DUKE MATHEMATICAL JOURNAL
Vol. 115, No. 3, © 2002

THE WIENER TEST FOR HIGHER ORDER ELLIPTIC EQUATIONS

VLADIMIR MAZ'YA

Abstract
We deal with strongly elliptic differential operators of an arbitrary even order $2m$ *with constant real coefficients and introduce a notion of the regularity of a boundary point with respect to the Dirichlet problem which is equivalent to that given by N. Wiener in the case of* $m = 1$. *It is shown that a capacitary Wiener's type criterion is necessary and sufficient for the regularity if* $n = 2m$. *In the case of* $n > 2m$, *the same result is obtained for a subclass of strongly elliptic operators.*

1. Introduction

Wiener's criterion for the regularity of a boundary point with respect to the Dirichlet problem for the Laplace equation [W] has been extended to various classes of elliptic and parabolic partial differential equations. These include linear divergence and nondivergence equations with discontinuous coefficients, equations with degenerate quadratic form, quasilinear and fully nonlinear equations, as well as equations on Riemannian manifolds, graphs, groups, and metric spaces (see [LSW], [FJK], [DMM], [LM], [KM], [MZ], [AH], [AHe], [La], [TW], to mention only a few). A common feature of these equations is that all of them are of second order, and Wiener-type characterizations for higher order equations have been unknown so far. Indeed, the increase of the order results in the loss of the maximum principle, Harnack's inequality, barrier techniques, and level truncation arguments, which are ingredients in different proofs related to the Wiener test for the second-order equations.

In the present paper we extend Wiener's result to elliptic differential operators $L(\partial)$ of order $2m$ in the Euclidean space $\mathbf{R}^n$ with constant real coefficients

$$L(\partial) = (-1)^m \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}\partial^{\alpha+\beta}.$$

We assume without loss of generality that $a_{\alpha\beta} = a_{\beta\alpha}$ and $(-1)^m L(\xi) > 0$ for all nonzero $\xi \in \mathbf{R}^n$. In fact, the results of this paper can be extended to equations with

DUKE MATHEMATICAL JOURNAL
Vol. 115, No. 3, © 2002
Received 8 April 2001.
2000 *Mathematics Subject Classification*. Primary 35J40; Secondary 31B15, 31B25.
Author's work supported by a grant from the Swedish National Science Foundation.

variable (e.g., Hölder continuous) coefficients in divergence form, but we leave aside this generalization to make our exposition more lucid.

We use the notation ∂ for the gradient $(\partial_{x_1}, \dots, \partial_{x_n})$, where ∂_{x_k} is the partial derivative with respect to x_k. By Ω we denote an open set in $\mathbf{R}^n$, and by $B_\rho(y)$ we denote the ball $\{x \in \mathbf{R}^n : |x - y| < \rho\}$, where $y \in \mathbf{R}^n$. We write B_ρ instead of $B_\rho(O)$.

Consider the Dirichlet problem

$$L(\partial)u = f, \quad f \in C_0^\infty(\Omega),\ u \in \mathring{H}^m(\Omega), \tag{1}$$

where we use the standard notation $C_0^\infty(\Omega)$ for the space of infinitely differentiable functions in $\mathbf{R}^n$ with compact support in Ω as well as $\mathring{H}^m(\Omega)$ for the completion of $C_0^\infty(\Omega)$ in the energy norm.

We call the point $O \in \partial\Omega$ regular with respect to $L(\partial)$ if for any $f \in C_0^\infty(\Omega)$ the solution of (1) satisfies

$$\lim_{\Omega \ni x \to O} u(x) = 0. \tag{2}$$

For $n = 2, 3, \dots, 2m-1$, the regularity is a consequence of the Sobolev imbedding theorem. Therefore, we suppose that $n \geq 2m$. In the case of $m = 1$, the above definition of regularity is equivalent to that given by Wiener (see [M4]).

The following result, which coincides with Wiener's criterion in the case of $n = 2$ and $m = 1$, is obtained in Sections 8 and 9.

THEOREM 1
Let $2m = n$. Then O is regular with respect to $L(\partial)$ if and only if

$$\int_0^1 C_{2m}(B_\rho \backslash \Omega)\rho^{-1}\, d\rho = \infty. \tag{3}$$

Here and elsewhere C_{2m} is the potential-theoretic Bessel capacity of order $2m$ (see [AH], [AHe]). The case of $n > 2m$ is more delicate because no result of Wiener's type is valid for all operators $L(\partial)$ (see [MNP, Chap. 10]). To be more precise, even the vertex of a cone can be irregular with respect to $L(\partial)$ if the fundamental solution of $L(\partial)$,

$$F(x) = F\Big(\frac{x}{|x|}\Big)|x|^{2m-n}, \quad x \in \mathbf{R}^n \backslash O, \tag{4}$$

changes sign. Examples of operators $L(\partial)$ with this property were given in [MN] and [D]. In the sequel Wiener's type characterization of regularity for $n > 2m$ is given for a subclass of the operators $L(\partial)$ called *positive with the weight F*. This means that for all real-valued $u \in C_0^\infty(\mathbf{R}^n \backslash O)$,

$$\int_{\mathbf{R}^n} L(\partial)u(x) \cdot u(x)F(x)\, dx \geq c \sum_{k=1}^m \int_{\mathbf{R}^n} |\nabla_k u(x)|^2 |x|^{2k-n}\, dx, \tag{5}$$

where ∇_k is the gradient of order k, that is, where $\nabla_k = \{\partial^\alpha\}$ with $|\alpha| = k$.

In Sections 5 and 7, we prove the following result.

THEOREM 2

Let $n > 2m$, and let $L(\partial)$ be positive with weight F. Then O is regular with respect to $L(\partial)$ if and only if

$$\int_0^1 C_{2m}(B_\rho \backslash \Omega)\rho^{2m-n-1}\, d\rho = \infty. \tag{6}$$

Note that in direct analogy with the case of the Laplacian we could say, in Theorems 1 and 2, that O is irregular with respect to $L(\partial)$ if and only if the set $\mathbf{R}^n \backslash \Omega$ is $2m$-thin in the sense of linear potential theory (see [L], [AH], [AHe]).

Since, obviously, the second-order operator $L(\partial)$ is positive with the weight F, Wiener's result for $n > 2$ is contained in Theorem 2. Moreover, one can notice that the same proof, with $F(x)$ being replaced by Green's function of the uniformly elliptic operator $u \to -\partial_{x_i}(a_{ij}(x)\partial_{x_j} u)$ with bounded measurable coefficients, leads to the main result in [LSW]. We also note that the pointwise positivity of F follows from (5), but the converse is not true. In particular, the m-harmonic operator with $2m < n$ satisfies (5) if and only if $n = 5, 6, 7$ for $m = 2$ and $n = 2m + 1, 2m + 2$ for $m > 2$ (see [M3], where the proof of the sufficiency of (6) is given for $(-\Delta)^m$ with m and n as above, and also [E] dealing with the sufficiency for noninteger powers of the Laplacian in the intervals $(0, 1)$ and $[n/2 - 1, n/2)$).

It is shown in [MP2] that the vertices of n-dimensional cones are regular with respect to Δ^2 for all dimensions. In Theorem 3 we consider the Dirichlet problem (1) for $n \geq 8$ and for the n-dimensional biharmonic operator with O being the vertex of an inner cusp. We show that condition (6), where $m = 2$, guarantees that $u(x) \to 0$ as x approaches O along any nontangential direction. This does not mean, of course, that Theorem 2 for the biharmonic operator can be extended to higher dimensions, but the domain Ω providing the corresponding counterexample should be more complicated than a cusp.

There are some auxiliary assertions of independent interest proved in this paper which concern the so-called L-capacitary potential U_K of the compact set $K \subset \mathbf{R}^n$, $n > 2m$, that is, the solution of the variational problem

$$\inf\Big\{ \int_{\mathbf{R}^n} L(\partial)u \cdot u\, dx : u \in C_0^\infty(\mathbf{R}^n) : u = 1 \text{ in vicinity of } K \Big\}.$$

We show, in particular, that for an arbitrary operator $L(\partial)$, the potential U_K is subject to the estimate

$$|U_K(y)| \leq c\ \operatorname{dist}(y, K)^{2m-n} C_{2m}(K) \quad \text{for all } y \in \mathbf{R}^n \backslash K,$$

where the constant c does not depend on K (see Prop. 1). The natural analogue of this estimate in the theory of Riesz potentials is quite obvious, and, as a matter of fact, our L-capacitary potential is representable as the Riesz potential $F * T$. However, one cannot rely upon methods of classical potential theory when studying U_K because, in general, T is only a distribution and not a positive measure. Among the properties of U_K resulting from the assumption of weighted positivity of $L(\partial)$ are the inequalities $0 < U_K < 2$ on $\mathbf{R}^n \backslash K$, which holds for an arbitrary compact set K of positive capacity C_{2m}. Generally, the upper bound 2 cannot be replaced by 1 if $m > 1$.

In conclusion, it is perhaps worth mentioning that the present paper gives answers to some questions posed in [M3].

2. Capacities and the L-capacitary potential

Let Ω be arbitrary if $n > 2m$ and bounded if $n = 2m$. By Green's m-harmonic capacity $\mathrm{cap}_m(K, \Omega)$ of a compact set $K \subset \Omega$ we mean

$$\inf\Big\{ \sum_{|\alpha|=m} \frac{m!}{\alpha!} ||\partial^\alpha u||^2_{L_2(\mathbf{R}^n)} : u \in C_0^\infty(\Omega),\ u = 1 \text{ in vicinity of } K \Big\}. \tag{7}$$

We omit the reference to Green and write $\mathrm{cap}_m(K)$ if $\Omega = \mathbf{R}^n$. It is well known that $\mathrm{cap}_m(K) = 0$ for all K if $n = 2m$.

Let $n > 2m$. One of equivalent definitions of the potential-theoretic Riesz capacity of order $2m$ is as follows:

$$c_{2m}(K) = \inf\Big\{ \sum_{|\alpha|=m} \frac{m!}{\alpha!} ||\partial^\alpha u||^2_{L_2(\mathbf{R}^n)} : u \in C_0^\infty(\mathbf{R}^n),\ u \geq 1 \text{ on } K \Big\}.$$

The capacities $\mathrm{cap}_m(K)$ and $c_{2m}(K)$ are equivalent; that is, their ratio is bounded and separated from zero by constants depending only on n and m (see [M2, Sec. 9.3.2]).

We use the notation $C_{2m}(K)$ for the potential-theoretic Bessel capacity of order $2m \leq n$ which can be defined by

$$\inf\Big\{ \sum_{0 \leq |\alpha| \leq m} \frac{m!}{\alpha!} ||\partial^\alpha u||^2_{L_2(\mathbf{R}^n)} : u \in C_0^\infty(\mathbf{R}^n),\ u \geq 1 \text{ on } K \Big\}.$$

Here also the replacement of the condition $u \geq 1$ on K by $u = 1$ in a neighbourhood of K leads to an equivalent capacity. Furthermore, if $n > 2m$ and $K \subset B_1$, the Riesz and Bessel capacities of K are equivalent.

We use the bilinear form

$$\mathscr{B}(u, v) = \int_\Omega \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta} \partial^\alpha u \cdot \partial^\beta v \, dx. \tag{8}$$

The solution $U_K \in \mathring{H}^m(\Omega)$ of the variational problem

$$\inf\left\{\mathscr{B}(u,u) : u \in C_0^\infty(\Omega),\ u = 1 \text{ on a neighbourhood of } K\right\} \tag{9}$$

is called Green's L-capacitary potential of the set K with respect to Ω, and the L-capacitary potential of K in the case of $\Omega = \mathbf{R}^n$.

We check that the m-capacitary potential of the unit ball B_1 in $\mathbf{R}^n$, where $n > 2m$, is given for $|x| > 1$ by

$$U_{B_1}(x) = \frac{\Gamma(n/2)}{\Gamma(m)\Gamma(-m+n/2)} \int_0^{|x|^{-2}} (1-\tau)^{m-1}\tau^{-m-1+n/2}\, d\tau. \tag{10}$$

This function solves the m-harmonic equation in $\mathbf{R}^n\backslash\overline{B_1}$ because the last integral is equal to

$$2\sum_{j=1}^{m} \frac{(-1)^{m-j}\Gamma(m)}{\Gamma(j)\Gamma(m-j+1)(n-2j)}|x|^{2j-n}.$$

Differentiating the integral in (10), we obtain

$$\partial^k_{|x|} U_{B_1}(x)\big|_{\partial B_1} = 0 \quad \text{for } k = 1, \dots, m-1.$$

The coefficient at the integral in (10) is chosen to satisfy the boundary condition

$$U_{B_1}(x) = 1 \quad \text{on } \partial B_1.$$

Owing to (10), we see that

$$0 < U_{B_1}(x) < 1 \quad \text{on } \mathbf{R}^n\backslash B_1$$

and that U_{B_1} is a decreasing function of $|x|$.

