Fast cubature of volume potentials over rectangular domains

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Abstract

In the present paper we study high-order cubature formulas for the computation of advection-diffusion potentials over boxes. By using the basis functions introduced in the theory of approximate approximations, the cubature of a potential is reduced to the quadrature of one dimensional integrals. For densities with separated approximation, we derive a tensor product representation of the integral operator which admits efficient cubature procedures in very high dimensions. Numerical tests show that these formulas are accurate and provide approximation of order $\mathcal{O}(h^6)$ up to dimension 10^8.

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1 Introduction

High-dimensional volume potentials arise in many mathematical models in the field of physics, chemistry, biology, financial mathematics and many others. In recent years, tensor product approximation has been recognized as a successful tool to overcome the "curse of dimensionality" and treat high-dimensional integral operators as described, for example, in [3, 4, 6, 2].

In the present paper we propose to combine high-order semi-analytic cubature formulas, obtained by using the method of approximate approximations (see [11] and the reference therein), with tensor product approximations.

Cubature formulas based on approximate approximations for volume potentials over $\mathbb{R}^n$ and over bounded domains have been considered in [10] and [9], respectively (see also [11]). The cubature of high-dimensional volume potentials over the full space and over half-spaces has been studied in [7] and [8]. Now we consider the volume potential

$$\mathcal{K}_\lambda f(\mathbf{x}) = \int_{[P,Q]} \kappa_\lambda(\mathbf{x}-\mathbf{y}) f(\mathbf{y}) d\mathbf{y}, \tag{1.1}$$

with the fundamental solution

$$\kappa_\lambda(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}} \Big(\frac{|\mathbf{x}|}{\lambda}\Big)^{1-n/2} K_{n/2-1}(\lambda|\mathbf{x}|)\,, \lambda \in \mathbb{C} \setminus (-\infty, 0],$$

over rectangular domains $[P,Q] = \prod_{j=1}^n [P_j, Q_j] \subset \mathbb{R}^n$. Here K_ν is the modified Bessel function of the second kind (see [1, 9.6, p.374]).

The function $u = \mathcal{K} f$ provides a solution of the modified Helmholtz equation

$$(-\Delta + \lambda^2) u = \begin{cases} f(\mathbf{x}) & \mathbf{x} \in [P,Q] \\ \\ 0 & \mathit{otherwise.} \end{cases}$$

For $\lambda = 0$, then

$$\kappa_0(\mathbf{x}) = \begin{cases} \dfrac{1}{2\pi} \log \dfrac{1}{|\mathbf{x}|}\,, & n = 2\,, \\ \dfrac{\Gamma(\frac{n}{2}-1)}{4\pi^{n/2}} \dfrac{1}{|\mathbf{x}|^{n-2}}\,, & n \geq 3 \end{cases}$$

is the fundamental solution of the Laplacian.

The theory of approximate approximations proposes semi-analytic cubature formulas for volume potentials by using quasi-interpolation of the density f by functions for which the integral operator can be taken analytically. Approximate quasi-interpolant has the form

$$\mathcal{M}_{h,\mathcal{D}} f(\mathbf{x}) = \mathcal{D}^{-n/2} \sum_{\mathbf{m} \in \mathbb{Z}^n} f(h\mathbf{m}) \eta \left(\frac{\mathbf{x} - h\mathbf{m}}{h\sqrt{\mathcal{D}}} \right)$$

where h and $\mathcal{D}$ are positive parameters and η is a smooth and rapidly decaying function which satisfies the moment conditions of order N

$$\int_{\mathbb{R}^n} \eta(\mathbf{x})\, \mathbf{x}^\alpha d\mathbf{x} = \delta_{0,\alpha}, \quad 0 \leq |\alpha| < N. \tag{1.2}$$

If $f \in C_0^N(\mathbb{R}^n)$, it is known ([11]) that

$$|f(\mathbf{x}) - \mathcal{M}_{h,\mathcal{D}} f(\mathbf{x})| \leq c(\sqrt{\mathcal{D}} h)^N \|\nabla_N f\|_{L^\infty} + \sum_{k=0}^{N-1} \varepsilon_k (\sqrt{\mathcal{D}} h)^k \big|\nabla_k f(\mathbf{x})\big|$$

with

$$\varepsilon_k \leq \sum_{\mathbf{m} \in \mathbb{Z}^n \setminus \{0\}} \big|\nabla_k \mathcal{F}\eta(\sqrt{\mathcal{D}}\mathbf{m})\big|\,; \lim_{\mathcal{D}\to\infty} \sum_{\mathbf{m} \in \mathbb{Z}^n \setminus \{0\}} \big|\nabla_k \mathcal{F}\eta(\sqrt{\mathcal{D}}\mathbf{m})\big| = 0.$$

If we replace f in (1.1) by the quasi-interpolant

$$\mathcal{D}^{-n/2} \sum_{h\,\mathbf{m}\in[P,Q]} f(h\mathbf{m})\eta\left(\frac{\mathbf{x}-h\mathbf{m}}{h\sqrt{\mathcal{D}}}\right) \tag{1.3}$$

we don't obtain good approximations because (1.3) approximates f only in a subdomain of $[P,Q]$ with positive distance from the boundary. To avoid this difficulty we extend f with preserved smoothness in a larger domain. Obviously the quasi-interpolant of the continuation $\widetilde{f}$ approximates f in $[P,Q]$. Assume that there exists $C>0$ such that

$$||\widetilde{f}||_{W^N_\infty(\mathbb{R}^n)} \leq C\,||f||_{W^N_\infty([P,Q])}.$$

Since η is a smooth and rapidly decaying function, for any error $\epsilon>0$ one can fix $r>0$ and the parameter $\mathcal{D}>0$ such that the quasi-interpolant with nodes in a neighborhood of $[P,Q]$

$$\mathcal{M}^r_{h,\mathcal{D}}\widetilde{f}(\mathbf{x}) = \mathcal{D}^{-n/2} \sum_{d(h\mathbf{m},[P,Q])\leq r\,h\sqrt{\mathcal{D}}} \widetilde{f}(h\mathbf{m})\eta\left(\frac{\mathbf{x}-h\mathbf{m}}{h\sqrt{\mathcal{D}}}\right)$$

approximates f with

$$|f(\mathbf{x})-\mathcal{M}^r_{h,\mathcal{D}}\widetilde{f}(\mathbf{x})| = \mathcal{O}((\sqrt{\mathcal{D}}h)^N+\epsilon)\|f\|_{W^N_\infty} \tag{1.4}$$

for all $\mathbf{x}\in[P,Q]$.

