathbf{x}, t) = \mathscr{P} g_j(\mathbf{x}, t),$$

where $\mathscr{P} g_j$ denotes the Poisson integral

$$\mathscr{P} g_j(\mathbf{x}, t) = \int_{\mathbb{R}^3} \Gamma(\mathbf{x} - \mathbf{y}, t) \, g_j(\mathbf{y}) d\mathbf{y} = \frac{1}{(4\nu\pi \, t)^{3/2}} \int_{\mathbb{R}^3} \mathrm{e}^{-|\mathbf{x} - \mathbf{y}|^2/(4\nu t)} \, g_j(\mathbf{y}) d\mathbf{y} \, .$$

We approximate the functions g_j, $j = 1, 2, 3$, by the approximate quasiinterpolants

$$\mathscr{N}_h g_j(\mathbf{x}) = \frac{1}{\mathscr{D}^{3/2}} \sum_{\mathbf{m}\in\mathbb{Z}^3} g_j(h\mathbf{m})\, \widetilde{\eta}_{2M}\left(\frac{\mathbf{x}-h\mathbf{m}}{h\sqrt{\mathscr{D}}}\right) \tag{2.1}$$

with the basis functions

$$\widetilde{\eta}_{2M}(\mathbf{x}) = \prod_{j=1}^{3} \eta_{2M}(x_j), \quad \eta_{2M}(x) = \frac{(-1)^{M-1}}{2^{2M-1}\sqrt{\pi}(M-1)!} \frac{H_{2M-1}(x)\mathrm{e}^{-x^2}}{x}, \tag{2.2}$$

where H_k are the Hermite polynomials

$$H_k(x) = (-1)^k \mathrm{e}^{x^2} \left(\frac{d}{dx}\right)^k \mathrm{e}^{-x^2}.$$

Here, $\mathscr{D}$ is a positive fixed parameter and h is the mesh size. The function $\widetilde{\eta}_{2M}$ satisfies the moment conditions of order $2M$

$$\int_{\mathbb{R}^3} \mathbf{x}^\alpha \widetilde{\eta}_{2M}(\mathbf{x}) d\mathbf{x} = \delta_{0,\alpha}, \qquad 0 \leqslant |\alpha| < N,$$

and the quasiinterpolant (2.1) provides an approximation of g_j with the general asymptotic error $\mathscr{O}(h^{2M} + \varepsilon)$ (cf. [11]). The saturation error ε does not converge to zero as $h \to 0$; however, it can be made arbitrary small if the parameter $\mathscr{D}$ is sufficiently large. Then the sum

$$(\mathscr{P}_M g_j)(\mathbf{x}, t) := (\mathscr{P}\mathscr{N}_h g_j)(\mathbf{x}, t) = \frac{1}{\mathscr{D}^{3/2}} \sum_{\mathbf{m}\in\mathbb{Z}^3} g_j(h\mathbf{m})\, \mathscr{P}\widetilde{\eta}_{2M}\left(\frac{\mathbf{x}-h\mathbf{m}}{h\sqrt{\mathscr{D}}}, \frac{t}{h^2\mathscr{D}}\right)$$

provides an approximation of $\mathscr{P} g_j(\mathbf{x}, t)$ with the error $\mathscr{O}((h\sqrt{\mathscr{D}})^{2M} + \varepsilon)$. Since the Poisson integral is a smoothing operator, one can prove that the saturation error also tends to 0 as $h \to 0$ and the approximate solution $\mathscr{P}_M g_j$ converges for any fixed $t > 0$ with order $\mathscr{O}(h^{2M})$ to $\mathscr{P} g_j$ (cf. [11, Theorem 6.1]).

Theorem 2.1 ([13, Theorem 3.1])**.** *Let $M \geqslant 1$. The Poisson integral applied to the generating function $\widetilde{\eta}_{2M}$ in (2.2) can be written as*

$$\mathscr{P}\widetilde{\eta}_{2M}(\mathbf{x}, t) = \frac{\mathrm{e}^{-|\mathbf{x}|^2/(1+4\nu t)}}{\pi^{3/2}} \prod_{j=1}^{3} \mathscr{Q}_M(x_j, 4\nu t),$$

where $\mathscr{Q}_M(x, r)$ is a polynomial in x of degree $2M-2$ of the form

$$\mathscr{Q}_M(x, r) = \sum_{k=0}^{M-1} \frac{1}{(1+r)^{k+1/2}} \frac{(-1)^k}{4^k k!} H_{2k}\Big(\frac{x}{\sqrt{1+r}}\Big). \tag{2.3}$$