By Green's formula

$$\begin{aligned}\sum_{|\alpha|=m} ||\partial^\alpha U_{B_1}||^2_{L_2(\mathbf{R}^n\backslash B_1)} &= -\int_{\partial B_1} U_{B_1}(x)\frac{\partial}{\partial|x|}(-\Delta)^{m-1}U_{B_1}(x)\, ds_x \\ &= \frac{-2\Gamma(n/2)}{(n-2m)\Gamma(m)\Gamma(-m+n/2)}\int_{\partial B_1}\frac{\partial}{\partial|x|}(-\Delta)^{m-1}|x|^{2m-n}\, ds_x\end{aligned}$$

and by

$$(-\Delta)^{m-1}|x|^{2m-n} = \frac{4^{m-1}\Gamma(m)\Gamma(-1+n/2)}{\Gamma(-m+n/2)}|x|^{2-n},$$

we obtain the value of the m-harmonic capacity of the unit ball:

$$\operatorname{cap}_m B_1 = \frac{4^m}{n-2m}\left(\frac{\Gamma(n/2)}{\Gamma(-m+n/2)}\right)^2\omega_{n-1} \tag{11}$$

with ω_{n-1} denoting the area of ∂B_1.

We recall that the Riesz capacitary measure of order $2m$, $2m < n$, is the normalized area on ∂B_1 (see [L, Chap. 2], Sec. 3). Hence, one can verify by direct computation that

$$c_{2m}(B_1) = \frac{2\sqrt{\pi}\,\Gamma(m)\Gamma(m-1+n/2)}{\Gamma(m-1/2)\Gamma(-m+n/2)}\,\omega_{n-1}. \tag{12}$$

LEMMA 1

For any $u \in C_0^\infty(\Omega)$ and any distribution $\Phi \in [C_0^\infty(\Omega)]^*$,

$$\mathscr{B}(u, u\Phi) = 2^{-1}\int_\Omega u^2 L(\partial)\Phi\,dx + \int_\Omega \sum_{j=1}^{m} \sum_{|\mu|=|\nu|=j} \partial^\mu u \cdot \partial^\nu u \cdot \mathscr{P}_{\mu\nu}(\partial)\Phi\,dx, \tag{13}$$

where $\mathscr{P}_{\mu\nu}(\zeta)$ are homogeneous polynomials of degree $2(m-j)$, $\mathscr{P}_{\mu\nu} = \mathscr{P}_{\nu\mu}$, and $\mathscr{P}_{\alpha\beta}(\zeta) = a_{\alpha\beta}$ for $|\alpha| = |\beta| = m$.

Proof

The left-hand side in (13) is equal to

$$\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}\int_\Omega u\partial^\alpha u \cdot \partial^\beta\Phi\,dx$$
$$+ \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}\Big(\int_\Omega \partial^\alpha u \cdot \partial^\beta u \cdot \Phi\,dx + \sum_{\beta>\gamma>0} \frac{\beta!}{\gamma!(\beta-\gamma)!}\int_\Omega \partial^\alpha u \cdot \partial^\gamma u \cdot \partial^{\beta-\gamma}\Phi\,dx\Big).$$

We have

$$\int_{\mathbf{R}^n} u\partial^\alpha u \cdot \partial^\beta \Phi\,dx$$
$$= 2^{-1}\int_\Omega \partial^\alpha(u^2)\partial^\beta\Phi\,dx - 2^{-1}\sum_{\alpha>\gamma>0}\frac{\alpha!}{\gamma!(\alpha-\gamma)!}\int_\Omega \partial^\gamma u \cdot \partial^{\alpha-\gamma}u \cdot \partial^\beta\Phi\,dx.$$

Hence and by $a_{\alpha\beta} = a_{\beta\alpha}$, we obtain the identity

$$\mathscr{B}(u, u\Phi) = 2^{-1}\int_\Omega u^2 L(\partial)\Phi\,dx$$
$$+ \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}\sum_{\beta>\gamma>0}\frac{\beta!}{\gamma!(\beta-\gamma)!}\int_\Omega \partial^\gamma u(\partial^\alpha u \cdot \partial^{\beta-\gamma}\Phi - 2^{-1}\partial^{\beta-\gamma}u \cdot \partial^\alpha\Phi)\,dx$$
$$+ \int_\Omega \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}\partial^\alpha u \cdot \partial^\beta u \cdot \Phi\,dx.$$

We need to prove that the second term can be written as

$$\int_\Omega \sum_{j=1}^{m-1} \sum_{|\mu|=|\nu|=j} \partial^\mu u \cdot \partial^\nu u \cdot \mathscr{P}_{\mu\nu}(\partial)\Phi\, dx.$$

It suffices to establish such a representation for the integral

$$i_{\alpha\beta\gamma} := \int_\Omega \partial^\alpha u \cdot \partial^\gamma u \cdot \partial^{\beta-\gamma}\Phi\, dx$$

with $|\alpha| > |\gamma|$. Let $|\alpha|+|\gamma|$ be even. We write $\alpha = \sigma+\tau$, where $|\sigma| = (|\alpha|+|\gamma|)/2$. After integrating by parts, we have

$$\begin{aligned} i_{\alpha\beta\gamma} &= (-1)^{|\tau|}\int_\Omega \partial^\sigma u \cdot \partial^{\gamma+\tau} u \cdot \partial^{\beta-\gamma}\Phi\, dx \\ &\quad + (-1)^{|\tau|}\sum_{0\le\delta<\tau} \frac{\tau!}{\delta!(\tau-\delta)!}\int_\Omega \partial^\sigma u \cdot \partial^{\gamma+\delta} u \cdot \partial^{\beta-\gamma+\tau-\delta}\Phi\, dx. \end{aligned}$$

The first integral on the right is in the required form because $|\sigma| = |\gamma| + |\tau| = (|\alpha|+|\gamma|)/2$. We have $|\gamma|+|\delta| < |\alpha|$ in the remaining terms. Therefore, these terms are subject to the induction hypothesis.

Now we let $|\alpha|+|\gamma|$ be odd. Then

$$\begin{aligned} i_{\alpha\beta\gamma} &= (-1)^{|\alpha|}\int_{\mathbf{R}^n} u\partial^\alpha(\partial^\gamma u \cdot \partial^{\beta-\gamma}\Phi)\, dx \\ &= (-1)^{|\alpha|}\int_{\mathbf{R}^n} u \sum_{0\le\delta\le\alpha} \frac{\alpha!}{\delta!(\alpha-\delta)!}\partial^{\gamma+\delta} u \cdot \partial^{\beta-\gamma+\alpha-\delta}\Phi\, dx. \end{aligned}$$

Integrating by parts, we obtain

$$\begin{aligned} i_{\alpha\beta\gamma} &= (-1)^{|\alpha|+|\gamma|}\int_{\mathbf{R}^n} u \sum_{0\le\delta\le\alpha} \frac{\alpha!}{\delta!(\alpha-\delta)!}\partial^\delta u \cdot \partial^\gamma(u\partial^{\beta-\gamma+\alpha-\delta}\Phi)\, dx \\ &= -\int_{\mathbf{R}^n} u \sum_{0\le\delta\le\alpha} \frac{\alpha!}{\delta!(\alpha-\delta)!}\sum_{0\le\kappa\le\gamma}\frac{\gamma!}{\kappa!(\gamma-\kappa)!}\partial^\delta u \cdot \partial^\kappa u \cdot \partial^{\alpha+\beta-\delta-\kappa}\Phi\, dx. \end{aligned}$$

Hence,

$$i_{\alpha\beta\gamma} = -2^{-1}\sum_{\substack{0\le\delta\le\alpha,\ 0\le\kappa\le\gamma \\ |\delta|+|\kappa|<|\alpha|+|\gamma|}} \frac{\alpha!\gamma!}{\delta!(\alpha-\delta)!\kappa!(\gamma-\kappa)!}\int_{\mathbf{R}^n} \partial^\delta u \cdot \partial^\kappa u \cdot \partial^{\alpha+\beta-\delta-\kappa}\Phi\, dx.$$

Every integral on the right is subject to the induction hypothesis. The result follows. □

As in the introduction, by $F(x)$ we denote the fundamental solution of $L(\partial)$ in $\mathbf{R}^n$ subject to (4). Setting $\Phi(x) = F(x-y)$ in (13), we conclude that for all $u \in C_0^\infty(\mathbf{R}^n)$,

$$\int_{\mathbf{R}^n} L(\partial)u(x) \cdot u(x) F(x-y)\, dx$$
$$= 2^{-1}u(y)^2 + \int_{\mathbf{R}^n} \sum_{j=1}^{m} \sum_{|\mu|=|\nu|=j} \partial^\mu u(x) \cdot \partial^\nu u(x) \cdot \mathscr{P}_{\mu\nu}(\partial) F(x-y)\, dx. \quad (14)$$

LEMMA 2
Let $\Omega = \mathbf{R}^n$, $2m < n$. For all $y \in \mathbf{R}^n \backslash K$,

$$U_K(y) = 2^{-1}U_K(y)^2$$
$$+ \int_{\mathbf{R}^n} \sum_{m \geq j \geq 1} \sum_{|\mu|=|\nu|=j} \partial^\mu U_K(x) \cdot \partial^\nu U_K(x) \cdot \mathscr{P}_{\mu\nu}(\partial) F(x-y)\, dx, \quad (15)$$

where the same notation as in Lemma 1 is used.

Proof
We fix an arbitrary point y in $\mathbf{R}^n \backslash K$. Let $\{u_s\}_{s \geq 1}$ be a sequence of functions in $C_0^\infty(\mathbf{R}^n)$ such that $u_s = U_K$ on a neighbourhood of y independent of s and $u_s \to U_K$ in $\mathring{H}^m(\mathbf{R}^n)$. Since U_K is smooth on $\mathbf{R}^n \backslash K$ and since the function F is smooth on $\mathbf{R}^n \backslash O$ and vanishes at infinity, we can pass to the limit in (14), where $u = u_s$. This implies

$$\lim_{s \to \infty} \int_{\mathbf{R}^n} L(\partial)U_K(x) \cdot u_s(x) F(x-y)\, dx = 2^{-1}U_K(y)^2$$
$$+ \int_{\mathbf{R}^n} \sum_{j=1}^{m} \sum_{|\mu|=|\nu|=j} \partial^\mu U_K(x) \cdot \partial^\nu U_K(x) \cdot \mathscr{P}_{\mu\nu}(\partial) F(x-y)\, dx, \quad (16)$$

where $L(\partial)U_K$ is an element of the space $H^{-m}(\mathbf{R}^n)$ dual to $\mathring{H}^m(\mathbf{R}^n)$, and the integral on the left is understood in the sense of distributions. Taking into account that $L(\partial)U_K = 0$ on $\mathbf{R}^n \backslash K$ and that u_s can be chosen to satisfy $u_s = 1$ on a neighbourhood of K, we write the left-hand side in (16) as

$$\int_{\mathbf{R}^n} L(\partial)U_K(x) \cdot F(x-y)\, dx = U_K(y). \quad (17)$$

The result follows. □

COROLLARY 1

Let $2m < n$. For for almost all $y \in \mathbf{R}^n$,

$$|\nabla_l U_K(y)| \le c\Big(|\nabla_l U_K(y)^2| + \int_{\mathbf{R}^n} \sum_{\substack{1\le r,s\le m\\ r+s>l}} \frac{|\nabla_r U_K(x)||\nabla_s U_K(x)|}{|x-y|^{n-r-s+l}}\, dx\Big), \tag{18}$$

where $l = 0, \dots, m$.

Proof

Since $\nabla_l U_K$ vanishes almost everywhere on K, it is enough to check (18) for $y \in \mathbf{R}^n \backslash K$. By (15), it suffices to estimate

$$\Big|\nabla_l \int_{\mathbf{R}^n} \partial^\mu U_K(x) \cdot \partial^\nu U_K(x) \cdot \mathscr{P}_{\mu\nu}(\partial) F(x-y)\, dx\Big|, \tag{19}$$

where $|\mu| = |\nu| = j$ and $j = 1, \dots, m$. Let $2j \le l$. Since ord $\mathscr{P}_{\mu\nu}(\partial) = 2(m-j)$, we have $|\nabla_l \mathscr{P}_{\mu\nu}(\partial) F(x-y)| \le c\,|x-y|^{-n+2j-l}$, and we may take

$$c \int_{\mathbf{R}^n} \frac{|\nabla_j U_K(x)|^2}{|x-y|^{n-2j+l}}\, dx \tag{20}$$

as a majorant for (19). In the case of $2j > l$, integrating by parts we estimate (19) by

$$c \int_{\mathbf{R}^n} \big|\nabla_{m-j}\big(\partial^\mu U_K(x) \cdot \partial^\nu U_K(x)\big)\big|\, \big|\nabla_{m-j+l} F(x-y)\big|\, dx \\ \le c_1 \int_{\mathbf{R}^n} \sum_{i=0}^{m-j} \frac{|\nabla_{i+j} U_K(x)|\, |\nabla_{m-i} U_K(x)|}{|x-y|^{n-m-j+l}}\, dx.$$

Since $m + j \ge 2j > l$, the sum of the last majorant and (20) is dominated by the right-hand side in (18). The proof is complete. □

PROPOSITION 1

Let $\Omega = \mathbf{R}^n$ and $2m < n$. For all $y \in \mathbf{R}^n \backslash K$, the following estimate holds:

$$|\nabla_j U_K(y)| \le c_j\, \mathrm{dist}(y, K)^{2m-n-j}\, \mathrm{cap}_m\, K, \tag{21}$$

where $j = 0, 1, 2, \dots$ and c_j does not depend on K and y.