Then the integral

$$\mathcal{K}_{\lambda,h}\widetilde{f}(\mathbf{x}) = \mathcal{K}_\lambda(\mathcal{M}^r_{h,\mathcal{D}}\widetilde{f})(\mathbf{x}) = \mathcal{D}^{-n/2} \sum_{d(h\mathbf{m},[P,Q])\leq r\,h\sqrt{\mathcal{D}}} \widetilde{f}(h\mathbf{m}) \int_{[P,Q]} \kappa_\lambda(\mathbf{x}-\mathbf{y})\,\eta\left(\frac{\mathbf{y}-h\mathbf{m}}{h\sqrt{\mathcal{D}}}\right) d\mathbf{y}$$

gives a cubature of (1.1).

Since $\mathcal{K}_\lambda$ is a bounded mapping between suitable function spaces, the differences $\mathcal{K}_{\lambda,h}\widetilde{f}(\mathbf{x}) - \mathcal{K}_\lambda f(\mathbf{x})$ behave like estimate (1.4). Therefore, to construct high order cubature formulas for (1.1), it remains to compute the integrals

$$\int_{[P,Q]} \kappa_\lambda(\frac{\mathbf{x}-h\mathbf{m}}{h\sqrt{\mathcal{D}}}-\mathbf{y})\,\eta(\mathbf{y})d\mathbf{y}$$

for nodes with $d(h\mathbf{m},[P,Q]) \leq r\,h\sqrt{\mathcal{D}}$. This is performed by using one-dimensional integral representations. As basis functions we take the tensor products of univariate basis functions

$$\widetilde{\eta}(\mathbf{x}) = \prod_{j=1}^{2M} \widetilde{\eta}_{2M}(x_j); \quad \widetilde{\eta}_{2M}(x_j) = \pi^{-1/2} L^{(1/2)}_{M-1}(x_j^2)\,\mathrm{e}^{-x_j^2} \tag{1.5}$$

which satisfies the moment condition (1.2) of order $N=2M$ (cf. [11]), where $L^{(\gamma)}_k$ are the generalized Laguerre polynomials

$$L^{(\gamma)}_k(y) = \frac{\mathrm{e}^{y}\,y^{-\gamma}}{k!}\left(\frac{d}{dy}\right)^k\left(\mathrm{e}^{-y}\,y^{k+\gamma}\right), \quad \gamma>-1\,.$$

Using the representation with a tensor product integrand

$$\int_{[P,Q]} \mathcal{K}_\lambda(\mathbf{x}-\mathbf{y}) \mathrm{e}^{-|\mathbf{y}|^2} d\mathbf{y} =$$

$$\frac{1}{4}\int_0^\infty \mathrm{e}^{-\lambda^2 t/4} \prod_{j=1}^n \frac{\mathrm{e}^{-x_j^2/(1+t)}}{2\sqrt{\pi}} \Big(\operatorname{erfc}\big(\sqrt{\frac{1+t}{t}}\Big(P_j - \frac{x_j}{1+t}\Big)\big) - \operatorname{erfc}\big(\sqrt{\frac{1+t}{t}}\Big(Q_j - \frac{x_j}{1+t}\Big)\big)\Big) dt \quad (1.6)$$

we derive a tensor product representation of the integral operator which admits efficient cubature procedures for densities with separated approximation (Section 2). We will consider quasi-interpolants (2.1) on anisotropic grids which use different step size $h_j > 0, j = 1, ..., n$ along different space dimensions. If $h_j = \tau h$, $0 < \tau \le 1$ the error of the quasi-interpolant (2.1) is always $\mathcal{O}(h^N)$. In Section 3 we provide numerical tests, showing that these formulas are accurate and provide approximation of order $\mathcal{O}(h^6)$ up to dimension 10^8.

2 Higher order cubature formula based on (1.6)

In this section we describe a high order cubature of $\mathcal{K}_\lambda f$ in the case of rectangular domain in $\mathbb{R}^n$. Let

$$[P,Q] = \{\mathbf{x} = (x_1, \ldots, x_n) : P_j \le x_j \le Q_j, j = 1, ..., n\} = \prod_{j=1}^n [P_j, Q_j] \, .$$

As basis functions we use (1.5).

In order to apply also quasi-interpolants on rectangular grids $(h_1 m_1, \ldots, h_n m_n), h_j > 0$, shortly denoted by $\{\mathbf{hm}\}$,

$$\mathcal{M}_{h,\mathcal{D}} \widetilde{f}(\mathbf{x}) = \mathcal{D}^{-n/2} \sum_{\mathbf{m}\in\mathbb{Z}^n} \widetilde{f}(\mathbf{hm}) \prod_{j=1}^n \widetilde{\eta}_{2M}\left(\frac{x_j - h_j m_j}{h_j\sqrt{\mathcal{D}}}\right) , \quad (2.1)$$

we define the basis function $\eta(\mathbf{x}) = \prod \widetilde{\eta}_{2M}(a_j x_j)$, $a_j > 0$, and look for integral representations of the solution of

$$(-\Delta + \lambda^2)\, u = \prod_{j=1}^n \chi_{(p_j,q_j)}(x_j)\, \widetilde{\eta}_{2M}(a_j x_j) \, . \quad (2.2)$$

Here $\chi_{(p_j,q_j)}$ is the characteristic function of the interval (p_j, q_j) with $-\infty \le p_j < q_j \le +\infty$, $j = 1, \ldots, n$.

Theorem 2.1. *Let* $\operatorname{Re}\lambda^2 \ge 0$ *and* $n \ge 3$. *The solution of equation* (2.2) *in* $\mathbb{R}^n$ *can be expressed by the one-dimensional integral*

$$u(\mathbf{x}) = \frac{1}{4}\int_0^\infty \mathrm{e}^{-\lambda^2 t/4} \prod_{j=1}^n \Big(\Phi_M(a_j x_j, a_j^2 t, a_j p_j) - \Phi_M(a_j x_j, a_j^2 t, a_j q_j)\Big)\, dt \quad (2.3)$$

where the function Φ_M *is given by*

$$\Phi_M(x,t,p) = \frac{\mathrm{e}^{-x^2/(1+t)}}{2\sqrt{\pi}} \Big(\operatorname{erfc}\big(F(t,x,p)\big)\mathcal{P}_M(t,x) - \frac{\mathrm{e}^{-F^2(t,x,p)}}{\sqrt{\pi}} \mathcal{Q}_M\big(t,x,p\big)\Big)$$

with the function

$$F(t,x,y)=\sqrt{\frac{1+t}{t}}\Big(y-\frac{x}{1+t}\Big),$$

and $\mathcal{P}_M,\mathcal{Q}_M$ *are polynomials in* x *of degree* $2M-2$ *and* $2M-3$*, respectively:*

$$\mathcal{P}_M(t,x)=\sum_{k=0}^{M-1}\frac{1}{(1+t)^{k+1/2}}L_k^{(-1/2)}\Big(\frac{x^2}{1+t}\Big),$$

$$\mathcal{Q}_M(t,x,y)=2\sum_{k=1}^{M-1}\frac{(-1)^k}{k!\,4^k}\sum_{\ell=1}^{2k}\frac{(-1)^\ell}{t^{\ell/2}}\Bigg(H_{2k-\ell}(y)H_{\ell-1}\Big(\frac{y-x}{\sqrt{t}}\Big)$$
$$-\binom{2k}{\ell}H_{2k-\ell}\Big(\frac{x}{\sqrt{1+t}}\Big)\frac{H_{\ell-1}\big(F(t,x,y)\big)}{(1+t)^{k+1/2}}\Bigg).$$