By Theorem 2.1, the sum

$$(\mathscr{P}_M g_j)(\mathbf{x}, t) = \frac{1}{(\pi\mathscr{D})^{3/2}} \sum_{\mathbf{m}\in\mathbb{Z}^3} g_j(h\mathbf{m}) \mathrm{e}^{-|\mathbf{x}-h\mathbf{m}|^2/(h^2\mathscr{D}+4\nu t)} \prod_{i=1}^{3} \mathscr{Q}_M\Big(\frac{x_i - hm_i}{h\sqrt{\mathscr{D}}}, 4\nu\frac{t}{h^2\mathscr{D}}\Big) \tag{2.4}$$

is a semianalytic cubature formula for $\mathscr{P}g_j$. For example, for $M=1$

$$\mathscr{Q}_1(x,r) = \frac{1}{\sqrt{1+r}}$$

and then

$$(\mathscr{P}_2 g_j)(\mathbf{x},t) = \frac{h^3}{\pi^{3/2}} \sum_{\mathbf{m}\in\mathbb{Z}^3} g_j(h\mathbf{m}) \frac{\mathrm{e}^{-|\mathbf{x}-h\mathbf{m}|^2/(h^2\mathscr{D}+4\nu t)}}{(h^2\mathscr{D}+4\nu t)^{3/2}}.$$

The computation of (2.4) is very efficient if the functions $g_j(\mathbf{x}), j=1,2,3$, admit a separated representation, i.e., within a prescribed accuracy ε, they can be represented as the sum of products of univariate functions

$$g_j(\mathbf{x}) = \sum_{s=1}^{S} \prod_{i=1}^{3} g_{j,i}^{(s)}(x_i) + \mathscr{O}(\varepsilon)$$

with suitable functions $g_{j,i}^{(s)}$. Then, at a point of a uniform grid $\{h\mathbf{k}\}$,

$$(\mathscr{P}g_j)(h\mathbf{k},t) \approx \sum_{s=1}^{S} \prod_{i=1}^{3} S_{j,s}^{(i)}\Big(k_i, \frac{4\nu t}{h^2\mathscr{D}}\Big), \tag{2.5}$$

where $S_{j,s}^{(i)}$ denotes the one-dimensional convolution

$$S_{j,s}^{(i)}(k,t) = (\pi\mathscr{D})^{-1/2} \sum_{m\in\mathbb{Z}} g_{j,i}^{(s)}(hm)\mathrm{e}^{-\frac{(k-m)^2}{\mathscr{D}(1+t)}} \mathscr{Q}_M\Big(\frac{k-m}{\sqrt{\mathscr{D}}}, t\Big).$$

3 Approximation of $P(\mathbf{x},t)$

For fixed $t>0$ we consider approximation of $P(\cdot,t) = \mathscr{L}F(\cdot,t)$ given in (1.6). We approximate F by the quasiinterpolant

$$\mathscr{M}_h F(\mathbf{x},t) = \mathscr{D}^{-3/2} \sum_{\mathbf{m}\in\mathbb{Z}^3} F(h\mathbf{m},t)\widetilde{\eta}_{2M}\left(\frac{\mathbf{x}-h\mathbf{m}}{h\sqrt{\mathscr{D}}}\right) \tag{3.1}$$

with the basis functions (2.2). The quasiinterpolant (3.1) provides an approximation of F with the general asymptotic error $\mathscr{O}(h^{2M}+\varepsilon)$ (cf. [11, Theorem 4.2]). Then the sum

$$\mathscr{L}_M F(\mathbf{x},t) := \mathscr{L}\mathscr{M}_h F(\mathbf{x},t) = \frac{h^2}{\mathscr{D}^{1/2}} \sum_{\mathbf{m}\in\mathbb{Z}^3} F(h\mathbf{m},t)(\mathscr{L}\widetilde{\eta}_{2M})\Big(\frac{\mathbf{x}-h\mathbf{m}}{h\sqrt{\mathscr{D}}}\Big)$$

$$= -\frac{h^2}{\mathscr{D}^{1/2}} \sum_{i=1}^{3} \sum_{\mathbf{m}\in\mathbb{Z}^3} \partial_{x_i} f_i(h\mathbf{m},t)(\mathscr{L}\widetilde{\eta}_{2M})\Big(\frac{\mathbf{x}-h\mathbf{m}}{h\sqrt{\mathscr{D}}}\Big)$$

is a semianalytic cubature formula for $P=\mathscr{L}F$. Moreover, by the smoothing properties of the harmonic potential, the corresponding small saturation error converges with the rate h^2 as $h\to 0$. Hence for sufficiently smooth functions $\mathscr{L}_M F$ approximates $\mathscr{L}F$ with the error

$\mathscr{O}(h^{2M} + h^2 \mathrm{e}^{-\mathscr{D}\pi^2})$ (cf. [11, Theorem 4.10]). Therefore, in numerical computations with $\mathscr{D} \geqslant 3$, $\mathscr{L}_M F(\mathbf{x}, t)$ behaves like a high order cubature formula. The function

$$v = \mathscr{L}\Big(\prod_{j=1}^{3} \eta_{2M} \Big)$$

is a solution to the problem

$$-\Delta\, v = \prod_{j=1}^{3} \eta_{2M}(x_j)\,, \ \mathbf{x} \in \mathbb{R}^3, \quad |v(\mathbf{x})| \to 0 \text{ as } |\mathbf{x}| \to \infty\,. \tag{3.2}$$