Proof
In order to simplify the notation, we set $y = 0$ and $\delta = \mathrm{dist}(y, K)$. By the well-known local estimate for variational solutions of $L(\partial)u = 0$ (see [ADN, Chap. 3]),

$$|\nabla_j u(0)|^2 \le c_j \delta^{-n-2j} \int_{B_{\delta/2}} u(x)^2\, dx, \tag{22}$$

it suffices to prove (21) for $j = 0$. By (22) and by Hardy's inequality,

$$\begin{aligned} U_K(0)^2 &\le c\delta^{2m-n} \int_{\mathbf{R}^n} U_K(x)^2 \frac{dx}{|x|^{2m}} \\ &\le c\delta^{2m-n} \int_{\mathbf{R}^n} |\nabla_m U_K(x)|^2\, dx \le c_0 \delta^{2m-n} \operatorname{cap}_m K. \end{aligned} \tag{23}$$

If $\operatorname{cap}_m K \ge c_0^{-1}\delta^{n-2m}$, then estimate (21) follows from (23).

Now let $\operatorname{cap}_m K < c_0^{-1}\delta^{n-2m}$. We have $U_K(0)^2 \le |U_K(0)|$ because of (23). Hence and by (15),

$$|U_K(0)| \le c \sum_{j=1}^{m} \int_{\mathbf{R}^n} |\nabla_j U(x)|^2 \frac{dx}{|x|^{n-2(m-j)}}.$$

Since by Hardy's inequality all integrals on the right are estimated by the mth integral, we obtain

$$|U_K(0)| \le c\Big(\delta^{2m} \sup_{x\in B_{\delta/2}} |\nabla_m U_K(x)|^2 + \int_{\mathbf{R}^n\setminus B_{\delta/2}} |\nabla_m U_K(x)|^2 \frac{dx}{|x|^{n-2m}}\Big).$$

We estimate the above supremum using (22) with $j = 0$ and with u replaced by $\nabla_m U_K$. Then

$$|U_K(0)| \le c\delta^{2m-n}\Big(\int_{B_\delta} |\nabla_m U_K(x)|^2\, dx + \int_{\mathbf{R}^n\setminus B_{\delta/2}} |\nabla_m U_K(x)|^2\, dx\Big).$$

The result follows from the definition of U_K. □

By $\mathscr{M}$ we denote the Hardy-Littlewood maximal operator, that is,

$$\mathscr{M} f(x) = \sup_{\rho>0} \frac{n}{\omega_{n-1}\rho^n} \int_{|y-x|<\rho} |f(y)|\, dy.$$

PROPOSITION 2
Let $2m < n$, and let $0 < \theta < 1$. Also, let K be a compact subset of $\overline{B_\rho}\setminus B_{\theta\rho}$. Then the L-capacitary potential U_K satisfies

$$\mathscr{M}\nabla_l U_K(0) \le c_\theta \rho^{2m-l-n} \operatorname{cap}_m K, \tag{24}$$

where $l = 0, 1, \ldots, m$ and c_θ does not depend on K and ρ.

Proof
Let $r > 0$. We have

$$\int_{B_r} |\nabla_l U_K(y)|\, dx \le c \Big(\int_{B_r \cap B_{\theta\rho/2}} |\nabla_l U_K(y)|\, dx + \int_{B_r \setminus B_{2\rho}} |\nabla_l U_K(y)|\, dx + \int_{B_r \cap (B_{2\rho} \setminus B_{\theta\rho/2})} |\nabla_l U_K(y)|\, dx \Big).$$

Since $\operatorname{dist}(y; K) \ge c\rho$ for $y \in B_{\theta\rho/2} \cap (B_r \setminus B_{2\rho})$, the first and second integrals on the right do not exceed $cr^n \rho^{2m-l-n} \operatorname{cap}_m K$ in view of (21). Hence, for $r \le \theta\rho/2$, the mean value of $|\nabla_l U_K|$ on B_r is dominated by $c\rho^{2m-l-n} \operatorname{cap}_m K$. Let $r > \theta\rho/2$. It follows from Corollary 1 that the integral

$$I_l(\rho) := \int_{B_{2\rho} \setminus B_{\theta\rho/2}} |\nabla_l U_K(y)|\, dx$$

is majorized by

$$c\Big(\int_{B_{2\rho} \setminus B_{\theta\rho/2}} |\nabla_l U_K(y)^2|\, dy + \int_{B_{2\rho} \setminus B_{\theta\rho/2}} dy \int_{\mathbf{R}^n} \sum_{\substack{1 \le r,s \le m \\ r+s>l}} \frac{|\nabla_r U_K(x)||\nabla_s U_K(x)|}{|x-y|^{n-r-s+l}}\, dx \Big)$$
$$\le c_1 \rho^n \sum_{1 \le r,s \le m} \int_{\mathbf{R}^n} \frac{|\nabla_r U_K|\, |\nabla_s U_K|}{(\rho + |x|)^{n-r-s+l}}\, dx$$
$$\le c_2 \rho^{2m-l} \sum_{1 \le r,s \le m} \int_{\mathbf{R}^n} \frac{|\nabla_r U_K|\, |\nabla_s U_K|}{|x|^{2m-r-s}}\, dx.$$

Hence and by Hardy's inequality, we obtain

$$I_l(\rho) \le c\, \rho^{2m-l} \int_{\mathbf{R}^n} |\nabla_m U_K(x)|^2\, dx \le c\, \rho^{2m-l} \operatorname{cap}_m K.$$

The proof is complete. □

3. Weighted positivity of $L(\partial)$

Let $2m < n$. It follows from (14) that the condition of weighted positivity (5) is equivalent to the inequality

$$\int_{\mathbf{R}^n} \sum_{j=1}^{m} \sum_{|\mu|=|\nu|=j} \partial^\mu u(x) \cdot \partial^\nu u(x) \cdot P_{\mu\nu}(\partial) F(x)\, dx \ge c \sum_{k=1}^{m} \int_{\mathbf{R}^n} \frac{|\nabla_k u(x)|^2}{|x|^{n-2k}}\, dx \qquad (25)$$

for all $u \in C_0^\infty(\mathbf{R}^n \setminus O)$. Since the restriction of F to ∂B_1 is a smooth function of the coefficients of $L(\partial)$, the last inequality implies that the set of the operators $L(\partial)$ which are positive with the weight F is open.

PROPOSITION 3
Inequality (5), valid for all $u \in C_0^\infty(\mathbf{R}^n \backslash O)$, implies

$$\mathscr{B}(u, uF) \geq 2^{-1} u(0)^2 + c \sum_{j=1}^{m} \int_{\mathbf{R}^n} \frac{|\nabla_j u(x)|^2}{|x|^{n-2j}}\, dx \tag{26}$$

for all $u \in C_0^\infty(\mathbf{R}^n)$.

Proof
Let $u \in C_0^\infty(\mathbf{R}^n)$, $0 < \varepsilon < 1/2$, and let $\eta_\varepsilon(x) = \eta\big((\log\varepsilon)^{-1}\log|x|\big)$, where $\eta \in C^\infty(\mathbf{R}^1)$, $\eta(t) = 0$ for $t \geq 2$, and $\eta(t) = 1$ for $t \leq 1$. Clearly, $\eta_\varepsilon(x) = 0$ for $x \in \mathbf{R}^n \backslash B_\varepsilon$, all derivatives of η_ε vanish outside $B_\varepsilon \backslash B_{\varepsilon^2}$, and $|\nabla_j \eta_\varepsilon(x)| \leq c_j |\log\varepsilon|^{-1} |x|^{-j}$.

By (5), the bilinear form $\mathscr{B}$ defined by (8) satisfies

$$\mathscr{B}\big((1-\eta_\varepsilon)u, (1-\eta_\varepsilon)uF\big) \geq c \sum_{j=1}^{m} \int_{\mathbf{R}^n} \big|\nabla_j\big((1-\eta_\varepsilon)u\big)\big|^2 \frac{dx}{|x|^{n-2j}}. \tag{27}$$

Using the just mentioned properties of η_ε, we see that

$$\begin{aligned} &\left|\Big(\int_{\mathbf{R}^n} \big|\nabla_j\big((1-\eta_\varepsilon)u\big)\big|^2 \frac{dx}{|x|^{n-2j}}\Big)^{1/2} - \Big(\int_{\mathbf{R}^n} (1-\eta_\varepsilon)^2 |\nabla_j u|^2 \frac{dx}{|x|^{n-2j}}\Big)^{1/2}\right| \\ &\quad \leq \Big(\int_{\mathbf{R}^n} |[\nabla_j, 1-\eta_\varepsilon]u|^2 \frac{dx}{|x|^{n-2j}}\Big)^{1/2} \leq c(u) \sum_{k=1}^{j} \int_{\mathbf{R}^n} |\nabla_k \eta_\varepsilon|^2 \frac{dx}{|x|^{n-2j}} \\ &\quad = O(|\log\varepsilon|^{-1}), \end{aligned}$$

where $[S, T]$ stands for the commutator $ST - TS$. Hence and by (27),

$$\liminf_{\varepsilon\to 0} \mathscr{B}\big((1-\eta_\varepsilon)u, (1-\eta_\varepsilon)uF\big) \geq c \sum_{j=1}^{m} \int_{\Omega} |\nabla_j u|^2 \frac{dx}{|x|^{n-2j}}. \tag{28}$$

Since, clearly,

$$\big|\mathscr{B}\big(\eta_\varepsilon(u-u(0)), \eta_\varepsilon(u-u(0))F\big)\big| \leq c \sum_{j=1}^{m} \int_{B_\varepsilon} \frac{|\nabla_j(\eta_\varepsilon(u-u(0)))|^2}{|x|^{n-2j}}\, dx = O(\varepsilon),$$

one can replace $(1-\eta_\varepsilon)u$ in the left-hand side of (28) by $u - u(0)\eta_\varepsilon$. We use the identity

$$\begin{aligned} \mathscr{B}\big((u-u(0)\eta_\varepsilon), (u-u(0)\eta_\varepsilon)F\big) &= \mathscr{B}(u, uF) + u(0)^2\big(\mathscr{B}(\eta_\varepsilon, \eta_\varepsilon F) - \mathscr{B}(\eta_\varepsilon, F)\big) \\ &\quad - u(0)\big(\mathscr{B}\big(\eta_\varepsilon, (u-u(0))F\big) + \mathscr{B}(u, \eta_\varepsilon F)\big). \end{aligned}$$

It is straightforward that $|\mathscr{B}(\eta_\varepsilon, (u-u(0))F)| + |\mathscr{B}(u, \eta_\varepsilon F)| \le c\varepsilon$. Therefore,

$$\liminf_{\varepsilon\to 0} \mathscr{B}\big(\eta_\varepsilon(u-u(0)), \eta_\varepsilon(u-u(0))F\big) = \mathscr{B}(u, uF) + u(0)^2\big(\mathscr{B}(\eta_\varepsilon, \eta_\varepsilon F) - \mathscr{B}(\eta_\varepsilon, F)\big).$$

Since $\mathscr{B}(\eta_\varepsilon, F) = 1$ and since it follows from (14) that

$$\big|2\mathscr{B}(\eta_\varepsilon, \eta_\varepsilon F) - 1\big| \le c\sum_{j=1}^{m}\int_{B_\varepsilon\setminus B_{\varepsilon^2}} |\nabla_j \eta_\varepsilon|^2 \frac{dx}{|x|^{n-2j}} = O(|\log\varepsilon|^{-1}),$$

we arrive at (26). □

PROPOSITION 4
The positivity of $L(\partial)$ with the weight F implies $F(x) > 0$.

Proof
Let $u_\varepsilon(x) = \varepsilon^{-n/2}\eta(\varepsilon^{-1}(x-\omega))|\xi|^{-m}\exp(i(x,\xi))$, where η is a nonzero function in $C_0^\infty(\mathbf{R}^n)$, ε is a positive number, $\omega \in \partial B_1$, and $\xi \in \mathbf{R}^n$. We put u_ε into the inequality

$$\operatorname{Re}\int_{\mathbf{R}^n}\sum_{j=1}^{m}\sum_{|\mu|=|\nu|=j}\partial^\mu u(x)\cdot\partial^\nu\overline{u(x)}P_{\mu\nu}(\partial)F(x)\,dx \ge c\sum_{j=1}^{m}\int_{\mathbf{R}^n}|\nabla_j u(x)|^2\frac{dx}{|x|^{n-2j}},$$

which is equivalent to (25). Taking the limits as $|\xi|\to\infty$, we obtain

$$\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}\Big(\frac{\xi}{|\xi|}\Big)^{\alpha+\beta}\varepsilon^{-n}\int_{\mathbf{R}^n}\big|\eta\big(\varepsilon^{-1}(x-\omega)\big)\big|^2 F(x)\,dx \le c\,\varepsilon^{-n}\int_{\mathbf{R}^n}\big|\eta\big(\varepsilon^{-1}(x-\omega)\big)\big|^2\,dx.$$

Now the positivity of F follows by the limit passage as $\varepsilon\to 0$. □

4. More properties of the L-capacitary potential

Let $L(\partial)$ be positive with the weight F. Then identity (15) implies that the L-capacitary potential of a compact set K with positive m-harmonic capacity satisfies

$$0 < U_K(x) < 2 \quad \text{on } \mathbf{R}^n\setminus K. \tag{29}$$

We show that, in general, the bound 2 in (29) cannot be replaced by 1.