If $\operatorname{Re}\lambda^2>0$*, then the representation* (2.3) *is valid for all* $n\geq 1$*.*

By H_k *we denote the Hermite polynomials*

$$H_k(x)=(-1)^k\,\mathrm{e}^{x^2}\,\frac{d^k}{dx^k}\,\mathrm{e}^{-x^2}\;. \tag{2.4}$$

Proof. The solution of (2.2) can be obtained explicitly by using the parabolic equation

$$\partial_t w-\Delta w+\lambda^2 w=0\,,\quad t\geq 0\,, \tag{2.5}$$

with the initial condition

$$w(\mathbf{x},0)=\prod_{j=1}^{n}\chi_{(p_j,q_j)}(x_j)\,\widetilde{\eta}_{2M}(a_jx_j)\,.$$

Integrating (2.5) in t we derive

$$w(\mathbf{x},T)-w(\mathbf{x},0)-(\Delta-\lambda^2)\int_0^T w(\mathbf{x},t)\,dt=0\,,$$

hence the solution of (2.2) is expressed as the one-dimensional integral

$$u(\mathbf{x})=\int_0^\infty w(\mathbf{x},t)\,dt$$

provided it exists. Obviously, if w solves (2.5), then $z=w\,\mathrm{e}^{\lambda^2 t}$ is the solution of the initial value problem for the heat equation

$$\partial_t z-\Delta z=0\,,\quad z(\mathbf{x},0)=\prod_{j=1}^{n}\chi_{(p_j,q_j)}(x_j)\,\widetilde{\eta}_{2M}(a_jx_j)\,,$$

which has, by Poisson's formula, the solution

$$
\begin{aligned}
z(\mathbf{x},t) &= \frac{1}{(4\pi t)^{n/2}} \int\limits_{\prod(p_j,q_j)} \mathrm{e}^{-|\mathbf{x}-\mathbf{y}|^2/(4t)} \prod_{j=1}^{n} \widetilde{\eta}_{2M}(a_j y_j)\, d\mathbf{y} \\
&= \prod_{j=1}^{n} \frac{1}{\pi^{1/2}(4a_j^2 t)^{1/2}} \int\limits_{a_j p_j}^{a_j q_j} \mathrm{e}^{-(a_j x_j - y_j)^2/(4a_j^2 t)}\, \widetilde{\eta}_{2M}(y_j)\, dy_j
\end{aligned}
$$

where $\prod(p_j,q_j)$ is the Cartesian product of the intervals (p_j,q_j). Denoting

$$
\Phi_M(x,t,p) = \frac{1}{\sqrt{\pi t}} \int\limits_{p}^{\infty} \mathrm{e}^{-(x-y)^2/t}\, \widetilde{\eta}_{2M}(y)\, dy
$$

we get the one-dimensional integral representation (2.3) of the solution of (2.2), provided this integral exists. Denoting

$$
\varphi_k(x,t,p) = \int\limits_{p}^{\infty} \mathrm{e}^{-(x-y)^2/t}\, \frac{d^{2k}}{dy^{2k}}\, \mathrm{e}^{-y^2}\; dy
$$

and using the general representation [11, p.55]

$$
\eta_{2M}(\mathbf{x}) = \pi^{-n/2} \sum_{j=0}^{M-1} \frac{(-1)^j}{j!4^j} \Delta^j\, \mathrm{e}^{-|\mathbf{x}|^2},
$$

we have

$$
\Phi_M(x,t,p) = \frac{1}{\pi\sqrt{t}} \sum_{k=0}^{M-1} \frac{(-1)^k}{k!\,4^k}\, \varphi_k(x,t,p)\,.
$$

From

$$
\varphi_0(x,t,p) = \int\limits_{p}^{\infty} \mathrm{e}^{-(x-y)^2/t}\, \mathrm{e}^{-y^2}\; dy = \frac{\sqrt{\pi}}{2}\sqrt{\frac{t}{1+t}}\, \mathrm{e}^{-x^2/(1+t)}\, \mathrm{erfc}\,\big(F(t,x,p)\big)\,,
$$

for $k \ge 1$, integration by parts leads to

$$
\varphi_k(x,t,p) = \frac{\partial^{2k}\varphi_0(x,t,p)}{\partial x^{2k}} - \sum_{\ell=0}^{2k-1} (-1)^\ell \frac{\partial^\ell}{\partial y^\ell}\, \mathrm{e}^{-(x-y)^2/t}\, \frac{d^{2k-\ell-1}}{dy^{2k-\ell-1}}\, \mathrm{e}^{-y^2}\, \Bigg|_{y=p}
$$

and the definition (2.4) gives

$$
\begin{aligned}
\frac{d^{2k-\ell-1}}{dy^{2k-\ell-1}}\, \mathrm{e}^{-y^2} &= (-1)^{2k-\ell-1}\, \mathrm{e}^{-y^2}\, H_{2k-\ell-1}(y)\,, \\
\frac{\partial^\ell}{\partial y^\ell}\, \mathrm{e}^{-(x-y)^2/t} &= \frac{(-1)^\ell\, \mathrm{e}^{-(x-y)^2/t}}{t^{\ell/2}}\, H_\ell\Big(\frac{y-x}{\sqrt{t}}\Big).
\end{aligned}
$$

In view of

$$\frac{d^\ell}{dx^\ell}\,\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}}(-1)^\ell\,\mathrm{e}^{-x^2}\,H_{\ell-1}(x), \quad \ell \geq 1\,,$$

one gets for $\ell < 2k$

$$\begin{aligned}\frac{\partial^{2k-\ell}}{\partial x^{2k-\ell}}\,\mathrm{erfc}\big(F(t,x,p)\big) &= \frac{(-1)^{2k-\ell}}{(t(1+t))^{k-\ell/2}}\left[\frac{d^{2k-\ell}}{dz^{2k-\ell}}\,\mathrm{erfc}(z)\right]_{z=F(t,x,p)} \\ &= \frac{2\,\mathrm{e}^{-F^2(t,x,p)}}{\sqrt{\pi}(t(1+t))^{k-\ell/2}}\,H_{2k-\ell-1}(F(t,x,p))\,.\end{aligned}$$