Theorem 3.1 ([17, Theorem 1])**.** *The solution to the problem* (3.2) *can be expressed by the one-dimensional integral*

$$v(\mathbf{x}) = \frac{1}{4\pi^{3/2}} \int_0^\infty \mathrm{e}^{-|\mathbf{x}|^2/(1+r)} \prod_{j=1}^{3} \mathscr{Q}_M(x_j, r) dr\,, \tag{3.3}$$

where $\mathscr{Q}_M(x, r)$ *is defined in* (2.3)*.*

For example, for $M = 1$

$$\mathscr{Q}_1(x, r) = \frac{1}{\sqrt{1+r}}, \quad \mathscr{L}(\mathrm{e}^{-|\cdot|^2})(\mathbf{x}) = \frac{1}{4\pi^{3/2}} \int_0^\infty \frac{\mathrm{e}^{-|\mathbf{x}|^2/(1+r)}}{(1+r)^{3/2}} dr$$

and

$$P(\mathbf{x}, t) \approx \mathscr{L}_M F(\mathbf{x}, t) = \frac{h^2 \mathscr{D}}{4(\pi\mathscr{D})^{3/2}} \sum_{\mathbf{m}\in\mathbb{Z}^3} F(h\mathbf{m}, t) \int_0^\infty \frac{\mathrm{e}^{-|\mathbf{x} - h\mathbf{m}|^2/(h^2\mathscr{D}(1+r))}}{(1+r)^{3/2}} dr$$

provides a second order approximation formula.

The gradient $\nabla P = \nabla \mathscr{L} F$ of the harmonic potential can be approximated by the gradient of the cubature of $\mathscr{L} F$, which is also a semianalytic formula. We have

$$\nabla P(\mathbf{x}, t) \approx \nabla(\mathscr{L}_M F)(\mathbf{x}, t) = \frac{h}{\mathscr{D}} \sum_{\mathbf{m}\in\mathbb{Z}^3} F(h\mathbf{m}, t) \bigg(\nabla \mathscr{L}\Big(\prod_{j=1}^{3} \eta_{2M}(x_j) \Big) \bigg) \Big(\frac{\mathbf{x} - h\mathbf{m}}{h\sqrt{\mathscr{D}}} \Big)\,.$$

Moreover, $\nabla(\mathscr{L}_M F)$ approximates $\nabla(\mathscr{L} F)$ with the error $\mathscr{O}(h^{2M} + h\mathrm{e}^{-\pi^2\mathscr{D}})$ (cf. [11, Theorem 4.11].) The function

$$v = \partial_{x_i} \mathscr{L}\Big(\prod_{j=1}^{3} \eta_{2M} \Big), \quad i = 1, 2, 3$$

is a solution to the problem

$$-\Delta\, v = \partial_{x_i} \prod_{j=1}^{3} \eta_{2M}(x_j)\,, \ \mathbf{x} \in \mathbb{R}^3, \quad |v(\mathbf{x})| \to 0 \text{ as } |\mathbf{x}| \to \infty. \tag{3.4}$$

Theorem 3.2. *The solution to the problem* (3.4) *can be expressed by the one-dimensional integral*

$$\partial_{x_i}\mathscr{L}\Big(\prod_{j=1}^{3}\eta_{2M}\Big)(\mathbf{x}) = -\frac{x_i}{2\pi^{3/2}}\int_0^\infty \frac{\mathrm{e}^{-|\mathbf{x}|^2/(1+r)}}{1+r}\prod_{j=1}^{3}\mathscr{Q}_M(x_j,r)dr$$

$$+\frac{1}{4\pi^{3/2}}\int_0^\infty \mathrm{e}^{-|\mathbf{x}|^2/(1+r)}\mathscr{R}_M(x_i,r)\prod_{\substack{j=1\\ j\neq i}}^{3}\mathscr{Q}_M(x_j,r)dr\,, \tag{3.5}$$

where $\mathscr{Q}_M(x,r)$ *is defined in* (2.3), $\mathscr{R}_1(x,r)=0$, *and*

$$\mathscr{R}_M(x,r)=\partial_x\mathscr{Q}_M(x,r)=\sum_{k=1}^{M-1}\frac{1}{(1+r)^{k+1}}\frac{(-1)^k}{4^{k-1}(k-1)!}H_{2k-1}\Big(\frac{x}{\sqrt{1+r}}\Big), M>1.$$

Proof. Formula (3.5) can be obtained by direct differentiation of (3.3) keeping in mind the identity $H'_{2k}(x)=4kH_{2k-1}(x)$ (cf. [18, formula (14), p. 193]). □

For example, for $M=1$

$$\mathscr{R}_1(x,r)=\partial_x\mathscr{Q}_1(x,r)=0\,,\quad \nabla\mathscr{L}(\mathrm{e}^{-|\cdot|^2})(\mathbf{x})=-\frac{\mathbf{x}}{2\pi^{3/2}}\int_0^\infty\frac{\mathrm{e}^{-|\mathbf{x}|^2/(1+r)}}{(1+r)^{5/2}}dr$$

and

$$\nabla P(\mathbf{x},t)\approx\nabla\mathscr{L}_M F(\mathbf{x},t)=-\frac{h\sqrt{\mathscr{D}}}{2(\pi\mathscr{D})^{3/2}}\sum_{\mathbf{m}\in\mathbb{Z}^3}F(h\mathbf{m},t)\frac{\mathbf{x}-h\mathbf{m}}{h\sqrt{\mathscr{D}}}\int_0^\infty\frac{\mathrm{e}^{-|\mathbf{x}-h\mathbf{m}|^2/(h^2\mathscr{D}(1+r)}}{(1+r)^{5/2}}dr.$$