PROPOSITION 5
If $L = \Delta^{2m}$, then there exists a compact set K such that $(U_K - 1)\big|_{\mathbf{R}^n\setminus K}$ changes sign in any neighbourhood of a point of K.

Proof
Let C be an open cone in $\mathbf{R}^n_+ = \{x = (x', x_n) : x_n > 0\}$, and let $C_\varepsilon = \{x : (\varepsilon^{-1}x', x_n) \in C\}$ with sufficiently small $\varepsilon > 0$. We define the compact set K as $\overline{B_1}\backslash C_\varepsilon$. Suppose that $U_K(x) - 1$ does not change sign on a δ-neighbourhood of the origin. Then either $U_K - 1$ or $1 - U_K$ is a nontrivial nonnegative $2m$-harmonic function on $B_\delta \cap C_\varepsilon$ subject to zero Dirichlet conditions on $B_\delta \cap \partial C_\varepsilon$, which contradicts [KKM, Lem. 1]. The result follows. □

We give a lower pointwise estimate for U_K stated in terms of capacity (cf. the upper estimate (21)).

PROPOSITION 6
Let $n > 2m$, and let $L(\partial)$ be positive with the weight F. If K is a compact subset of B_d and $y \in \mathbf{R}^n\backslash K$, then $U_K(y) \geq c(|y| + d)^{2m-n} \operatorname{cap}_m K$.

Proof
Let a be a point in the semiaxis $(2, \infty)$ which is specified later. By (26),

$$\begin{aligned} U_K(y) &\geq c(|y| + ad)^{2m-n} \int_{B_{ad}} |\nabla_m u|^2 \, dx \\ &\geq c(|y| + ad)^{2m-n} \Big(\operatorname{cap}_m K - \int_{\mathbf{R}^n\backslash B_{ad}} |\nabla_m u|^2 \, dx \Big). \end{aligned} \tag{30}$$

It follows from Proposition 1 that for $x \in \mathbf{R}^n\backslash B_{ad}$,

$$|\nabla_m U_K(x)| \leq c_0 \frac{\operatorname{cap}_m K}{(|x| - d)^{n-2m}} \leq 2^{n-2m} c_0 \frac{\operatorname{cap}_m K}{|x|^{n-m}}.$$

Hence,

$$\int_{\mathbf{R}^n\backslash B_{ad}} |\nabla_m u|^2 \, dx \leq c(\operatorname{cap}_m K)^2 \int_{\mathbf{R}^n\backslash B_{ad}} \frac{dx}{|x|^{2n-2m}} = c_1 \frac{(\operatorname{cap}_m K)^2}{(ad)^{n-2m}},$$

and by (30),

$$U_K(y) \geq \frac{\operatorname{cap}_m K}{(|y| + d)^{n-2m}} \Big(1 - c \frac{\operatorname{cap}_m K}{(ad)^{n-2m}} \Big).$$

Choosing a to make the difference in braces positive, we complete the proof. □

5. Proof of sufficiency in Theorem 2

In the next lemma and henceforth, we use the notation

$$M_\rho(u) = \rho^{-n} \int_{\Omega \cap S_\rho} u(x)^2 \, dx, \quad S_\rho = \{x : \rho < |x| < 2\rho\}.$$

LEMMA 3
Let $2m < n$, and let $L(\partial)$ be positive with the weight F. Further, let $u \in \mathring{H}^m(\Omega)$ be a solution of

$$L(\partial)u = 0 \quad \text{on } \Omega \cap B_{2\rho}. \tag{31}$$

Then $\mathscr{B}(u\eta_\rho, u\eta_\rho F_y) \le cM_\rho(u)$ for an arbitrary point $y \in B_\rho$, where $\eta_\rho(x) = \eta(x/\rho)$, $\eta \in C_0^\infty(B_2)$, $\eta = 1$ on $B_{3/2}$, $F_y(x) = F(x-y)$.

Proof
By definition of $\mathscr{B}$,

$$\begin{aligned}&\mathscr{B}(u\eta_\rho, u\eta_\rho F_y) - \mathscr{B}(u, u\eta_\rho^2 F_y)\\ &\quad = \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta} \int_\Omega \big([\partial^\alpha, \eta_\rho]u \cdot \partial^\beta(u\eta_\rho F_y) - \partial^\alpha u \cdot [\partial^\beta, \eta_\rho](u\eta_\rho F_y)\big)\, dx.\end{aligned} \tag{32}$$

It follows from (31) that $\mathscr{B}(u, u\eta_\rho^2 F_y) = 0$. The absolute value of the right-hand side in (32) is majorized by

$$c \sum_{j=0}^m \rho^{2j-n} \int_\Omega \zeta_\rho |\nabla_j u|^2\, dx, \tag{33}$$

where $\zeta_\rho(x) = \zeta(x/\rho)$, $\zeta \in C_0^\infty(S_1)$, and $\zeta = 1$ on $\operatorname{supp}|\nabla\eta|$. The result follows by the well-known local energy estimate (see [ADN, Chap. 3])

$$\int_\Omega \zeta_\rho |\nabla_j u|^2\, dx \le c\rho^{-2j} \int_{\Omega\cap S_\rho} u^2\, dx. \tag{34}$$

□

Combining Proposition 3 and Lemma 3, we arrive at the following local estimate.

COROLLARY 2
Let the conditions of Lemma 3 be satisfied. Then

$$u(y)^2 + \int_{\Omega\cap B_\rho} \sum_{k=1}^m \frac{|\nabla_k u(x)|^2}{|x-y|^{n-2k}} \le cM_\rho(u), \quad y \in \Omega \cap B_\rho. \tag{35}$$

We need the following Poincaré-type inequality proved in [M1] (see also [M2, Sec. 10.1.2]).

LEMMA 4
Let $u \in \mathring{H}^m(\Omega)$. Then for all $\rho > 0$,

$$M_\rho(u) \le \frac{c\,\rho^{n-2m}}{\operatorname{cap}_m(\bar{S}_\rho \backslash \Omega)} \int_{\Omega\cap S_\rho} \sum_{k=1}^m \frac{|\nabla_k u(x)|^2}{\rho^{n-2k}}\, dx. \tag{36}$$

COROLLARY 3
Let the conditions of Lemma 3 be satisfied. Then, for all points $y \in \Omega \cap B_\rho$, the following estimate holds:

$$u(y)^2 + \int_{\Omega\cap B_\rho} \sum_{k=1}^{m} \frac{|\nabla_k u(x)|^2}{|x-y|^{n-2k}}\,dx \le \frac{c\rho^{n-2m}}{\operatorname{cap}_m(\bar{S}_\rho, \Omega)} \int_{\Omega\cap S_\rho} \sum_{k=1}^{m} \frac{|\nabla_k u(x)|^2}{\rho^{n-2k}}\,dx.$$

Proof
We combine Corollary 2 with inequality (36). □

LEMMA 5
Let $2m < n$, and let $L(\partial)$ be positive with the weight F. Also, let $u \in \mathring{H}^m(\Omega)$ satisfy $L(\partial)u = 0$ on $\Omega \cap B_{2R}$. Then, for all $\rho \in (0, R)$,

$$\sup\big\{|u(p)|^2 : p \in \Omega \cap B_\rho\big\} + \int_{\Omega\cap B_\rho} \sum_{k=1}^{m} \frac{|\nabla_k u(x)|^2}{|x|^{n-2k}}\,dx$$
$$\le c M_R(u) \exp\Big(-c \int_\rho^R \operatorname{cap}_m(\bar{B}_\tau \setminus \Omega) \frac{d\tau}{\tau^{n-2m+1}}\Big). \quad (37)$$

Proof
Let us use the notation

$$\gamma_m(r) = r^{2m-n} \operatorname{cap}_m(\bar{S}_r \setminus \Omega). \quad (38)$$

It is sufficient to prove (37) only for $\rho \le R/2$ because in the opposite case the result follows from Corollary 2. Denote the first and second terms on the left in (37) by φ_ρ and ψ_ρ, respectively. From Corollary 3, it follows that for $r \le R$,

$$\varphi_r + \psi_r \le \frac{c}{\gamma_m(r)}(\psi_{2r} - \psi_r) \le \frac{c}{\gamma_m(r)}(\psi_{2r} - \psi_r + \varphi_{2r} - \varphi_r).$$

This along with the obvious inequality $\gamma_m(r) \le c$ implies

$$\varphi_r + \psi_r \le c\, \exp\big(-c_0\gamma_m(r)\big)(\varphi_{2r} + \psi_{2r}).$$

By setting $r = 2^{-j}R,\ j = 1, \dots,$ we arrive at the estimate

$$\varphi_{2^{-l}R} + \psi_{2^{-l}R} \le c \exp\Big(-c \sum_{j=1}^{l} \gamma_m(2^{-j}R)\Big)(\varphi_R + \psi_R).$$

We choose l so that $l < \log_2(R/\rho) \le l+1$ in order to obtain

$$\varphi_\rho + \psi_\rho \le c \exp\Big(-c_0 \sum_{j=1}^{l} \gamma_m(2^{-j}R)\Big)(\varphi_R + \psi_R).$$

Now we notice that by Corollary 2, $\varphi_R + \psi_R \le cM_R(u)$. Assuming that cap_m is replaced in definition (38) by the equivalent Riesz capacity c_{2m} and using the subadditivity of this capacity, we see that

$$\varphi_\rho + \psi_\rho \le cM_R(u)\exp\Big(-c_0 \sum_{j=1}^{l} \frac{c_{2m}(\bar{B}_{2^{1-j}R} \setminus \Omega) - c_{2m}(\bar{B}_{2^{-j}R} \setminus \Omega)}{(2^{1-j}R)^{n-2m}}\Big). \tag{39}$$

Noting that the last sum is equal to

$$-\frac{c_{2m}(\bar{B}_{2^{-l}R} \setminus \Omega)^{n-2m}}{(2^{-l}R)^{n-2m}} + (1 - 2^{-n+2m}) \sum_{j=0}^{l-1} \frac{c_{2m}(\bar{B}_{2^{-j}R} \setminus \Omega)}{(2^{-j}R)^{n-2m}}$$
$$\ge c_1 \int_\rho^R \mathrm{cap}_m(\bar{B}_\tau \setminus \Omega) \frac{d\tau}{\tau^{n-2m+1}} - c_2,$$

we obtain the result from (39). □

By (37), we conclude that (6) is sufficient for the regularity of O.

6. Regularity as a local property

We show that the regularity of a point O does not depend on the geometry of Ω at any positive distance from O.

LEMMA 6

Let $n > 2m$, and let $L(\partial)$ be positive with the weight F. If O is regular for the operator L on Ω, then the solution $u \in \mathring{H}^m(\Omega)$ of

$$L(\partial)u = \sum_{\{\alpha:\, |\alpha| \le m\}} \partial^\alpha f_\alpha \quad \text{on } \Omega,$$

with $f_\alpha \in L_2(\Omega) \cap C^\infty(\Omega)$ and $f_\alpha = 0$ in a neighbourhood of O, satisfies (2).

Proof

Let $\zeta \in C_0^\infty(\Omega)$. We represent u as the sum $v + w$, where $w \in \mathring{H}^m(\Omega)$ and

$$L(\partial)v = \sum_{\{\alpha:\, |\alpha| \le m\}} \partial^\alpha(\zeta f_\alpha).$$

By the regularity of O, we have $v(x) = o(1)$ as $x \to O$. We verify that w can be made arbitrarily small by making the Lebesgue measure of the support of $1 - \zeta$ sufficiently small. Let $f_\alpha = 0$ on B_δ, and let $y \in \Omega$, $|y| < \delta/2$. By definition of w and by (26),

$$\sum_{\{\alpha:\, |\alpha| \le m\}} \int_\Omega (1-\zeta) f_\alpha (-\partial)^\alpha (wF_y)\, dx \ge 2^{-1} w^2(p) + c \sum_{k=1}^{m} \int_\Omega \frac{|\nabla_k w(x)|^2}{|x-y|^{n-2k}}\, dx,$$

where $F_y(x) = F_y(x-y)$ and c does not depend on Ω. The proof is complete. $\square$

LEMMA 7

Let O be a regular point for the operator $L(\partial)$ on Ω, and let Ω' be a domain such that $\Omega' \cap B_{2\rho} = \Omega \cap B_{2\rho}$ for some $\rho > 0$. Then O is regular for the operator $L(\partial)$ on Ω'.