Therefore, since

$$\frac{d^\ell}{dx^\ell}\,\mathrm{e}^{-x^2/(1+t)} = \frac{(-1)^\ell\,\mathrm{e}^{-x^2/(1+t)}}{(1+t)^{\ell/2}}H_\ell\Big(\frac{x}{\sqrt{1+t}}\Big)\,,$$

we obtain

$$\begin{aligned}\frac{\partial^{2k}}{\partial x^{2k}}\,\varphi_0(x,t,p) &= \frac{\sqrt{\pi t}}{2}\,\frac{\mathrm{e}^{-x^2/(1+t)}}{(1+t)^{k+1/2}}H_{2k}\Big(\frac{x}{\sqrt{1+t}}\Big)\,\mathrm{erfc}(F(t,x,p)) \\ &\quad - \frac{\sqrt{t}\,\mathrm{e}^{-x^2/(1+t)}\,\mathrm{e}^{-F^2(t,x,p)}}{(1+t)^{k+1/2}}\sum_{\ell=0}^{2k-1}\binom{2k}{\ell}\frac{(-1)^\ell}{t^{k-\ell/2}}H_\ell\Big(\frac{x}{\sqrt{1+t}}\Big)H_{2k-\ell-1}(F(t,x,p))\,.\end{aligned}$$

Thus simple transformations give

$$\begin{aligned}\varphi_k(x,t,p) = \mathrm{e}^{-x^2/(1+t)}\Bigg(&\mathrm{erfc}\big(F(t,x,p)\big)H_{2k}\Big(\frac{x}{\sqrt{1+t}}\Big)\frac{\sqrt{\pi\,t}}{2(1+t)^{k+1/2}} \\ &+ \mathrm{e}^{-F^2(t,x,p)}\sum_{\ell=1}^{2k}\frac{(-1)^\ell}{t^{(\ell-1)/2}} \\ &\times\bigg(\binom{2k}{\ell}H_{2k-\ell}\Big(\frac{x}{\sqrt{1+t}}\Big)\frac{H_{\ell-1}(F(t,x,p))}{(1+t)^{k+1/2}} - H_{\ell-1}\Big(\frac{p-x}{\sqrt{t}}\Big)H_{2k-\ell}(p)\bigg)\Bigg).\end{aligned}$$

Using the relation $H_{2k}(x) = (-1)^k\,4^k\,k!\,L_k^{(-1/2)}(x^2)$ we find therefore

$$\begin{aligned}\Phi_M(x,t,p) &= \frac{\mathrm{e}^{-x^2/(1+t)}\,\mathrm{erfc}\big(F(t,x,p)\big)}{2\sqrt{\pi}}\sum_{k=0}^{M-1}\frac{1}{(1+t)^{k+1/2}}L_k^{(-1/2)}\Big(\frac{x^2}{1+t}\Big) \\ &\quad + \frac{\mathrm{e}^{-x^2/(1+t)}\,\mathrm{e}^{-F^2(t,x,p)}}{\pi}\sum_{k=0}^{M-1}\frac{(-1)^k}{k!\,4^k}\sum_{\ell=1}^{2k}\frac{(-1)^\ell}{t^{\ell/2}} \\ &\quad \times\bigg(\binom{2k}{\ell}H_{2k-\ell}\Big(\frac{x}{\sqrt{1+t}}\Big)\frac{H_{\ell-1}\big(F(t,x,p)\big)}{(1+t)^{k+1/2}} - H_{\ell-1}\Big(\frac{p-x}{\sqrt{t}}\Big)H_{2k-\ell}(p)\bigg) \\ &= \frac{\mathrm{e}^{-x^2/(1+t)}}{2\sqrt{\pi}}\Big(\mathrm{erfc}\big(F(t,x,p)\big)\mathcal{P}_M(t,x) - \frac{\mathrm{e}^{-F^2(t,x,p}}{\sqrt{\pi}}\mathcal{Q}_M\big(t,x,p\big)\Big).\end{aligned}$$

$\square$

The polynomials $\mathcal{P}_M(t,x)$ and $\mathcal{Q}_M(t,x,p)$ for $M=1,2,3$ are given by

$$\mathcal{P}_1(t,x)=\frac{1}{(1+t)^{1/2}}\,,\qquad \mathcal{P}_2(t,x)=\mathcal{P}_1(t,x)+\frac{1}{2(1+t)^{3/2}}-\frac{x^2}{(1+t)^{5/2}}\,,$$
$$\mathcal{P}_3(t,x)=\mathcal{P}_2(t,x)+\frac{3}{8(1+t)^{5/2}}-\frac{3\,x^2}{2(1+t)^{7/2}}+\frac{x^4}{2(1+t)^{9/2}}\,,$$
$$\mathcal{Q}_1(t,x,p)=0,\qquad \mathcal{Q}_2(t,x,p)=\frac{\sqrt{t}}{(1+t)}\Big(\frac{x}{1+t}+p\Big),$$
$$\mathcal{Q}_3(t,x,p)=-\frac{\sqrt{t}}{4(1+t)}\Big(\frac{2x^3}{(1+t)^3}+\frac{2px^2-5x}{(1+t)^2}+\frac{(2p^2-5)x-3p}{1+t}+p(2p^2-7)\Big).$$

Remark 2.1. *Since for positive r*

$$0<\operatorname{erfc}(r)\le \mathrm{e}^{-r^2}\quad \textit{and}\quad 2-\mathrm{e}^{-r^2}<\operatorname{erfc}(-r)<2$$

from the relation

$$F^2(t,x,p)=p^2+\frac{(x-p)^2}{t}-\frac{x^2}{1+t}$$

we get

$$|\,\mathrm{e}^{-x^2/(1+t)}\operatorname{erfc}\big(F(t,x,p)\big)|\le \mathrm{e}^{-p^2}\quad \textit{if}\quad p>0$$

and

$$|\,\mathrm{e}^{-x^2/(1+t)}\operatorname{erfc}\big(F(t,x,p)\big)-2\,\mathrm{e}^{-x^2/(1+t)}\,|<\mathrm{e}^{-p^2}\quad \textit{if}\quad p<0\,.$$

Thus for sufficiently large $|p|$

$$\Phi_M(x,t,p)=\begin{cases}\pi^{-1/2}\,\mathrm{e}^{-x^2/(1+t)}\,\mathcal{P}_M(t,x)+\mathcal{O}(\mathrm{e}^{-p^2}) & \textit{if } p<0\,,\\ \mathcal{O}(\mathrm{e}^{-p^2}) & \textit{if } p>0\,,\end{cases}$$

and therefore, for sufficiently large r one can use the approximation

$$\Phi_M(x,t,p)-\Phi_M(x,t,q)\approx\begin{cases}0\,, & p,q\ge r\ \textit{or}\ p,q\le -r\,,\\ \pi^{-1/2}\,\mathrm{e}^{-x^2/(1+t)}\,\mathcal{P}_M(t,x), & p\le -r\ \textit{and}\ q\ge r\,,\end{cases}$$

with the error $\mathcal{O}(\mathrm{e}^{-r^2})$. Similarly, if $q-p\ge 2r$, then

$$\Phi_M(x,t,p)-\Phi_M(x,t,q)\approx\begin{cases}\Phi_M(x,t,p)\,, & -r<p<r\,,\\ \pi^{-1/2}\ \ \mathrm{e}^{-x^2/(1+t)}\,\mathcal{P}_M(t,x)-\Phi_M(x,t,q), & -r<q<r\,.\end{cases}$$