The quadrature of the integrals (3.3) and (3.5) with certain quadrature weights ω_p and nodes r_p gives the separated representations

$$P(\mathbf{x},t)\approx\frac{h^2\mathscr{D}}{4(\pi\mathscr{D})^{3/2}}\sum_{\mathbf{m}\in\mathbb{Z}^3}F(h\mathbf{m},t)\sum_p\omega_p\mathrm{e}^{-|\mathbf{x}-h\mathbf{m}|^2/(h^2\mathscr{D}(1+r_p))}\prod_{j=1}^{3}\mathscr{Q}_M\Big(\frac{x_j-hm_j}{h\sqrt{\mathscr{D}}},r_p\Big),\tag{3.6}$$

$$\partial_{x_i}P(\mathbf{x},t)\approx-\frac{h\sqrt{\mathscr{D}}}{2(\pi\mathscr{D})^{3/2}}\sum_{\mathbf{m}\in\mathbb{Z}^3}F(h\mathbf{m},t)$$

$$\times\sum_p\omega_p\Bigg(\frac{x_i-hm_i}{h\sqrt{\mathscr{D}}}\mathrm{e}^{-|\mathbf{x}-h\mathbf{m}|^2/(h^2\mathscr{D}(1+r_p))}\prod_{j=1}^{3}\mathscr{Q}_M\Big(\frac{x_j-hm_j}{h\sqrt{\mathscr{D}}},r_p\Big)$$

$$+\mathrm{e}^{-|\mathbf{x}-h\mathbf{m}|^2/(h^2\mathscr{D}(1+r_p))}\mathscr{R}_M\Big(\frac{x_i-hm_i}{h\sqrt{\mathscr{D}}},r_p\Big)\prod_{j=1,\,j\neq i}^{3}\mathscr{Q}_M\Big(\frac{x_j-hm_j}{h\sqrt{\mathscr{D}}},r_p\Big)\Bigg).\tag{3.7}$$

The approximation formulas (3.6) and (3.7) are very efficient if $F(\mathbf{x},t)=-\nabla\cdot\mathbf{f}(\mathbf{x},t)$ has separated representation, i.e., for a given accuracy ε it can be represented as the sum of products of vectors in dimension 1

$$F(\mathbf{x},t)=\sum_{s=1}^{S}a_s\prod_{j=1}^{3}F_j^{(s)}(x_j,t)+\mathscr{O}(\varepsilon).$$

Then an approximate value of $P(h\mathbf{k},t)$ and $\nabla P(h\mathbf{k},t)$ at a point of a uniform grid can be computed by the sum of products of one-dimensional convolutions

$$P(h\mathbf{k},t) \approx \frac{h^2\mathscr{D}}{4(\pi\mathscr{D})^{3/2}} \sum_{s=1}^{S} a_s \sum_p \omega_p \prod_{j=1}^{3} \Sigma_j^{(s)}(k_j, r_p, t)\,, \tag{3.8}$$

$$\partial_{x_i} P(h\mathbf{k},t) \approx \frac{h\sqrt{\mathscr{D}}}{4(\pi\mathscr{D})^{3/2}} \sum_{s=1}^{S} a_s \sum_p \omega_p \Bigg(T_i^{(s)}(k_i, r_p, t) \prod_{\substack{j=1\\ j\neq i}}^{3} \Sigma_j^{(s)}(k_j, r_p, t)$$

$$+ R_i^{(s)}(k_i, r_p, t) \prod_{\substack{j=1\\ j\neq i}}^{3} \Sigma_j^{(s)}(k_j, r_p, t)\Bigg), \quad i = 1,2,3 \tag{3.9}$$

with the one-dimensional convolutions

$$\Sigma_j^{(s)}(k,r,t) = \sum_{m\in\mathbb{Z}} F_j^{(s)}(hm,t)\mathscr{Q}_M\Big(\frac{k-m}{\sqrt{\mathscr{D}}}, r\Big) \mathrm{e}^{-(k-m)^2/(\mathscr{D}(1+r))},$$

$$T_j^{(s)}(k,r,t) = -2\sum_{m\in\mathbb{Z}} F_j^{(s)}(hm,t)\frac{k-m}{\sqrt{\mathscr{D}}}\mathscr{Q}_M\Big(\frac{k-m}{\sqrt{\mathscr{D}}}, r\Big) \mathrm{e}^{-(k-m)^2/(\mathscr{D}(1+r))},$$

$$R_j^{(s)}(k,r,t) = \sum_{m\in\mathbb{Z}} F_j^{(s)}(hm,t)\mathscr{R}_M\Big(\frac{k-m}{\sqrt{\mathscr{D}}}, r\Big) \mathrm{e}^{-(k-m)^2/(\mathscr{D}(1+r))}.$$