Proof

Let $u \in \mathring{H}^m(\Omega')$ satisfy $L(\partial)u = f$ on Ω' with $f \in C_0^\infty(\Omega')$. We introduce $\eta_\rho(x) = \eta(x/\rho)$, $\eta \in C_0^\infty(B_2)$, $\eta = 1$ on $B_{3/2}$. Then $\eta_\rho u \in \mathring{H}^m(\Omega)$ and $L(\partial)(\eta_\rho u) = \eta_\rho f + [L(\partial), \eta_\rho]u$ on Ω. Since the commutator $[L(\partial), \eta_\rho]$ is a differential operator of order $2m-1$ with smooth coefficients supported by $B_{2\rho} \setminus \overline{B_{3\rho/2}}$, it follows that

$$L(\partial)(\eta_\rho u) = \sum_{\{\alpha\colon |\alpha| \le m\}} \partial^\alpha f_\alpha \quad \text{on } \Omega,$$

where $f_\alpha \in L_2(\Omega) \cap C^\infty(\Omega)$ and $f_\alpha = 0$ in a neighbourhood of O. Therefore, $(\eta_\rho u)(x) = o(1)$ as x tends to O by Lemma 6 and by the regularity of O with respect to $L(\partial)$ on Ω. $\square$

7. Proof of necessity in Theorem 2

Let $n > 2m$, and let condition (6) be violated. We fix a sufficiently small $\varepsilon > 0$ depending on the operator $L(\partial)$, and we choose a positive integer N in order to have

$$\sum_{j=N}^{\infty} 2^{(n-2m)j} \operatorname{cap}_m(\overline{B_{2^{-j}}} \setminus \Omega) < \varepsilon. \tag{40}$$

By Lemma 7, it suffices to show that O is irregular with respect to the domain $\mathbf{R}^n \setminus K$, where $K = \overline{B}_{2^{-N}} \setminus \Omega$. Denote by U_K the L-capacitary potential of K. By subtracting a cutoff function $\eta \in C_0^\infty(\mathbf{R}^n)$ used in the proof of Lemma 7 from U_K and noting that η is equal to 1 in a neighbourhood of K, we obtain a solution of $Lu = f$ on $\mathbf{R}^n \setminus K$ with $f \in C_0^\infty(\mathbf{R}^n)$ and zero Dirichlet data on $\partial(\mathbf{R}^n \setminus K)$. Therefore, it is sufficient to show that $U_K(x)$ does not tend to 1 as $x \to O$. This statement results from (40) and the inequality

$$\mathscr{M} U_K(0) \le c \sum_{j \ge N} 2^{(n-2m)j} \operatorname{cap}_m(\overline{B_{2^{-j}}} \setminus \Omega), \tag{41}$$

which is obtained in what follows.

We introduce the L-capacitary potential $U^{(j)}$ of the set $K^{(j)} = K \cap (\overline{B_{2^{1-j}} \setminus B_{2^{-1-j}}})$, $j = N, N+1, \dots$. We also need a partition of unity $\{\eta^{(j)}\}_{j \ge N}$ subordinate to the covering of K by the sets $B_{2^{1-j}} \setminus \overline{B_{2^{-1-j}}}$. One can construct this partition of unity so that $|\nabla_k \eta^{(j)}| \le c_k 2^{kj}$, $k = 1, 2, \ldots$. We now define the function

$$V = \sum_{j \ge N} \eta^{(j)} U^{(j)} \tag{42}$$

satisfying the same Dirichlet conditions as U_K. Let $Q_u(y)$ denote the quadratic form

$$\sum_{k=1}^{m} \int_{\mathbf{R}^n} \frac{|\nabla_k u(x)|^2}{|x-y|^{n-2k}}\, dx,$$

and let $I_\lambda f$ be the Riesz potential $|x|^{\lambda-n} * f$, $0 < \lambda < n$. It is standard that $\mathscr{M} I_\lambda f(0) \le c I_\lambda f(0)$ if $f \ge 0$ (see the proof of [L, Th. 1.11]). Hence,

$$\mathscr{M} Q_u(0) \le c \sum_{k=1}^{m} \int_{\mathbf{R}^n} |\nabla_k u(x)|^2 \frac{dx}{|x|^{n-2k}}.$$

This inequality and definition (42) show that

$$\begin{aligned}\mathscr{M} Q_V(O) &\le \sum_{j\ge N} \sum_{k=0}^{m} \int_{B_{2^{1-j}} \setminus B_{2^{-1-j}}} |\nabla_k U^{(j)}(x)|^2 \frac{dx}{|x|^{n-2k}} \\ &\le c \sum_{j \ge N} 2^{(n-2m)j} \int_{\mathbf{R}^n} |\nabla_k U^{(j)}(x)|^2 \frac{dx}{|x|^{2(m-k)}} \\ &\le c \sum_{j\ge N} 2^{(n-2m)j} \int_{\mathbf{R}^n} |\nabla_m U^{(j)}(x)|^2\, dx,\end{aligned}$$

the last estimate being based on Hardy's inequality. Therefore,

$$\mathscr{M} Q_V(0) \le c \sum_{j \ge N} 2^{(n-2m)j} \operatorname{cap}_m K^{(j)}. \tag{43}$$

Furthermore, by Proposition 2,

$$\mathscr{M} V(0) \le c \sum_{j\ge N} 2^{(n-2m)j} \operatorname{cap}_m K^{(j)}. \tag{44}$$

We deduce similar inequalities for $W = U_K - V$. Note that W solves the Dirichlet problem with zero boundary data for the equation $L(\partial)W = -L(\partial)V$ on $\mathbf{R}^n \setminus K$. Hence and by (26), we conclude that for $y \in \mathbf{R}^n \setminus K$,

$$2^{-1} W(y)^2 + c Q_W(y) \le \Big| \int_{\mathbf{R}^n} \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta} \partial^\alpha V(x) \cdot \partial^\beta \big(W(x) F(x-y)\big)\, dx \Big|, \tag{45}$$

which implies

$$2^{-1}\mathscr{M}W^2(0)+c\mathscr{M}Q_W(0)\le c\sum_{k=0}^{m}\int_{\mathbf{R}^n}|\nabla_k W(x)||\nabla_m V(x)|\,\frac{dx}{|x|^{n-m-k}}. \tag{46}$$

Since $0<U<2$ and $0<V<2$, we have $|W|<2$, and so the term in (46) corresponding to $k=0$ does not exceed

$$2\int_{\mathbf{R}^n}|\nabla_m V(x)|\frac{dx}{|x|^{n-m}}\le c\sum_{j\ge N}\int_{\mathbf{R}^n}\left|\nabla_m\eta^{(j)}(x)U^{(j)}(x)\right|\frac{dx}{|x|^{n-m}}.$$

Applying Proposition 2 to each potential $U^{(j)}$, we obtain

$$\int_{\mathbf{R}^n}|\nabla_m V(x)|\frac{dx}{|x|^{n-m}}\le c\sum_{j\ge N}2^{(n-2m)j}\operatorname{cap}_m K^{(j)}. \tag{47}$$

The terms with $k>0$ in the right-hand side of (46) do not exceed the value $cQ_W(0)^{1/2}Q_V(0)^{1/2}$. This, along with (47) and (43), leads to the estimate

$$2^{-1}\mathscr{M}W^2(0)+c\mathscr{M}Q_W(0)\le c\sum_{j\ge N}2^{(n-2m)j}\operatorname{cap}_m K^{(j)}. \tag{48}$$

We are ready to obtain (41). Owing to (15), $\mathscr{M}U_K(0)\le 2^{-1}\mathscr{M}U_K^2(0)+c\mathscr{M}Q_{U_K}(0)$, and since $U_K=V+W$, inequality (41) follows from (43), (44), and (48). The proof is complete.

8. Proof of sufficiency in Theorem 1

In the case of $n=2m$, the operator $L(\partial)$ is arbitrary. We introduce a sufficiently large positive constant C subject to a condition specified later. We also need a fundamental solution

$$F(x)=\varkappa\log|x|^{-1}+\Psi\Big(\frac{x}{|x|}\Big) \tag{49}$$

of $L(\partial)$ in $\mathbf{R}^n$ (see [J]). Here $\varkappa=\text{const}$, and we assume that the function Ψ, which is defined up to a constant term, is chosen so that

$$F(x)\ge\varkappa\log(4|x|^{-1})+C\quad\text{on } B_2. \tag{50}$$

PROPOSITION 7

Let Ω be an open set in $\mathbf{R}^n$ with diameter d_Ω. Then for all $u\in C_0^\infty(\Omega)$ and $y\in\Omega$,

$$\int_\Omega L(\partial)u(x)\cdot u(x)F\Big(\frac{x-y}{d_\Omega}\Big)\,dx-2^{-1}u(y)^2 \ge c\sum_{j=1}^{m}\int_\Omega\frac{|\nabla_j u(x)|^2}{|x-y|^{2(m-j)}}\log\frac{4d_\Omega}{|x-y|}\,dx. \tag{51}$$

Everywhere in this section, by c we denote positive constants independent of Ω.

Proof

It suffices to assume $d_\Omega = 1$. By Lemma 1, the left-hand side in (51) is equal to the quadratic form

$$\mathscr{H}_u(y) = \int_\Omega \sum_{j=1}^{m} \sum_{|\mu|=|\nu|=j} \partial^\mu u \cdot \partial^\nu u \cdot P_{\mu\nu}(\partial) F(x-y)\, dx.$$

By Hardy's inequality,

$$\Big| \mathscr{H}_u(y) - \int_\Omega \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta} \partial^\alpha u(x) \cdot \partial^\beta u(x) \cdot F(x-y)\, dx \Big| \\ \leq c \sum_{j=1}^{m-1} \int_\Omega \frac{|\nabla_j u(x)|^2}{|x-y|^{2(m-j)}}\, dx \leq c \int_\Omega |\nabla_m u(x)|^2\, dx.$$

Hence, there exist constants c_1 and c_2 such that

$$c_1 \mathscr{H}_u(y) \leq \int_\Omega |\nabla_m u(x)|^2 \log(4|x-y|^{-1})\, dx \leq c_2 \mathscr{H}_u(y). \tag{52}$$

(Here we used the fact that the constant C in (50) is sufficiently large in order to obtain the right-hand inequality.) By the Hardy-type inequality

$$\int_\Omega \frac{|\nabla_j u(x)|^2}{|x-y|^{2(m-j)}} \log(4|x-y|^{-1})\, dx \leq c \int_\Omega |\nabla_m u(x)|^2 \log(4|x-y|^{-1})\, dx, \tag{53}$$

we can also write

$$\int_\Omega \frac{|\nabla_j u(x)|^2}{|x-y|^{2(m-j)}} \log(4|x-y|^{-1})\, dx \leq c\, \mathscr{H}_u(y). \tag{54}$$

The proof is complete. □

LEMMA 8

Let $n = 2m$, and let $u \in \mathring{H}^m(\Omega)$ be subject to (31). Then for an arbitrary point $y \in B_\rho$, $\rho \leq 1$,

$$u(y)^2 + \mathscr{B}(u\eta_\rho, u\eta_\rho F_{y,\rho}) \leq c\, M_\rho(u),$$

where $\mathscr{B}$, η_ρ, and $M_\rho(u)$ are the same as in Lemma 3, $F_{y,\rho}(x) = F((x-y)/2\rho)$, and F is given by (49).

Proof

We majorize the second term by repeating the proof of Lemma 3. Then the first term is estimated by (51), where the role of Ω is played by $\Omega \cap B_{2\rho}$ and u is replaced by $u\eta_\rho$. The result follows. □

Combining Proposition 7 with $\Omega \cap B_{2\rho}$ and $u\eta_\rho$ instead of Ω and u, with Lemma 8 we obtain the following local estimate similar to (35).

LEMMA 9
Let the conditions of Lemma 8 be satisfied. Then, for all $y \in \Omega \cap B_\rho$, $\rho \leq 1$, the following estimate holds:

$$u(y)^2 + \int_{\Omega\cap B_\rho} \sum_{k=1}^{m} \frac{|\nabla_k u(y)|^2}{|x-y|^{n-2k}} \log(4\rho|x-y|^{-1})\, dx \leq c\, M_\rho(u). \tag{55}$$

Now we are in a position to finish the proof of sufficiency in Theorem 1.