3 Implementation and numerical results

We compute the cubature formula

$$\mathcal{K}_{\lambda,\mathbf{h}}\widetilde{f}(\mathbf{x})=\mathcal{D}^{-n/2}\sum_{\mathbf{hm}\in\widetilde{\Omega}_{r\mathbf{h}}}\widetilde{f}(\mathbf{hm})\int\limits_{[P,Q]}\kappa_\lambda(\mathbf{x}-\mathbf{y})\prod_{j=1}^{n}\widetilde{\eta}_{2M}\left(\frac{y_j-h_jm_j}{h_j\sqrt{\mathcal{D}}}\right)d\mathbf{y}$$

where $\widetilde{\Omega}_{r\mathbf{h}} = \prod_{j=1}^n (P_j - rh_j\sqrt{\mathcal{D}}, Q_j + rh_j\sqrt{\mathcal{D}})$, using the tensor product representation of Theorem 2.1. At the grid points $\mathbf{hk} = (h_1k_1, \dots, h_nk_n)$ we obtain

$$\int\limits_{[P,Q]} \kappa_\lambda(\mathbf{hk}-\mathbf{y}) \prod_{j=1}^n \widetilde{\eta}_{2M}\left(\frac{y_j - h_jm_j}{h_j\sqrt{\mathcal{D}}}\right) d\mathbf{y} = \frac{1}{4}\int\limits_0^\infty \mathrm{e}^{-\lambda^2 t/4}$$
$$\times \prod_{j=1}^n \Big(\Phi_M\big(\frac{k_j-m_j}{\sqrt{\mathcal{D}}}, \frac{t}{h_j^2\mathcal{D}}, \frac{P_j - h_jm_j}{h_j\sqrt{\mathcal{D}}}\big) - \Phi_M\big(\frac{k_j-m_j}{\sqrt{\mathcal{D}}}, \frac{t}{h_j^2\mathcal{D}}, \frac{Q_j - h_jm_j}{h_j\sqrt{\mathcal{D}}}\big)\Big)\, dt$$

and therefore

$$\mathcal{K}_{\lambda,\mathbf{h}}\widetilde{f}(\mathbf{x}) = \sum_{\mathbf{hm}\in\widetilde{\Omega}_{r\mathbf{h}}} \widetilde{f}(\mathbf{hm})\, \mathbf{b}_{\mathbf{k},\mathbf{m}}^{(M)}\,, \tag{3.1}$$

where we introduce the one-dimensional integral

$$\mathbf{b}_{\mathbf{k},\mathbf{m}}^{(M)} = \frac{1}{4\mathcal{D}^{n/2}} \int\limits_0^\infty \mathrm{e}^{-\lambda^2 t/4} \prod_{j=1}^n \Big(b_{k_j,m_j}^j(P_j) - b_{k_j,m_j}^j(Q_j)\Big)\, dt \tag{3.2}$$

and use the abbreviation

$$b_{k,m}^j(P) = \mathrm{e}^{-(k-m)^2/(\mathcal{D}(1+t))} \Bigg(\operatorname{erfc}\big(F(\frac{t}{h_j^2\mathcal{D}}, \frac{k-m}{\sqrt{\mathcal{D}}}, \frac{P-h_jm}{h_j\sqrt{\mathcal{D}}})\big)\mathcal{P}_M\big(\frac{t}{h_j^2\mathcal{D}}, \frac{k-m}{\sqrt{\mathcal{D}}}\big)$$
$$-\pi^{-1/2}\exp\Big(-F^2\big(\frac{t}{h_j^2\mathcal{D}}, \frac{k-m}{\sqrt{\mathcal{D}}}, \frac{P-h_jm}{h_j\sqrt{\mathcal{D}}}\big)\Big)\mathcal{Q}_M\big(\frac{t}{h_j^2\mathcal{D}}, \frac{k-m}{\sqrt{\mathcal{D}}}, \frac{P-h_jm}{h_j\sqrt{\mathcal{D}}}\big)\Bigg)/(2\sqrt{\pi}).$$

According to Remark 2.1, for appropriately chosen $r > 0$ we can set within a given accuracy

$$b_{k,m}^j(P) = a_{k-m}^j = \pi^{-1/2}\,\mathrm{e}^{-(k-m)^2/(\mathcal{D}(1+t))}\,\mathcal{P}_M\big(\frac{t}{h_j^2\mathcal{D}}, \frac{k-m}{\sqrt{\mathcal{D}}}\big) \qquad \text{if } P - h_jm \le -rh_j\sqrt{\mathcal{D}}\,,$$
$$b_{k,m}^j(P) = 0 \qquad \text{if } P - h_jm \ge rh_j\sqrt{\mathcal{D}}\,,$$

which speeds up the computation of (3.2). In particular, we can split (3.1) into

$$\mathcal{K}_{\lambda,\mathbf{h}}^{(M)} f(\mathbf{hk}) = \sum_{\mathbf{hm}\in\Omega_{r\mathbf{h}}} f(\mathbf{hm})\,\mathbf{a}_{\mathbf{k}-\mathbf{m}}^{(M)} + \sum_{\mathbf{hm}\in\widetilde{\Omega}_{r\mathbf{h}}\setminus\Omega_{r\mathbf{h}}} \widetilde{f}(\mathbf{hm})\mathbf{b}_{\mathbf{k},\mathbf{m}}^{(M)}\,, \tag{3.3}$$

where $\Omega_{r\mathbf{h}} = \prod_{j=1}^n (P_j + rh_j\sqrt{\mathcal{D}}, Q_j - rh_j\sqrt{\mathcal{D}})$, and the coefficients in the convolutional sum are given by

$$\mathbf{a}_{\mathbf{k}}^{(M)} = \frac{1}{4\mathcal{D}^{n/2}} \int\limits_0^\infty \mathrm{e}^{-\lambda^2 t/4} \prod_{j=1}^n a_{k_j}^j\, dt$$
$$= \frac{1}{4(\pi\mathcal{D})^{n/2}} \int\limits_0^\infty \mathrm{e}^{-\lambda^2 t/4}\,\mathrm{e}^{-|\mathbf{k}|^2/(\mathcal{D}(1+t))} \prod_{j=1}^n \mathcal{P}_M\big(\frac{t}{h_j^2\mathcal{D}}, \frac{k_j}{\sqrt{\mathcal{D}}}\big)\, dt\,.$$