4 Approximation of $\mathbf{u}_2$

In this section, we consider approximation of the solution $\mathbf{u}_2 = (u_{21}, u_{22}, u_{23})$ to Equations (1.7). Each component u_{2j}, $j = 1,2,3$, satisfies the nonhomogeneous heat equation

$$\frac{\partial u_{2j}}{\partial t} - \nu \Delta u_{2j} = \varphi_j, \quad u_{2j}(\mathbf{x},0) = 0, \quad \mathbf{x}\in\mathbb{R}^3, \quad t>0\,, \tag{4.1}$$

where $\varphi_j = f_j - \partial_{x_j} P$. Hence

$$u_{2j}(\mathbf{x},t) = \mathscr{H}\varphi_j(\mathbf{x},t),$$

where

$$\mathscr{H}\varphi_j(\mathbf{x},t) = \int_0^t (\mathscr{P}\varphi_j(\cdot,s))(\mathbf{x},t-s)ds.$$

We replace the density φ_j by the quasiinterpolant on the rectangular grid $\{(h\mathbf{m},\tau i)\}$

$$\mathscr{M}_{h,\tau}\varphi_j(\mathbf{x},t) = \frac{1}{\mathscr{D}_0^{1/2}\mathscr{D}^{3/2}} \sum_{\substack{i\in\mathbb{Z}\\ \mathbf{m}\in\mathbb{Z}^3}} \varphi_j(h\mathbf{m},\tau i)\,\eta_{2M}\left(\frac{t-\tau i}{\tau\sqrt{\mathscr{D}_0}}\right)\widetilde{\eta}_{2M}\left(\frac{\mathbf{x}-h\mathbf{m}}{h\sqrt{\mathscr{D}}}\right).$$

Here, τ and h are steps, $\mathscr{D}_0$ and $\mathscr{D}$ are positive fixed parameters, $\widetilde{\eta}_{2M}$ and η_{2M} are the generating functions given in (2.2). The sum $\mathscr{M}_{h,\tau}\varphi_j$ approximates φ_j with the error

$$|\varphi_j(\mathbf{x},t) - \mathscr{M}_{h,\tau}\varphi_j(\mathbf{x},t)| = \mathscr{O}((h\sqrt{\mathscr{D}})^{2M} + (\tau\sqrt{\mathscr{D}_0})^{2M}) + \varepsilon$$

for all $(\mathbf{x}, t) \in \mathbb{R}^3 \times [0, T], T > 0$ and the sum

$$\mathscr{H}\mathscr{M}_{h,\tau}\varphi_j(\mathbf{x}, t) = \frac{1}{\mathscr{D}_0^{1/2}\mathscr{D}^{3/2}} \sum_{i\in\mathbb{Z},\, \mathbf{m}\in\mathbb{Z}^3} \varphi_j(h\mathbf{m}, \tau i) \int_0^t \eta_{2M}\Big(\frac{s-\tau i}{\tau\sqrt{\mathscr{D}_0}}\Big)\, \mathscr{P}\widetilde{\eta}_{2M}\Big(\frac{\mathbf{x}-h\mathbf{m}}{h\sqrt{\mathscr{D}}}, \frac{t-s}{h^2\mathscr{D}}\Big) ds$$

approximates $\mathscr{H}\varphi_j$ in $\mathbb{R}^3 \times [0, T]$ with the error estimate

$$|\mathscr{H}\mathscr{M}_{h,\tau}\varphi_j(\mathbf{x}, t) - \mathscr{H}\varphi_j(\mathbf{x}, t)| = \mathscr{O}((h\sqrt{\mathscr{D}})^{2M} + (\tau\sqrt{\mathscr{D}_0})^{2M}) + \varepsilon$$

(cf. [17, Theorem. 2.1]). Here,

$$\varphi_j(h\mathbf{m}, \tau i) = f_j(h\mathbf{m}, \tau i) - \partial_{x_j} P(h\mathbf{m}, \tau i)$$

and $\partial_{x_j} P(h\mathbf{m}, \tau i)$ is given in (3.9).

At a point of a rectangular grid $\{(h\mathbf{k}, \tau\ell)\}$, we get

$$\mathscr{H}\mathscr{M}_{h,\tau}\varphi_j(h\mathbf{k}, \tau\ell) = \frac{1}{\mathscr{D}_0^{1/2}\mathscr{D}^{3/2}} \sum_{i\in\mathbb{Z},\, \mathbf{m}\in\mathbb{Z}^3} \varphi_j(h\mathbf{m}, \tau i) K_M(h\mathbf{k}, h\mathbf{m}, \tau\ell, \tau i), \tag{4.2}$$

where

$$K_M(h\mathbf{k}, h\mathbf{m}, \tau\ell, \tau i) = \int_0^{\tau\ell} \eta_{2M}\Big(\frac{\sigma-\tau i}{\tau\sqrt{\mathscr{D}_0}}\Big)\, \mathscr{P}\widetilde{\eta}_{2M}\Big(\frac{\mathbf{k}-\mathbf{m}}{\sqrt{\mathscr{D}}}, \frac{\tau\ell-\sigma}{h^2\mathscr{D}}\Big) d\sigma.$$