Let $n = 2m$, and let $u \in \mathring{H}^m(\Omega)$ and $L(\partial)u = 0$ on $\Omega \cap B_{2\rho}$. We diminish the right-hand side in (55) replacing B_ρ by $B_\rho \backslash B_\varepsilon$ with an arbitrarily small $\varepsilon > 0$. The integral obtained is continuous at $y = 0$. Hence,

$$\int_{\Omega\cap B_\rho} \sum_{k=1}^{m} \frac{|\nabla_k u(x)|^2}{|x|^{n-2k}} \log(4\rho|x|^{-1})\, dx \leq c\, M_\rho(u). \tag{56}$$

Putting here $\rho = 1$ and $\gamma_m(r) = \mathrm{cap}_m(\overline{S_r}\backslash\Omega, B_{4r})$, we estimate the left-hand side from below by using the estimate

$$M_\rho(u) \leq \frac{c}{\gamma_m(r)} \int_{\Omega\cap S_r} \sum_{k=1}^{m} \frac{|\nabla_k u(x)|^2}{\rho^{n-2k}}\, dx$$

proved in [M1] (see also [M2, Sec. 10.1.2]). We have

$$\sum_{j\geq 1} j\gamma_m(2^{-j}) M_{2^{-j}}(u) \leq c\, M_1(u).$$

Hence and by (55),

$$\sum_{j=1}^{\infty} j\gamma_m(2^{-j}) \sup_{\Omega\cap B_{2^{-j}}} u^2 \leq c\, M_1(u).$$

Suppose that O is irregular. Assuming that $\lim_{j\to\infty} \sup_{\Omega\cap B_{2^{-j}}} u^2 > 0$, we have

$$\sum_{j=1}^{\infty} j\gamma_m(2^{-j}) < \infty. \tag{57}$$

Since $\mathrm{cap}_m(\overline{S_r}\backslash\Omega, B_{4r}) \geq \mathrm{cap}_m(\overline{S_r}\backslash\Omega) \geq c\; C_{2m}(\overline{S_r}\backslash\Omega)$ for $r \leq 1$ (see Sec. 2) and since the Bessel capacity is subadditive, we obtain the estimate $\gamma_m(2^{-j}) \geq$

$c(C_{2m}(\bar{B}_{2^{1-j}}\backslash\Omega) - C_{2m}(\bar{B}_{2^{-j}}\backslash\Omega))$. Hence and by Abel's summation, we conclude that

$$\sum_{j=1}^{\infty} C_{2m}(\bar{B}_{2^{-j}}\backslash\Omega) < \infty;$$

that is, condition (57) is violated. The result follows.

9. Proof of necessity in Theorem 1

By $G(x, y)$, we denote Green's function of the Dirichlet problem for $L(\partial)$ on the ball B_1. Also, we use the fundamental solution F given by (49). As is well known and easily checked, for all x and y in $B_{4/5}$,

$$\Big|G(x, y) - F(x - y)\Big| \leq c, \tag{58}$$

where c is a constant depending on $L(\partial)$. Hence, there exists a sufficiently small κ such that for all y in the ball $B_{3/4}$ and for all x subject to $|x - y| \leq \kappa$,

$$c_1 \log(2\kappa|x - y|^{-1}) \leq G(x, y) \leq c_2 \log(2\kappa|x - y|^{-1}), \tag{59}$$

and for all multi-indices α, β with $|\alpha| + |\beta| > 0$,

$$|\partial_x^\alpha \partial_y^\beta G(x, y)| \leq c_{\alpha,\beta}|x - y|^{-|\alpha|-|\beta|}. \tag{60}$$

Moreover, $G(x, y)$ and its derivatives are uniformly bounded for all x and y in B_1 with $|x - y| > \kappa$. By Lemma 1, for all $u \in C_0^\infty(B_1)$,

$$\int_{B_1} L(\partial)u \cdot uG_y\, dx = 2^{-1}u(y)^2 + \int_{B_1} \sum_{j=1}^{m} \sum_{|\mu|=|\nu|=j} \partial^\mu u \cdot \partial^\nu u \cdot P_{\mu\nu}(\partial)G_y\, dx,$$

where $y \in B_1$ and $G_y(x) = G(x, y)$. Hence, using the same argument as in Lemma 2, we see that for an arbitrary compact set K in $\bar{B}_1$ and for all $y \in B_1\backslash K$ the L-capacitary potential with respect to B_1 satisfies

$$U_K(y) = \frac{1}{2}U_K(y)^2 + \int_{B_1} \sum_{j=1}^{m} \sum_{|\mu|=|\nu|=j} \partial^\mu U_K \cdot \partial^\nu U_K \cdot P_{\mu\nu}(\partial)G_y\, dx. \tag{61}$$

(Note that the notation U_K was used in the case of $n < 2m$ in a different sense.)

LEMMA 10
Let K be a compact subset of $\bar{B}_{1/2}$. For all $y \in B_1\backslash K$, the following inequality holds:

$$|U_K(y) - 1| \leq 1 + c\ \operatorname{cap}_m(K, B_1), \tag{62}$$

where (and in the sequel) by c we denote positive constants independent of K.

Proof

Since $L(\partial)U_K = 0$ on $B_1\backslash B_{1/2}$ and since U_K satisfies zero Dirichlet conditions on ∂B_1, it is standard that $\sup_{B_1\backslash B_{3/4}} |U_K| \le c\ \sup_{B_{3/4}\backslash B_{1/2}} |U_K|$ (see [ADN, Chap. 3]). So we only need to check (62) for $y \in B_{3/4}\backslash K$. By (61) and (60),

$$\big(U_K(y) - 1\big)^2 \le 1 - \int_{B_1} a_{\alpha\beta}\partial^\alpha U_K \cdot \partial^\beta U_K \cdot G_y\, dx \\ + c\sum_{j=1}^{m-1} \int_{B_1} |\nabla_j U_K(x)|^2 |x-y|^{2j-n}\, dx.$$

From (59) and Hardy's inequality

$$\int_{B_1} |\nabla_j U_K(x)|^2 |x-y|^{2j-n}\, dx \le c\int_{B_1} |\nabla_m U_K(x)|^2\, dx, \quad 1 \le j \le m,$$

it follows that

$$\big(U_K(y) - 1\big)^2 \le 1 - c_1\int_{B_\kappa(y)} |\nabla_m U_K(x)|^2 \log(4\kappa|x-y|^{-1})\, dx \\ + c\int_{B_1} |\nabla_m U_K(x)|^2\, dx \le 1 + c_2\,\mathrm{cap}_m(K, B_1),$$

which is equivalent to (62). □

LEMMA 11

Let $n = 2m$, and let K be a compact subset of $\bar{B}_1\backslash B_{1/2}$. Then the L-capacitary potential U_K with respect to B_2 satisfies

$$\mathscr{M}\nabla_l U_K(0) \le c\ \mathrm{cap}_m(K, B_2) \quad \textit{for } l = 0, 1, \ldots, m.$$

Proof

It follows from (61) and (53) that U_K satisfies the inequalities

$$|U_K(y)| \le c\Big(U_K(y)^2 + \int_{B_2} |\nabla_m U_K(x)|^2 \log(4|x-y|^{-1})\, dx\Big),$$
$$|\nabla_l U_K(y)| \le c\Big(|\nabla_l U_K(y)^2| + \int_{B_2} \sum_{\substack{1\le r,s\le m\\ r+s>l}} \frac{|\nabla_r U_K(x)||\nabla_s U_K(x)|}{|x-y|^{n-r-s+l}}\, dx\Big)$$

(cf. the proof of Cor. 1). It remains to repeat the proof of Proposition 2 with the above inequalities playing the role of (18). □

LEMMA 12

Let $n = 2m$, and let K be a compact subset of $\bar{B}_\delta$, $\delta < 1$, subject to

$$C_{2m}(K) \leq \frac{\varepsilon(m)}{\log(2/\delta)}, \tag{63}$$

where $\varepsilon(m)$ is a sufficiently small constant independent of K and δ. Then there exists a constant $c(m)$ such that $\operatorname{cap}_m(K, B_{2\delta}) \leq c(m)\, C_{2m}(K)$.

Proof

Let $\delta^{-1}K$ denote the image of K under the δ^{-1}-dilation. Clearly, $\operatorname{cap}_m(K, B_{2\delta}) = \operatorname{cap}_m(\delta^{-1}K, B_2)$. By using a cutoff function, one shows that $\operatorname{cap}_m(\delta^{-1}K, B_2)$ does not exceed $c\ \inf\{\sum_{0\leq k\leq m} ||\nabla_k u||^2_{L_2(\mathbf{R}^n)} :\ u \in C_0^\infty(\mathbf{R}^n),\ u = 1$ in a neighbourhood of $\delta^{-1}K\}$. Now we recall that by allowing the admissible functions to satisfy the inequality $U \geq 1$ on K in the last infimum, one arrives at the capacity of $\delta^{-1}K$ equivalent to $C_{2m}(\delta^{-1}K)$. Hence, it is enough to verify that

$$C_{2m}(\delta^{-1}K) \leq c\, C_{2m}(K). \tag{64}$$

We denote by $P\mu$ the $2m$-order Bessel potential of a measure μ and by G_{2m} the kernel of the integral operator P. Let μ_K be the corresponding equilibrium measure of K. Since $K \subset \bar{B}_\delta$ and $\delta < 1$, we obtain for all $y \in K$ except for a subset of K with zero capacity C_{2m},

$$\begin{aligned}
\int_K G_{2m}\big(\delta^{-1}(x-y)\big)\, d\mu_K(x) &\geq c \int_K \log(\delta|x-y|^{-1})\, d\mu_K(x) \\
&\geq c\Big(\int_K \log(2|x-y|^{-1})\, d\mu_K(x) - C_{2m}(K)\log(2\delta^{-1})\Big) \\
&\geq c\Big(\int_K G_{2m}(x-y)\, d\mu_K(x) - \varepsilon(m)\Big) \geq c_0\big(1-\varepsilon(m)\big).
\end{aligned}$$

Thus, for the measure $\mu^{(\delta)}(\xi) = c_0^{-1}(1-\varepsilon(m))^{-1}\mu_K(\delta\xi)$ which is supported by $\delta^{-1}K$, we have $P\mu^{(\delta)} \geq 1$ on $\delta^{-1}K$ outside a subset with zero capacity C_{2m}. Therefore,

$$\begin{aligned}
C_{2m}(\delta^{-1}K) &\leq \langle P\mu^{(\delta)}, \mu^{(\delta)}\rangle \\
&= c_0^{-2}\big(1-\varepsilon(m)\big)^{-2} \int_K\int_K G_{2m}\big(\delta^{-1}(x-y)\big)\, d\mu_K(x)\, d\mu_K(y),
\end{aligned} \tag{65}$$

where $\langle P\mu^{(\delta)}, \mu^{(\delta)}\rangle$ denotes the energy of $\mu^{(\delta)}$. Now we note that

$$G_{2m}\big(\delta^{-1}(x-y)\big) \leq c\log(4\delta|x-y|^{-1}) < c\log(4|x-y|^{-1}) \leq c_1 G_{2m}(x-y)$$

for x and y in K. This and (65), combined with the fact that the energy of μ_K is equal to $C_{2m}(K)$, complete the proof of the lemma. □

Suppose that O is regular with respect to the set Ω. Assuming that

$$\int_0^1 C_{2m}(\bar{B}_r\backslash\Omega)\frac{dr}{r} < \infty, \tag{66}$$

we arrive at a contradiction. We fix a sufficiently small $\varepsilon > 0$ and choose a positive integer N so that

$$\sum_{j=N}^{\infty} C_{2m}(\bar{B}_{2^{-j}}\backslash\Omega) < \varepsilon. \tag{67}$$

Let $K = \bar{B}_{2^{-N}}\backslash\Omega$, and let U_K denote the L-capacitary potential of K with respect to B_1. We note that using (51) one can literally repeat the proof of locality of the regularity property given in Lemma 8. Therefore, O is regular with respect to $B_1\backslash K$, which implies $U_K(x) \to 1$ as $x \to O$, $x \in B_1\backslash K$. It suffices to show that this is not the case. It is well known that (67) implies

$$\sum_{j\geq N} jC_{2m}(K^{(j)}) \leq c\,\varepsilon,$$

where $K^{(j)} = \{x \in K : 2^{-1-j} \leq |x| \leq 2^{1-j}\}$ and c depends only on n. A proof can be found in [H, p. 240] for $m = 1$, and no changes are necessary to apply the argument for $m > 1$. Hence and by Lemma 12, we obtain

$$\sum_{j\geq N} j\,\mathrm{cap}_m(K^{(j)}, B_{2^{2-j}}) \leq c\,\varepsilon. \tag{68}$$

We use the partition of unity $\{\eta^{(j)}\}_{j\geq N}$ introduced at the beginning of Section 9, and by $U^{(j)}$ we denote the L-capacitary potential of $K^{(j)}$ with respect to $B_{2^{2-j}}$. We also need the function V defined by (42) with the new $U^{(j)}$. Let

$$T^{(j)}(y) = \sum_{k=1}^{m}\int_{B_1}\frac{|\nabla_k U^{(j)}(x)|^2}{|x-y|^{n-2k}}\log\frac{2^{4-j}}{|x-y|}\,dx.$$

By (53),

$$T^{(j)}(y) = c\int_{B_1}|\nabla_m U^{(j)}(x)|^2\log\frac{2^{4-j}}{|x-y|}\,dx,$$

and therefore for $r \leq 1$,

$$\begin{aligned} r^{-n}\int_{B_r} T^{(j)}(y)\,dy &\leq c\int_{B_{2^{2-j}}}|\nabla_m U^{(j)}(x)|^2\log\frac{2^{4-j}}{r+|x|}\,dx \\ &\leq c\log\Big(\frac{2^{4-j}}{r}\Big)\,\mathrm{cap}(K^{(j)}, B_{2^{2-j}}). \end{aligned}$$