Following [12] the one-dimensional integrals of $\mathbf{a}_{\mathbf{k}}^{(M)}$ and $\mathbf{b}_{\mathbf{k},\mathbf{m}}^{(M)}$ are transformed to integrals over $\mathbb{R}$ with integrands decaying doubly exponentially by making the substitutions

$$t = \mathrm{e}^{\xi}, \quad \xi = \alpha(\sigma + \mathrm{e}^{\sigma}), \quad \sigma = \beta(u - \mathrm{e}^{-u}) \tag{3.4}$$

with certain positive constants α, β, and the computation is based on the classical trapezoidal rule. Then the tensor product structure of the integrands allows the efficient computation of the coefficients $\mathbf{b}_{\mathbf{k},\mathbf{m}}^{(M)}$ and $\mathbf{a}_{\mathbf{k}}^{(M)}$. Moreover, the computation of the convolutional sum is very efficient for integrands, which allow a separated representation, i.e., for given accuracy ϵ they can be represented as a sum of products of vectors in dimension 1

$$f(h_1 m_1, \ldots, h_m m_n) = \sum_{p=1}^{R} r_p \prod_{j=1}^{n} f_j^{(p)}(h_j m_j) + \mathcal{O}(\epsilon)\,.$$

In [7] we have described this approach to the fast computation of high dimensional volume potentials for compactly supported integrands. To compute the convolutional sum

$$\sum_{\mathbf{hm}\in\Omega_{r\mathbf{h}}} \mathbf{a}_{\mathbf{k}-\mathbf{m}}^{(M)} f(\mathbf{hm})$$

we get after the substitutions

$$\mathbf{a}_{\mathbf{k}}^{(M)} = \frac{1}{4(\pi\mathcal{D})^{n/2}} \int_{-\infty}^{\infty} \mathrm{e}^{-\lambda^2\Phi(u)/4}\, \mathrm{e}^{-|\mathbf{k}|^2/(\mathcal{D}(1+\Phi(u))} \prod_{j=1}^{n} \mathcal{P}_M\Big(\frac{\Phi(u)}{h_j^2\mathcal{D}}, \frac{k_j}{\sqrt{\mathcal{D}}}\Big)\, \Phi'(u)\, du\,,$$

where we set

$$\begin{aligned} \Phi(u) &= \exp(\alpha\beta(u - \exp(-u)) + \alpha\exp(\beta(u - \exp(-u)))),\\ \Phi'(u) &= \Phi(u)\alpha\beta(1 + \mathrm{e}^{-u})(1 + \exp(\beta(u - \exp(-u)))). \end{aligned}$$

The quadrature with the trapezoidal rule with step size τ

$$\mathbf{a}_{\mathbf{k}}^{(M)} \approx \frac{\tau}{4(\pi\mathcal{D})^{n/2}} \sum_{s=-N_0}^{N_1} \mathrm{e}^{-\lambda^2\Phi(s\,\tau)/4}\, \mathrm{e}^{-|\mathbf{k}|^2/(\mathcal{D}(1+\Phi(s\,\tau)))} \prod_{j=1}^{n} \mathcal{P}_M\Big(\frac{\Phi(s\,\tau)}{h_j^2\mathcal{D}}, \frac{k}{\sqrt{\mathcal{D}}}\Big)\Phi'(s\,\tau)$$

provides the approximation via one-dimensional discrete convolutions

$$\begin{aligned} \sum_{\mathbf{hm}\in\Omega_{r\mathbf{h}}} \mathbf{a}_{\mathbf{k}-\mathbf{m}}\, f(\mathbf{hm}) &\approx \frac{\tau}{4(\pi\mathcal{D})^{n/2}} \sum_{p=1}^{R} r_p \sum_{s=-N_0}^{N_1} \mathrm{e}^{-\lambda^2\Phi(s\,\tau)/4}\, \Phi'(s\,\tau)\\ \times \prod_{j=1}^{n} \sum_{m_j} \mathrm{e}^{-(k_j-m_j)^2/(\mathcal{D}(1+\Phi(s\,\tau)))}\, & P_M\Big(\frac{\Phi(s\,\tau)}{h_j^2\mathcal{D}}, \frac{k_j - m_j}{\sqrt{\mathcal{D}}}\Big) f_j^{(p)}(h_j m_j)\,. \end{aligned}$$

We provide some numerical tests to the approximation of the potential $\mathcal{K}_\lambda f$ over the cube $[-1,1]^n$, $n \geq 3$, with the density

$$f(\mathbf{x}) = (-\Delta + \lambda^2) \prod_{j=1}^{n} u(x_j) = \sum_{p=1}^{n} \prod_{j=1}^{n} f_j^{(p)}(x_j),\ \mathbf{x} = (x_1, ..., x_n) \in [-1,1]^n; \tag{3.5}$$

$$f_j^{(p)}(x) = u(x) \quad \text{if} \quad j \neq p; \; f_j^{(p)}(x) = -u''(x) + \frac{\lambda^2}{n} u(x) \quad \text{if} \quad j = p.$$

Let $\widetilde{f}_j^{(p)}$ be an extension of $f_j^{(p)}$ outside the interval $[-1,1]$ with preserved smoothness and

$$\widetilde{f}(\mathbf{x}) = \sum_{p=1}^{n} \prod_{j=1}^{n} \widetilde{f}_j^{(p)}(x_j), \; \mathbf{x} \in \mathbb{R}^n.$$

By using Hestenes reflection principle ([5]) we construct an extension of $f_j^{(p)}$ outside the interval $[-1,1]$ as

$$\widetilde{f}_j^{(p)}(x) = \begin{cases} \sum_{s=1}^{N+1} c_s f_j^{(p)}(-a_s(x+1)-1), & x < -1 \\ f_j^{(p)}(x), & -1 \leq x \leq 1 \\ \sum_{s=1}^{N+1} c_s f_j^{(p)}(-a_s(x-1)+1), & x > 1 \end{cases}$$

where $a_1, ..., a_{N+1}$ are different positive constants and the coefficients $\mathbf{c}_N = \{c_1, ..., c_{N+1}\}$ are the unique solution of the $(N+1) \times (N+1)$ system of linear equations

$$\sum_{s=1}^{N+1} c_s(-a_s)^k = 1, \quad k = 0, ..., N.$$

We provide results for $\widetilde{f}_j^{(p)} = f_j^{(p)}$ and three different Hestenes extensions corresponding to $a_s = 2^{-s}$ (Extension 1) , $a_s = s^{-1}$ (Extension 2), $a_s = s$ (Extension 3).

The approximation values are computed by the cubature formula (3.3) for $h_j = h$, $j = 1, ..., n$. To have the saturation error comparable with the double precision rounding errors, we have chosen the parameter $\mathcal{D} = 4$.