By Theorem 2.1,

$$K_M(h\mathbf{k}, h\mathbf{m}, \tau\ell, \tau i)$$
$$= \pi^{-3/2} \int_0^{\tau\ell} \eta_{2M}\Big(\frac{\sigma-\tau i}{\tau\sqrt{\mathscr{D}_0}}\Big) \prod_{j=1}^{3} \mathrm{e}^{-(k_j-m_j)^2/(\mathscr{D}(1+4\nu\frac{\tau\ell-\sigma}{h^2\mathscr{D}}))}\, \mathscr{Q}_M\Big(\frac{k_j-m_j}{\sqrt{\mathscr{D}}}, 4\nu\frac{\tau\ell-\sigma}{h^2\mathscr{D}}\Big) d\sigma$$

and the integrals cannot be taken analytically. The computation of the sum (4.2) involves additionally an integration which must be approximated by an efficient quadrature rule with certain quadrature weights ω_p and nodes r_p

$$K_M(h\mathbf{k}, h\mathbf{m}, \tau\ell, \tau i)$$
$$\approx \pi^{-3/2} \sum_p \omega_p\, \eta_{2M}\Big(\frac{\tau_p-\tau i}{\tau\sqrt{\mathscr{D}_0}}\Big) \prod_{j=1}^{3} \mathrm{e}^{-(k_j-m_j)^2/(\mathscr{D}(1+4\nu\frac{\tau\ell-\tau_p}{h^2\mathscr{D}}))}\, \mathscr{Q}_M\Big(\frac{k_j-m_j}{\sqrt{\mathscr{D}}}, 4\nu\frac{\tau\ell-\tau_p}{h^2\mathscr{D}}\Big).$$

If $\Phi = (\varphi_1, \varphi_2, \varphi_3)$ admits a separate representation, then the sum (4.2) gives an efficiently computable high order approximation of the initial value problem (4.1) based on the computation of one-dimensional sums.

5 Implementation and Numerical Experiments

5.1. Homogeneous heat equation. We assume that $\mathbf{f} = \mathbf{0}$ and $\mathbf{g}(\mathbf{x}) = \nabla \times (0, 0, \mathrm{e}^{-|\mathbf{x}|^2}) = (-2x_2\mathrm{e}^{-|\mathbf{x}|^2}, 2x_1\mathrm{e}^{-|\mathbf{x}|^2}, 0)$. Then $\nabla \cdot \mathbf{g}(\mathbf{x}) = 0$ and the solution to the problem (1.4), (1.5) is provided by $\mathbf{u}_2 \equiv 0$, $P \equiv 0$, and $\mathbf{u} = \mathbf{u}_1 = (u_{11}, u_{12}, u_{13})$ with

$$u_{11}(\mathbf{x}, t) = (\mathscr{P} g_1)(\mathbf{x}, t) = \frac{-2}{(4\pi\nu t)^{3/2}} \int\limits_{\mathbb{R}^3} y_2 \mathrm{e}^{-|\mathbf{y}-\mathbf{x}|^2/(4\nu t)} \mathrm{e}^{-|\mathbf{y}|^2} d\mathbf{y} = -2x_2 \frac{\mathrm{e}^{-|\mathbf{x}|^2/(1+4\nu t)}}{(4\nu t+1)^{5/2}},$$

$$u_{12}(\mathbf{x}, t) = (\mathscr{P} g_2)(\mathbf{x}, t) = \frac{2}{(4\pi\nu t)^{3/2}} \int\limits_{\mathbb{R}^3} y_1 \mathrm{e}^{-|\mathbf{y}-\mathbf{x}|^2/(4\nu t)} \mathrm{e}^{-|\mathbf{y}|^2} d\mathbf{y} = 2x_1 \frac{\mathrm{e}^{-|\mathbf{x}|^2/(1+4\nu t)}}{(4\nu t+1)^{5/2}},$$

$$u_{13}(\mathbf{x}, t) = (\mathscr{P} g_3)(\mathbf{x}, t) = 0 .$$

We provide results of some experiments which show the accuracy and numerical convergence order of the approximation formulas (2.5). In Tables 1 and 2, we compare the exact solution $\mathscr{P} g_1$ with the coefficient of kinematic viscosity ν equals to 1 and the approximate solution in (2.5) at one fixed point, for $M = 1, 2, 3, 4$ and different values of h. We choose $\mathscr{D} = 4$ to have the saturation error comparable with the double precision rounding errors. Numerical experiments show that the predicted convergence order is obtained and for $M = 4$ and small h the saturation error is reached.

h^{-1}	$M=1$ error	rate	$M=2$ error	rate	$M=3$ error	rate	$M=4$ error	rate
10	0.235E-03		0.133E-05		0.742E-08		0.367E-10	
20	0.591E-04	1.99	0.841E-07	3.98	0.117E-09	5.98	0.145E-12	7.98
40	0.148E-04	1.99	0.841E-09	3.99	0.184E-11	5.99	0.586E-15	7.94
80	0.370E-05	1.99	0.329E-09	3.99	0.288E-13	5.99	0.694E-17	
160	0.925E-06	1.99	0.206E-10	3.99	0.545E-15	5.72	0.902E-16	