Hence and because supp $\eta^{(j)} \subset B_{2^{1-j}} \backslash \bar{B}_{2^{-1-j}}$, we have

$$\mathscr{M}(\eta^{(j)} T^{(j)})(0) \leq c \ \operatorname{cap}_m(K^{(j)}, B_{2^{2-j}}). \tag{69}$$

Furthermore, by (61) and Lemma 10,

$$\begin{aligned}\mathscr{M}(\eta^{(j)} U^{(j)})(0) \leq 2^{-1}\big(1 + c_0 \operatorname{cap}_m(K^{(j)}, B_{2^{2-j}})\big)\mathscr{M}(\eta^{(j)} U^{(j)})(0)\\ + c_1 \mathscr{M}(\eta^{(j)} T^{(j)})(0).\end{aligned}$$

Since we may have $\operatorname{cap}_m(K^{(j)}, B_{2^{2-j}}) \leq (2c_0)^{-1}$ by choosing a sufficiently small ε, we obtain $\mathscr{M}(\eta^{(j)} U^{(j)})(0) \leq 4c_1 \mathscr{M}(\eta^{(j)} T^{(j)})(0)$, and by (69),

$$\mathscr{M}(\eta^{(j)} U^{(j)})(0) \leq c \ \operatorname{cap}_m(K^{(j)}, B_{2^{2-j}}), \tag{70}$$

which implies

$$\mathscr{M} V(0) \leq c \sum_{j \geq N} \operatorname{cap}(K^{(j)}, B_{2^{2-j}}). \tag{71}$$

We introduce the function

$$T_u(y) = \sum_{k=1}^{m} \int_{B_1} \frac{|\nabla_k u(x)|^2}{|x-y|^{n-2k}} \log(4|x-y|^{-1})\, dy.$$

By (53),

$$\begin{aligned}T_V(y) &\leq c \int_{B_1} \big(\nabla_m V(x)\big)^2 \log(4|x-y|^{-1})\, dy\\ &\leq c \sum_{j \geq N} \int_{B_1} |\nabla_m(\eta^{(j)} U^{(j)})(x)|^2 \log(4|x-y|^{-1})\, dx.\end{aligned}$$

Hence, for $r \leq 1$,

$$\begin{aligned}r^{-n} \int_{B_r} T_V(y)\, dy &\leq c \sum_{j \geq N} \int_{B_{2^{1-j}} \backslash B_{2^{-1-j}}} |\nabla_m(\eta^{(j)} U^{(j)})(x)|^2 \log \frac{4}{|x| + r}\, dx\\ &\leq c \sum_{j \geq N} j \int_{B_1} |\nabla_m(\eta^{(j)} U^{(j)})(x)|^2\, dx.\end{aligned} \tag{72}$$

Clearly,

$$\begin{aligned}\int_{B_1} |\nabla_m(\eta^{(j)} U^{(j)})(x)|^2\, dx \leq c \int_{B_1} |\nabla_m \eta^{(j)}(x)|^2 U^{(j)}(x)^2\, dx\\ + c \sum_{k=1}^{m} \int_{B_1} \frac{|\nabla_k U^{(j)}(x)|^2}{|x|^{2(m-k)}}\, dx.\end{aligned} \tag{73}$$

Owing to Hardy's inequality, each term in the last sum is majorized by

$$c \int_{B_1} |\nabla_m U^{(j)}(x)|^2 \, dx = c \ \mathrm{cap}_m(K^{(j)}, B_{2^{-j}}).$$

By Lemma 9, the first integral in the right-hand side of (73) is dominated by

$$c\ 2^{2mj} \int_{\mathrm{supp}\, \eta^{(j)}} U^{(j)}(x)^2 \, dx \le c\ \mathscr{M}(\zeta^{(j)} U^{(j)})(0),$$

where $\zeta^{(j)}$ is a function in $C_0^\infty(B_{2^{1-j}} \backslash \bar{B}_{2^{-1-j}})$ equal to 1 on the support of $\eta^{(j)}$. Now we note that (70) is also valid with $\eta^{(j)}$ replaced by $\zeta^{(j)}$. Hence,

$$\int_{B_1} |\nabla_m (\eta^{(j)} U^{(j)})(x)|^2 \, dx \le c \ \mathrm{cap}_m(K^{(j)}, B_{2^{2-j}}), \tag{74}$$

which combined with (72) gives

$$\mathscr{M} T_V(0) \le c \sum_{j \ge N} j \ \mathrm{cap}(K^{(j)}, B_{2^{2-j}}). \tag{75}$$

We turn to estimating the function $W = U_K - V$, which solves the Dirichlet problem for the equation

$$L(\partial) W = -L(\partial) V \quad \text{on } B_1 \backslash K. \tag{76}$$

It follows from (51) that for $y \in B_1 \backslash K$,

$$\begin{aligned} 2^{-1} W(y)^2 + c \int_{B_1} \big(\nabla_m W(x)\big)^2 \log(4|x-y|^{-1}) \, dx \\ \le \int_{B_1} \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta} \partial^\alpha V(x) \cdot \partial^\beta \big(W(x) F(x-y)\big) \, dx. \end{aligned} \tag{77}$$

Hence and by (49),

$$\begin{aligned} W(y)^2 + \int_{B_1} \big(\nabla_m W(x)\big)^2 \log(4|x-y|^{-1}) \, dx \\ \le c \Big(\int_{B_1} |\nabla_m V(x)| |W(x)| \frac{dx}{|x-y|^{n-m}} \\ + \int_{B_1} |\nabla_m V(x)| \sum_{k=1}^{m-1} |\nabla_k W(x)| \frac{dx}{|x-y|^{n-m-k}} \\ + \int_{B_1} |\nabla_m V(x)| |\nabla_m W(x)| \log(4|x-y|^{-1}) \, dx \Big). \end{aligned} \tag{78}$$

Since both $|U_K|$ and $|V|$ are bounded by a constant depending on L, the same holds for $|W|$. Thus, the integral on the right containing $|W|$ is majorized by

$$c\int_{B_1} |\nabla_m V(x)| \frac{dx}{|x-y|^{n-m}}.$$

Obviously, two other integrals in the right-hand side of (78) are not greater than

$$c\, T_V(y)^{1/2}\Big(\sum_{k=1}^{m-1}\int_{B_1}\frac{(\nabla_k W(x))^2}{|x-y|^{n-2k}}\,dx+\int_{B_1}\big(\nabla_m W(x)\big)^2\log\frac{4}{|x-y|}\,dx\Big)^{1/2}.$$

By Hardy's inequality, we can remove the sum in k enlarging the constant c. Hence and by (78),

$$W(y)^2+\int_{B_1}\big(\nabla_m W(x)\big)^2\log\frac{4}{|x-y|}\,dx\le c\Big(\int_{B_1}|\nabla_m V(x)|\frac{dx}{|x-y|^{n-m}}+T_V(y)\Big).$$

Hence and by $U_K = V+W$, we arrive at

$$\begin{aligned}U_K(y)^2+c\int_{B_1}\big(\nabla_m U_K(x)\big)^2\log\frac{4}{|x-y|}\,dx\\ \le c\Big(V(y)^2+T_V(y)+\int_{B_1}|\nabla_m V(x)|\frac{dx}{|x-y|^{n-m}}\Big).\end{aligned}$$

The left-hand side is not less than $c|U_K(y)|$ by (61). Therefore,

$$\mathscr{M}U_K(0)\le c\Big(\mathscr{M}V^2(0)+\mathscr{M}T_V(0)+\int_{B_1}|\nabla_m V(x)|\frac{dx}{|x|^{n-m}}\Big).$$

By Lemma 10, $|V|\le c$. This, along with (71) and (75), implies

$$\mathscr{M}V^2(0)+\mathscr{M}T_V(0)\le\sum_{j\ge N} j\,\mathrm{cap}(K^{(j)}, B_{2^{2-j}}).$$

It follows from the definition of V and from Lemma 11 that

$$\begin{aligned}\int_{B_1}\frac{|\nabla_m V(x)|}{|x|^{n-m}}\,dx &\le c\sum_{j\ge N}2^{(n-m)j}\int_{B_{2^{2-j}}}|\nabla_m(\eta^{(j)}U^{(j)})(x)|\,dx\\ &\le c\sum_{j\ge N}\mathrm{cap}_m(K^{(j)}, B_{2^{2-j}}).\end{aligned}$$

Finally,

$$\mathscr{M}U_K(0)\le c\sum_{j\ge N} j\,\mathrm{cap}_m(K^{(j)}, B_{2^{2-j}}),$$

and the contradiction required is a consequence of (69). The necessity of (3) for the regularity of O follows.

10. The biharmonic equation in a domain with inner cusp ($n \geq 8$)
Let the bounded domain Ω be described by the inequality $x_n < f(x')$, $x' = (x_1, \dots, x_{n-1})$, on B_1, where f is a continuous function on the ball $\{x' : |x'| < 1\}$, subject to the conditions: $f(0) = 0$, f is smooth for $x' \neq 0$, and $\partial f/\partial|x'|$ is a decreasing function of $|x'|$ which tends to $+\infty$ as $|x'| \to 0$.

These conditions show that at the point O the surface $\partial\Omega$ has a cusp that is directed inside Ω.

THEOREM 3
Let $n \geq 8$, and let u solve the Dirichlet problem

$$\Delta^2 u = f, \quad u \in \mathring{H}^2(\Omega),$$

where $f \in C_0^\infty(\Omega)$. If

$$\int_0^1 C_4(\overline{B_\rho}\backslash\Omega)\frac{d\rho}{\rho^{n-3}} = \infty, \tag{79}$$

then $u(x) \to 0$ as x tends to O along any nontangential direction.

Proof
By ν_x we denote the exterior normal to $\partial\Omega$ at the point $x \in (B_1 \cap \partial\Omega)\backslash O$. We introduce the function family $\{f_\varepsilon\}$ by $f_\varepsilon(x') = (f(x') - \varepsilon)_+ + \varepsilon$. Replacing $x_n < f(x')$ in the definition of Ω by $x_n < f_\varepsilon(x')$, we obtain the family of domains Ω_ε such that $O \in \Omega_\varepsilon$ and $\Omega_\varepsilon \downarrow \Omega$ as $\varepsilon \downarrow 0$.

By the implicit function theorem, the set $E_\varepsilon = \{x : x_n = f(x') = \varepsilon\}$ is a smooth $(n-2)$-dimensional surface for sufficiently small ε. In a neighbourhood of any point of E_ε, the boundary of Ω_ε is diffeomorphic to a dihedral angle. It follows from our conditions on f that the two hyperplanes, which are tangent to $\partial\Omega$ at any point of the edge E_ε, form a dihedral angle with opening $> 3\pi/2$ (from the side of Ω). Then, as is well known, the solution of the Dirichlet problem

$$\Delta^2 u_\varepsilon = f, \quad u_\varepsilon \in \mathring{H}^m(\Omega_\varepsilon),$$

satisfies the estimate

$$|\nabla_j u_\varepsilon(x)| = O\big(\mathrm{dist}(x, E_\varepsilon)^{-j+\lambda}\big), \tag{80}$$

where $\lambda > 3/2$ (see, e.g., [MP1, Th. 10.5] combined with [KMR, Sec. 7.1]). The value of λ can be made more precise, but this is irrelevant for us. In fact, we only need (80) to justify the integration by parts in what follows.