TABLE 1. Absolute error and rate of convergence for $\mathscr{P} g_1$ at $\mathbf{x} = (1.2, 1.2, 1.2)$, $t = 1$ by using $\mathscr{P}_M g_1$.

h^{-1}	$M=1$ error	rate	$M=2$ error	rate	$M=3$ error	rate	$M=4$ error	rate
10	0.540E-03		0.321E-05		0.175E-07		0.809E-10	
20	0.136E-03	1.98	0.202E-06	3.98	0.276E-09	5.98	0.319E-12	7.98
40	0.341E-04	1.99	0.127E-07	3.99	0.432E-11	5.99	0.128E-14	7.96
80	0.852E-05	1.99	0.791E-09	3.99	0.676E-13	5.99	0.763E-16	
160	0.213E-05	1.99	0.494E-10	3.99	0.117E-14	5.85	0.125E-15	

TABLE 2. Absolute error and rate of convergence for $\mathscr{P} g_1$ at $\mathbf{x} = (0, 1.6, 0)$, $t = 1$ by using $\mathscr{P}_M g_1$.

5.2. Numerical results for approximation of P and ∇P. We assume that

$$\mathbf{f}(\mathbf{x}, t) = 2t\,\mathbf{x}\mathrm{e}^{-|\mathbf{x}|^2} = (2t\,x_1 \mathrm{e}^{-|\mathbf{x}|^2}, 2t\,x_2 \mathrm{e}^{-|\mathbf{x}|^2}, 2t\,x_3 \mathrm{e}^{-|\mathbf{x}|^2}),$$

$$\mathbf{g}(\mathbf{x}) = (0, 0, 0).$$

Then

$$\nabla \cdot \mathbf{f} = t(6 - 4|\mathbf{x}|^2)\mathrm{e}^{-|\mathbf{x}|^2}.$$

The unique solution to the problem (1.1), (1.2), (1.3) is

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{0}, \quad P(\mathbf{x}, t) = -t\,\mathrm{e}^{-|\mathbf{x}|^2}.$$

We report on the absolute errors and the approximation rates for the harmonic potential $P = \mathscr{L}F$ (cf. Tables 3 and 4) and $\partial_{x_2} P = 2tx_2\mathrm{e}^{-|\mathbf{x}|^2}$ (cf. Tables 5 and 6) at a fixed point. The approximate values are computed by the cubature formulas given in (3.8) having the approximation order $\mathscr{O}(h^{2M} + h^2\mathrm{e}^{-\mathscr{D}\pi^2})$ and in (3.9) having the approximation order $\mathscr{O}(h^{2M} + h\mathrm{e}^{-\mathscr{D}\pi^2})$ respectively, for $M = 1, 2, 3, 4$. We used the uniform grids size $h = 0.1 \cdot 2^{1-k}$, $k = 1, \ldots, 5$, and choose the parameter $\mathscr{D} = 5$. By [16], the one-dimensional integrals in (3.3) and (3.5) are transformed to integrals over $\mathbb{R}$ with integrands decaying doubly exponentially by making the substitutions

$$t = \mathrm{e}^{\xi}, \quad \xi = a(\tau + \mathrm{e}^{\tau}), \quad \tau = b(u - \mathrm{e}^{-u})$$

with certain positive constants a and b. The computation is based on the classical trapezoidal rule with step size $\varkappa$ exponentially converging for rapidly decaying smooth functions on the real line. In our computations, we assumed $a = 5$, $b = 6$, $\varkappa = 0.0009$ and $8 \cdot 10^2$ points in the quadrature formula in order to reach the saturation error with the approximation formula of order $N = 8$.

The numerical results show that higher order cubature formulas gives an essentially better approximation than the second order formulas, and the predicted convergence order is reached.

	$M=1$		$M=2$		$M=3$		$M=4$	
h^{-1}	error	rate	error	rate	error	rate	error	rate
10	0.188D-02		0.718D-04		0.731D-05		0.960D-08	
20	0.470D-03	2.00	0.462D-05	3.95	0.143D-06	5.68	0.783D-10	6.93
40	0.117D-03	2.00	0.291D-06	3.99	0.236D-08	5.92	0.352D-12	7.79
80	0.293D-04	2.00	0.182D-07	3.99	0.373D-10	5.98	0.116D-14	8.24
160	0.733D-05	2.00	0.114D-08	3.99	0.585D-12	5.99	0.218D-13	

Table 3. Absolute error and rate of convergence for $P(\mathbf{x}, t)$ at $\mathbf{x} = (1.2, 1.2, 1.2)$, $t = 1$ by using (3.8).

	$M=1$		$M=2$		$M=3$		$M=4$	
h^{-1}	error	rate	error	rate	error	rate	error	rate
10	0.386D-02		0.811D-04		0.127D-04		0.656D-06	
20	0.101D-02	1.93	0.633D-05	3.68	0.223D-06	5.83	0.284D-08	7.85
40	0.255D-03	1.98	0.417D-06	3.92	0.358D-08	5.96	0.114D-10	7.96
80	0.640D-04	1.99	0.264D-07	3.98	0.564D-10	5.99	0.445D-13	7.99
160	0.160D-04	1.99	0.165D-08	3.99	0.882D-12	5.99	0.194D-15	

Table 4. Absolute error and rate of convergence for $P(\mathbf{x}, t)$ at $\mathbf{x} = (0, 1.6, 0)$, $t = 1$ by using (3.8).

h^{-1}	$M=1$ error	rate	$M=2$ error	rate	$M=3$ error	rate	$M=4$ error	rate
10	0.279D-02		0.163D-03		0.584D-05		0.140D-06	
20	0.719D-03	1.96	0.107D-04	3.92	0.961D-07	5.92	0.543D-09	8.01
40	0.181D-03	1.99	0.678D-06	3.98	0.152D-08	5.98	0.211D-11	8.01
80	0.454D-04	2.00	0.425D-07	4.00	0.238D-10	6.00	0.826D-14	8.00
160	0.113D-04	2.00	0.266D-08	4.00	0.373D-12	6.00	0.139D-16	

TABLE 5. Absolute error and rate of convergence for $\partial_{x_2} P(\mathbf{x},t)$ at $\mathbf{x} = (1.2, 1.2, 1.2)$, $t=1$ by using (3.9).

h^{-1}	$M=1$ error	rate	$M=2$ error	rate	$M=3$ error	rate	$M=4$ error	rate
10	0.175D-04		0.319D-03		0.390D-05		0.141D-05	
20	0.750D-02	1.98	0.195D-04	4.03	0.382D-06	5.84	0.686D-08	7.68
40	0.188D-02	2.00	0.121D-05	4.01	0.613D-08	5.96	0.283D-10	7.92
80	0.471D-03	2.00	0.753D-07	4.00	0.964D-10	5.99	0.112D-12	7.97
160	0.118D-03	2.00	0.470D-08	4.00	0.151D-11	6.00	0.416D-15	

TABLE 6. Absolute error and rate of convergence for $\partial_{x_2} P(\mathbf{x},t)$ at $\mathbf{x} = (0, 1.6, 0)$, $t=1$ by using (3.9).

5.3. Nonhomogeneous heat equation. In Tables 7 and 8, we report on the absolute errors and approximation rates for the solution to Equations (1.7) with $\nu = 1$. We assumed $\Phi = (\varphi_1, \varphi_2, \varphi_3) = (\mathrm{e}^{-|\mathbf{x}|^2}(1+6t-4|\mathbf{x}|^2 t), 0, 0)$ which gives the exact solution $\mathbf{u}_2 = (t\mathrm{e}^{-|\mathbf{x}|^2}, 0, 0)$. The approximations were computed by $\mathscr{HM}_{h,\tau}$ in (4.2) for $M = 1,2,3,4$, with the parameters $\mathscr{D} = \mathscr{D}_0 = 4$.

Making the substitution

$$\sigma = \frac{\tau\ell}{2}\Big(1 + \tanh\Big(\frac{\pi}{2}\sinh\xi\Big)\Big) = \frac{\tau\ell}{1+\mathrm{e}^{-\pi\sinh\xi}}$$

introduced in [16], we transform K_M to an integral over $\mathbb{R}$ with doubly exponentially decaying integrand. Then we apply the classical trapezoidal rule with the parameters $\varkappa = 0.002$ and $2\cdot 10^3$ terms in the quadrature formula in order to reach the saturation error with the approximation formula of order $N = 8$.

h^{-1}	τ^{-1}	$M=1$ error	rate	$M=2$ error	rate	$M=3$ error	rate	$M=4$ error	rate
10	40	0.151E-02		0.463E-04		0.771E-06		0.497E-08	
20	80	0.376E-03	2.00	0.296E-05	3.97	0.118E-07	6.02	0.333E-10	7.22
40	160	0.938E-04	2.00	0.186E-06	3.99	0.183E-09	6.00	0.146E-12	7.84
80	320	0.234E-04	2.00	0.117E-07	3.99	0.286E-11	6.00	0.671E-15	7.76
160	640	0.586E-05	2.00	0.729E-09	3.99	0.445E-13	6.00	0.121E-15	

TABLE 7. Absolute error and rate of convergence for the approximation of $\mathscr{H}\varphi_1(\mathbf{x},t)$ defined in (1.8) at $\mathbf{x} = (1.2, 1.2, 1.2)$, $t=1$ by using (4.2).

h^{-1}	τ^{-1}	$M=1$ error	rate	$M=2$ error	rate	$M=3$ error	rate	$M=4$ error	rate
10	40	0.313E-02		0.552E-04		0.671E-05		0.276E-06	
20	80	0.810E-03	1.95	0.411E-05	3.75	0.115E-06	5.87	0.117E-08	7.88
40	160	0.204E-03	1.99	0.268E-06	3.94	0.184E-08	5.97	0.468E-11	7.97
80	320	0.512E-04	1.99	0.169E-07	3.98	0.289E-10	5.99	0.183E-13	7.99
160	640	0.128E-04	1.99	0.106E-08	3.99	0.452E-12	5.99	0.389E-15	

TABLE 8. Absolute error and rate of convergence for the approximation of $\mathscr{H}\varphi_1(\mathbf{x},t)$ defined in (1.8) at $\mathbf{x}=(0,1.6,0)$, $t=1$ by using (4.2).

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