By y, we denote a point on the semiaxis $x' = 0$, $x_n \leq 0$, at a small distance from O. Let (r, ω) be spherical coordinates centered at y, and let G denote the image of

Ω_ε under the mapping $x \to (t, \omega)$, where $t = -\log r$. For $u_\varepsilon(x)$ written in the coordinates (t, ω), we use the notation $v(t, \omega)$. Also, let δ_ω denote the Laplace-Beltrami operator on ∂B_1, and let ∂_t, ∂_t^2, and so on, denote partial derivatives with respect to t. Since $\Delta = e^{2t}(\partial_t^2 - (n-2)\partial_t + \delta_\omega)$, we have $\Delta^2 = e^{4t}\Lambda$, where

$$\begin{aligned}\Lambda &= \big((\partial_t+2)^2 - (n-2)(\partial_t+2) + \delta_\omega\big)\big(\partial_t^2 - (n-2)\partial_t + \delta_\omega\big)\\ &= \partial_t^4 + 2\partial_t^2\delta_\omega + \delta_\omega^2 - 2(n-4)(\partial_t^3 + \partial_t\delta_\omega) - 2(n-4)\delta_\omega\\ &\quad + (n^2 - 10n + 20)\partial_t^2 + 2(n-2)(n-4)\partial_t.\end{aligned}$$

Consider the integral

$$I_1 = \int_{\Omega_\varepsilon} \Delta^2 u_\varepsilon \cdot \frac{\partial u_\varepsilon}{\partial r}\frac{dx}{r^{n-5}} = \int_G \Lambda v \cdot \partial_t v \, dt\, d\omega.$$

Integrating by parts in the right-hand side, we obtain

$$\begin{aligned}I_1 &= 2(n-4)\int_G \big((\partial_t^2 v)^2 + (\operatorname{grad}_\omega \partial_t v)^2 + (n-2)(\partial_t v)^2\big)\, dt\, d\omega\\ &\quad - \frac{1}{2}\int_{\partial G}\big((\partial_t v)^2 + 2(\operatorname{grad}_\omega \partial_t v)^2 + (\delta_\omega v)^2\big)\cos(\nu, t)\, ds.\end{aligned}$$

Since the angle between ν and the vector $x - y$ does not exceed $\pi/2$, we have $\cos(\nu, t) \le 0$ and therefore

$$2(n-4)\int_G \big((\partial_t v)^2 + (\operatorname{grad}_\omega \partial_t v)^2 + (n-2)(\partial_t v)^2\big)\, dt\, d\omega \le I_1. \tag{81}$$

We make use of another integral

$$I_2 = \int_{\Omega_\varepsilon} \Delta^2 u_\varepsilon \cdot u_\varepsilon \frac{dx}{r^{n-4}} = \int_G \Lambda v \cdot v \, dt\, d\omega. \tag{82}$$

We remark that $y \in \Omega_\varepsilon$ implies

$$2\int_G \partial_t v \cdot v\, dt\, d\omega = \int_{\partial B_1}\big(v(+\infty, \omega)\big)^2 d\omega = \omega_{n-1}\big(u_\varepsilon(y)\big)^2.$$

After integrating by parts in (82), we obtain

$$\begin{aligned}&\int_G \big((\partial_t^2 v)^2 + (\delta_\omega v)^2 + 2(\operatorname{grad}_\omega v_t)^2 + 2(n-4)(\operatorname{grad}_\omega v)^2\\ &\qquad - (n^2 - 10n + 20)(\partial_t v)^2\big)\, dt\, d\omega + \omega_{n-1}(n-2)(n-4)\big(u_\varepsilon(y)\big)^2 \le I_2.\end{aligned}$$

Combining this inequality with (81), we arrive at

$$\begin{aligned}&\int_G \big(2(n-3)(\partial_t^2 v)^2 + 2(n-2)(\operatorname{grad}_\omega \partial_t v)^2 + 2(\delta_\omega v)^2 + 4(n-4)(\operatorname{grad}_\omega v)^2\\ &\qquad + 8(n-3)(\partial_t v)^2\big)\, dt\, d\omega + 2\omega_{n-1}(n-2)(n-4)\big(u_\varepsilon(y)\big)^2 \le I_1 + 2I_2.\end{aligned}$$

Coming back to the coordinates x, we obtain

$$(u_\varepsilon(y))^2 + \int_{\Omega_\varepsilon} \Big((\nabla_2 u_\varepsilon)^2 + \frac{(\nabla u_\varepsilon)^2}{r^2}\Big)\frac{dx}{r^{n-4}} \le c \int_{\Omega_\varepsilon} f\Big(r\frac{\partial u_\varepsilon}{\partial r} + 2u_\varepsilon\Big)\frac{dx}{r^{n-4}}. \tag{83}$$

Since $u_\varepsilon \to u$ in $H^m(\mathbf{R}^n)$, we can here replace u_ε by u and Ω_ε by Ω.

Now let η_ρ and ζ_ρ be the cutoff functions used in the proof of Lemma 3. Since $\Delta^2(u\eta_\rho) = f\eta_\rho + [\Delta^2, \eta_\rho]u$ and $f = 0$ near O, we see that for $y_n \in (-\rho/2, 0)$,

$$\big(u(y)\big)^2 + \int_\Omega \Big(\big(\nabla_2(u\eta_\rho)\big)^2 + \frac{(\nabla(u\eta_\rho))^2}{r^2}\Big)\frac{dx}{r^{n-4}} \le c\int_{\Omega_\varepsilon}\Big(r\frac{\partial(u\eta_\rho)}{\partial r} + 2u\eta_\rho\Big)[\Delta^2, \eta_\rho]u\frac{dx}{r^{n-4}}.$$

Integrating by parts in the right-hand side, we majorize it by (33), and therefore it follows from (34) that

$$\sup_{-\rho/2<y_n<0} |u(0, y_n)|^2 + \int_{B_\rho}\Big((\nabla_2 u)^2 + \frac{(\nabla u)^2}{r^2}\Big)\frac{dx}{r^{n-4}} < c\ M_\rho(u). \tag{84}$$

We fix a sufficiently small θ and introduce the cone $C_\theta = \{x : x_n > 0, |x'| \le \theta x_n\}$. Clearly, for all $r \in (0, \rho)$,

$$\sup_{(\partial B_r)\backslash C_\theta} |u|^2 \le c\Big(|u(0, -r)|^2 + r^2 \sup_{(\partial B_r)\backslash C_\theta} |\nabla u|^2\Big),$$

the function u being extended by zero outside Ω. Hence and by the well-known local estimate

$$r^2 \sup_{(\partial B_r)\backslash C_\theta} |\nabla u|^2 \le c\int_{(B_{2r}\backslash B_{r/2})\backslash C_{\theta/2}} |\nabla u(x)|^2 \frac{dx}{|x|^{n-2}},$$

we obtain

$$\sup_{B_{\rho/2}\backslash C_\theta} |u|^2 \le c\,\Big(\sup_{0>y_n>-\rho/2} |u(0, y_n)|^2 + \int_{B_\rho} |\nabla u(x)|^2 \frac{dx}{|x|^{n-2}}\Big).$$

Making use of (84), we arrive at

$$\sup_{B_{\rho/2}\backslash C_\theta} |u|^2 + \int_{B_\rho}\Big(|\nabla_2 u|^2 + \frac{|\nabla u|^2}{|x|^2}\Big)\frac{dx}{|x|^{n-4}} \le c\ M_\rho(u).$$

Repeating the proof of Lemma 5, we obtain that, for $\rho \in (0, R)$ and for small R, the following inequality holds:

$$\sup_{B_{\rho/2}\backslash C_\theta} |u|^2 + \int_{B_\rho}\Big(|\nabla_2 u|^2 + \frac{|\nabla u|^2}{|x|^2}\Big)\frac{dx}{|x|^{n-4}} \le c\ M_R(u)\ \exp\Big(-c\int_\rho^R \mathrm{cap}_2(\bar{B}_\tau\backslash\Omega)\frac{d\tau}{\tau^{n-3}}\Big).$$

The result follows. □

References

[AH] D. R. ADAMS and A. HEARD, *The necessity of the Wiener test for some semi-linear elliptic equations*, Indiana Univ. Math. J. **41** (1992), 109 – 124. MR 94c:35080 479, 480, 481

[AHe] D. R. ADAMS and L. I. HEDBERG, *Function Spaces and Potential Theory*, Grundlehren Math. Wiss. **314**, Springer, Berlin, 1996. MR 97j:46024 479, 480, 481

[ADN] S. AGMON, A. DOUGLIS, and L. NIRENBERG, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions, I*, Comm. Pure Appl. Math. **12** (1959), 623 – 727. MR 23:A2610 488, 493, 502

[DMM] G. DAL MASO and U. MOSCO, *Wiener criteria and energy decay for relaxed Dirichlet problems*, Arch. Rational Mech. Anal. **95** (1986), 345 – 387. MR 87m:35021 479

[D] E. B. DAVIES, *Limits on L^p regularity of self-adjoint elliptic operators*, J. Differential Equations **135** (1997), 83 – 102. MR 98i:47044 480

[E] S. EILERTSEN, *On weighted positivity and the Wiener regularity of a boundary point for the fractional Laplacian*, Ark. Mat. **38** (2000), 53 – 75. MR 2001f:31009 481

[FJK] E. FABES, D. JERISON, and C. KENIG, *The Wiener test for degenerate elliptic equations*, Ann. Inst. Fourier (Grenoble) **32** (1982), 151 – 182. MR 84g:35067 479

[H] L. I. HEDBERG, *Approximation by harmonic functions, and stability of the Dirichlet problem*, Exposition. Math. **11** (1993), 193 – 259. MR 94k:31001 504

[J] F. JOHN, *Plane Waves and Spherical Means Applied to Partial Differential Equations*, Interscience, New York, 1955. MR 17:746d 498

[KM] T. KILPELÄINEN and J. MALÝ, *The Wiener test and potential estimates for quasilinear elliptic equations*, Acta Math. **172** (1994), 137 – 161. MR 95a:35050 479

[KKM] V. A. KOZLOV, V. A. KONDRAT'EV, and V. G. MAZ'YA, *On sign variability and the absence of "strong" zeros of solutions of elliptic equations*, Math. USSR-Izv. **34** (1990), 337 – 353. MR 90i:35021 492

[KMR] V. A. KOZLOV, V. G. MAZ'YA, and J. ROSSMANN, *Spectral Problems Associated with Corner Singularities of Solutions to Elliptic Equations*, Math. Surveys Monogr. **85**, Amer. Math. Soc., Providence, 2001. MR 2001i:35069 508

[L] N. S. LANDKOF, *Foundations of Modern Potential Theory* (in Russian), "Nauka," Moscow, 1966, MR 35:5644; English translation in Grundlehren Math. Wiss. **180**, Springer, New York, 1972. MR 50:2520 481, 484, 497

[La] D. A. LABUTIN, *Potential estimates for a class of fully nonlinear elliptic equations*, Duke Math. J. **111** (2002), 1 – 49. CMP 1 876 440 479

[LM] P. LINDQVIST and O. MARTIO, *Two theorems of N. Wiener for solutions of quasilinear elliptic equations*, Acta Math. **155** (1985), 153 – 171. MR 87g:35074 479

[LSW] W. LITTMAN, G. STAMPACCHIA, and H. F. WEINBERGER, *Regular points for elliptic equations with discontinuous coefficients*, Ann. Scuola Norm. Sup. Pisa (3) **17** (1963), 43 – 77. MR 28:4228 479, 481

[MZ] J. MALÝ and W. P. ZIEMER, *Fine Regularity of Solutions of Elliptic Partial Differential Equations*, Math. Surveys Monogr. **51**, Amer. Math. Soc., Providence, 1997. MR 98h:35080 479

[M1] V. G. MAZ'YA [V. G. MAZ'JA], *The Dirichlet problem for elliptic equations of arbitrary order in unbounded domains* (in Russian), Dokl. Akad. Nauk SSSR **150** (1963), 1221 – 1224, MR 27:5026; English translation in Soviet Math. Dokl. **4** (1963), 860-863. 493, 500

[M2] ———, *Sobolev Spaces*, Springer Ser. Soviet Math., Springer, Berlin, 1985. MR 87g:46056 482, 493, 500

[M3] ———, "Unsolved problems connected with the Wiener criterion" in *The Legacy of Norbert Wiener: A Centennial Symposium (Cambridge, Mass., 1994)*, Proc. Sympos. Pure Math. **60**, Amer. Math. Soc., Providence, 1997, 199 – 208. MR 98e:35040 481, 482

[M4] ———, "On Wiener's type regularity of a boundary point for higher order elliptic equations" in *Nonlinear Analysis, Function Spaces and Applications (Prague, 1998), Vol. 6,* Acad. Sci. Czech Repub., Prague, 1999, 119 – 155. MR 2001m:35129 480

[MN] V. G. MAZ'YA and S. A. NAZAROV, *The apex of a cone can be irregular in Wiener's sense for a fourth-order elliptic equation* (in Russian), Mat. Zametki **39** (1986), 24 – 28; English translation in Math. Notes **39** no. 1, (1986), 14 – 16. MR 87h:35088 480

[MNP] V. MAZ'YA [W. G. MAZJA], S. NAZAROV [S. A. NASAROW], and B. PLAMENEVSKIJ [B. A. PLAMENEWSKI], *Asymptotische Theorie elliptischer Randwertaufgaben in singulär gestörten Gebieten, I: Störungen isolierter Randsingularitäten*, Math. Lehrbucher Monogr. I. Abt. Math. Lehrbucher **82**, Akademie, Berlin, 1991, MR 92g:35059; English translation in Oper. Theory Adv. Appl. **111**, Birkhäuser, Basel, 2000. MR 2001e:35044a 480

[MP1] V. G. MAZ'YA [V. G. MAZ'JA] and B. A. PLAMENEVSKIĬ, *L_p-estimates of solutions of elliptic boundary value problems in domains with ribs* (in Russian), Trudy Moskov. Mat. Obshch. **37** (1978), 49 – 93, MR 81b:35027; English translation in Trans. Moscow Math. Soc. **37** (1980), 49 – 97. 508

[MP2] ———, *On the maximum principle for the biharmonic equation in a domain with conical points* (in Russian), Izv. Vyssh. Uchebn. Zaved. Mat. **1981**, no. 2, 52 – 59. MR 84b:35037 481

[TW] N. S. TRUDINGER and X.-J. WANG, *On the weak continuity of elliptic operators and applications to potential theory,* Amer. J. Math. **124** (2002), 369 – 410. CMP 1 890 997 479

[W] N. WIENER, *The Dirichlet problem*, J. Math. Phys. Mass. Inst. Tech. **3** (1924), 127 – 146; reprinted in N. Wiener, *Collected Works with Commentaries, Vol. 1,* ed. P. Masani, Mathematicians of Our Time **10**, MIT Press, Cambridge, Mass., 1976, 394 – 413. MR 58:27161 479

Department of Mathematics, University of Linköping, SE-581 83 Linköping, Sweden;
vlmaz@mai.liu.se