In Tables 1, 2 and 3 we report on the absolute error and the approximation rate for the three-dimensional potential $\mathcal{K}_\lambda f$, when $u(x) = \cos^2(\pi x/2)$ (Table 1), $u(x) = (x^2-1)^3$ (Table 2) and $u(x) = (x^2-1)^2$ (Table 3), in the case $\lambda^2 = 1$ and $\lambda^2 = 1+i$. We have chosen the parameters $\alpha = 2, \beta = 2$ in the transformations (3.4) and $\tau = 0.005$, $N_1 = -N_0 = 300$ in the quadrature formula. The numerical results confirm the h^2-, h^4- and, respectively, h^6- convergence of the cubature formulas (3.3) when $M = 1, 2, 3$. For extensions 1, 2 and 3 the numerical results are similar with those if using $\widetilde{f}_j^{(p)} = f_j^{(p)}$. In Table 3 we see that the error of the approximate quasi-interpolant of order 6 has reached the saturation bound. This is a feature of the method that approximate quasi-interpolant of order N reproduces polynomials of degree $< N$ up to the saturation error.

To check the effectiveness of the method for very high dimension n we computed the potential over $[-1,1]^n$ of the density (3.5) with $u(x) = 1 - \sin(\pi x^2/2)$ (Table 4) and $u(x) = \mathrm{e}^x(1-x^2)^2$ (Table 5) in dimension $n = 10^i$, $i = 1, .., 8$ and different extensions. We have chosen $a = 6$, $b = 5$, $\tau = 0.003$, $N_0 = -40$, $N_1 = 200$. The results show that $\mathcal{K}_{\lambda,h}^{(3)}$ approximates with the predicted approximation rate 6, also for very large n and the error scales linearly in the space dimension.

References

[1] M. Abramowitz, I.A. Stegun, Handbook of mathematical functions with formulas, graphs, and mathematical tables, volume 55 of *National Bureau of Standards Applied Mathematics Series*, For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C., 1964.

[2] C. Bertoglio, B.N. Khoromskij, Low-rank quadrature-based tensor approximation of the Galerkin projected Newton/Yukawa kernels, Comput. Phys. Commun. 183 (2012) 904–912.

[3] W. Hackbusch, B.N. Khoromskij, Tensor-product approximation to operators and functions in high dimensions, J. Complexity 23 (2007) 697–714.

[4] W. Hackbusch, B.N. Khoromskij, Tensor-product approximation to multidimensional integral operators and Green's functions, SIAM J. Matrix Anal. Appl. 30 (2008) 1233–1253.

[5] M.R. Hestenes, Extension of the range of a differentiable function, Duke Math. J. 8 (1941) 183–192.

[6] B.N. Khoromskij, Fast and accurate tensor approximation of a multivariate convolution with linear scaling in dimension, J. Comput. Appl. Math. 234 (2010) 3122–3139.

[7] F. Lanzara, V. Maz'ya, G. Schmidt, On the fast computation of high dimensional volume potentials, Math. Comp. 80 (2011a) 887–904.

[8] F. Lanzara, V.G. Maz'ya, G. Schmidt, Accurate cubature of volume potentials over high-dimensional half-spaces, J. Math. Sci. (N. Y.) 173 (2011b) 683–700. Problems in mathematical analysis. No. 55.

[9] F. Lanzara, V.G. Maz'ya, G. Schmidt, Computation of volume potentials over bounded domains via approximate approximations, J. Math. Sci. (N. Y.) 187 (2013) to appear.

[10] V. Maz'ya, G. Schmidt, "Approximate approximations" and the cubature of potentials, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. 6 (1995) 161–184.

[11] V. Maz'ya, G. Schmidt, Approximate approximations, volume 141 of *Mathematical Surveys and Monographs*, American Mathematical Society, Providence, RI, 2007.

[12] J. Waldvogel, Towards a general error theory of the trapezoidal rule, in: Approximation and computation, volume 42 of *Springer Optim. Appl.*, Springer, New York, 2011, pp. 267–282.

$\lambda^2 = 1$:

		$M=1$		$M=2$		$M=3$	
$\widetilde{f}(\mathbf{x})$	h^{-1}	error	rate	error	rate	error	rate
$f(\mathbf{x})$	10	0.822E-01		0.414E-02		0.135E-03	
	20	0.219E-01	1.9062	0.272E-03	3.9267	0.223E-05	5.9201
	40	0.557E-02	1.9760	0.172E-04	3.9821	0.354E-07	5.9800
	80	0.140E-02	1.9940	0.108E-05	3.9955	0.555E-09	5.9950
	160	0.350E-03	1.9985	0.675E-07	3.9989	0.867E-11	5.9987
	320	0.875E-04	1.9996	0.422E-08	3.9997	0.136E-12	5.9994
ext 1	10	0.821E-01		0.413E-02		0.135E-03	
	20	0.219E-01	1.9057	0.272E-03	3.9265	0.223E-05	5.9201
	40	0.557E-02	1.9760	0.172E-04	3.9820	0.354E-07	5.9800
	80	0.140E-02	1.9940	0.108E-05	3.9955	0.554E-09	5.9961
	160	0.350E-03	1.9985	0.675E-07	3.9989	0.825E-11	6.0692
	320	0.875E-04	1.9996	0.422E-08	3.9997	0.789E-12	3.3868
ext 2	10	0.826E-01		0.422E-02		0.140E-03	
	20	0.219E-01	1.9138	0.273E-03	3.9520	0.224E-05	5.9686
	40	0.557E-02	1.9769	0.172E-04	3.9850	0.354E-07	5.9856
	80	0.140E-02	1.9941	0.108E-05	3.9959	0.554E-09	5.9967
	160	0.350E-03	1.9985	0.675E-07	3.9989	0.883E-11	5.9718
	320	0.875E-04	1.9996	0.422E-08	3.9997	0.120E-12	6.1971
ext 3	10	0.946E-01		0.139E-01		0.260E-01	
	20	0.224E-01	2.0769	0.771E-03	4.1771	0.871E-04	8.2194
	40	0.559E-02	2.0047	0.228E-04	5.0788	0.111E-05	6.2957
	80	0.140E-02	1.9977	0.113E-05	4.3396	0.341E-08	8.3438
	160	0.350E-03	1.9990	0.679E-07	4.0529	0.147E-10	7.8633
	320	0.875E-04	1.9997	0.422E-08	4.0067	0.147E-12	6.6382

$\lambda^2 = 1 + i$:

		$M=1$		$M=2$		$M=3$	
$\widetilde{f}(\mathbf{x})$	h^{-1}	error	rate	error	rate	error	rate
$f(\mathbf{x})$	10	0.815E-01		0.410E-02		0.134E-03	
	20	0.217E-01	1.9060	0.270E-03	3.9267	0.221E-05	5.9201
	40	0.553E-02	1.9760	0.171E-04	3.9821	0.351E-07	5.9800
	80	0.139E-02	1.9940	0.107E-05	3.9955	0.550E-09	5.9950
	160	0.347E-03	1.9985	0.669E-07	3.9989	0.860E-11	5.9987
	320	0.868E-04	1.9996	0.418E-08	3.9997	0.135E-12	5.9974
ext 1	10	0.814E-01		0.410E-02		0.134E-03	
	20	0.217E-01	1.9055	0.270E-03	3.9265	0.221E-05	5.9201
	40	0.553E-02	1.9759	0.171E-04	3.9820	0.351E-07	5.9800
	80	0.139E-02	1.9940	0.107E-05	3.9955	0.550E-09	5.9959
	160	0.347E-03	1.9985	0.669E-07	3.9989	0.826E-11	6.0555
	320	0.868E-04	1.9996	0.418E-08	3.9997	0.710E-12	3.5419
ext 2	10	0.819E-01		0.417E-02		0.139E-03	
	20	0.218E-01	1.9127	0.270E-03	3.9490	0.222E-05	5.9631
	40	0.553E-02	1.9768	0.171E-04	3.9846	0.351E-07	5.9849
	80	0.139E-02	1.9941	0.107E-05	3.9959	0.550E-09	5.9964
	160	0.347E-03	1.9985	0.669E-07	3.9989	0.873E-11	5.9767
	320	0.868E-04	1.9996	0.418E-08	3.9997	0.122E-12	6.1594
ext 3	10	0.924E-01		0.130E-01		0.238E-01	
	20	0.222E-01	2.0586	0.717E-03	4.1823	0.799E-04	8.2188
	40	0.554E-02	2.0011	0.220E-04	5.0283	0.101E-05	6.3010
	80	0.139E-02	1.9973	0.111E-05	4.3051	0.313E-08	8.3370
	160	0.347E-03	1.9989	0.673E-07	4.0461	0.139E-10	7.8181
	320	0.868E-04	1.9997	0.419E-08	4.0058	0.145E-12	6.5840

Table 1: Absolute errors and approximation rates for $\mathcal{K}_\lambda f(0.3, 0.3, 0)$ using $\mathcal{K}^{(M)}_{\lambda,h} f(0.3, 0.3, 0)$ with the density f given in (3.5) with $u(x) = \cos^2(\pi x/2)$ and different extensions, $M = 1, 2, 3$, $\lambda^2 = 1$ and $\lambda^2 = 1 + i$.

$\lambda^2 = 1$:

$\widetilde{f}(\mathbf{x})$	h^{-1}	$M=1$ error	$M=1$ rate	$M=2$ error	$M=2$ rate	$M=3$ error	$M=3$ rate
$f(\mathbf{x})$	10	0.673E-01		0.626E-02		0.427E-04	
	20	0.159E-01	2.0819	0.392E-03	3.9965	0.668E-06	5.9997
	40	0.391E-02	2.0238	0.246E-04	3.9970	0.104E-07	6.0000
	80	0.973E-03	2.0062	0.154E-05	3.9991	0.163E-09	6.0000
	160	0.243E-03	2.0016	0.960E-07	3.9998	0.255E-11	6.0000
	320	0.607E-04	2.0004	0.600E-08	3.9999	0.398E-13	6.0002
ext 1	10	0.637E-01		0.634E-02		0.427E-04	
	20	0.157E-01	2.0254	0.393E-03	4.0094	0.668E-06	5.9997
	40	0.389E-02	2.0075	0.246E-04	4.0003	0.104E-07	5.9995
	80	0.972E-03	2.0020	0.154E-05	3.9999	0.156E-09	6.0635
	160	0.243E-03	2.0005	0.961E-07	4.0000	0.389E-11	5.3255
	320	0.607E-04	2.0001	0.600E-08	4.0000	0.603E-12	2.6899
ext 2	10	0.603E-01		0.644E-02		0.427E-04	
	20	0.154E-01	1.9662	0.395E-03	4.0264	0.668E-06	5.9997
	40	0.388E-02	1.9925	0.246E-04	4.0052	0.104E-07	6.0003
	80	0.971E-03	1.9983	0.154E-05	4.0012	0.163E-09	6.0019
	160	0.243E-03	1.9996	0.961E-07	4.0003	0.224E-11	6.1838
	320	0.607E-04	1.9999	0.600E-08	4.0001	0.408E-12	2.4557
ext 3	10	0.291E-01		0.626E-02		0.427E-04	
	20	0.133E-01	1.1335	0.392E-03	3.9965	0.668E-06	5.9997
	40	0.374E-02	1.8264	0.246E-04	3.9970	0.104E-07	6.0000
	80	0.963E-03	1.9586	0.154E-05	3.9991	0.163E-09	6.0000
	160	0.224E-03	1.9894	0.960E-07	3.9998	0.255E-11	6.0000
	320	0.607E-04	1.9975	0.600E-08	3.9999	0.398E-13	6.0001

$\lambda^2 = 1 + i$:

$\widetilde{f}(\mathbf{x})$	h^{-1}	$M=1$ error	$M=1$ rate	$M=2$ error	$M=2$ rate	$M=3$ error	$M=3$ rate
$f(\mathbf{x})$	10	0.604E-01		0.572E-02		0.441E-04	
	20	0.142E-01	2.0834	0.358E-03	3.9963	0.690E-06	5.9997
	40	0.350E-02	2.0242	0.224E-04	3.9969	0.108E-07	6.0000
	80	0.872E-03	2.0062	0.140E-05	3.9991	0.168E-09	6.0000
	160	0.218E-03	2.0016	0.878E-07	3.9998	0.263E-11	6.0000
	320	0.544E-04	2.0004	0.548E-08	3.9999	0.410E-13	6.0025
ext 1	10	0.572E-01		0.579E-02		0.441E-04	
	20	0.140E-01	2.0271	0.360E-03	4.0096	0.690E-06	5.9997
	40	0.349E-02	2.0080	0.225E-04	4.0004	0.108E-07	5.9996
	80	0.871E-03	2.0021	0.140E-05	4.0000	0.163E-09	6.0465
	160	0.218E-03	2.0006	0.878E-07	4.0000	0.372E-11	5.4561
	320	0.544E-04	2.0001	0.548E-08	4.0000	0.539E-12	2.7853
ext 2	10	0.542E-01		0.589E-02		0.441E-04	
	20	0.138E-01	1.9681	0.361E-03	4.0272	0.690E-06	5.9997
	40	0.348E-02	1.9931	0.225E-04	4.0055	0.108E-07	6.0002
	80	0.870E-03	1.9984	0.140E-05	4.0013	0.168E-09	6.0014
	160	0.218E-03	1.9996	0.878E-07	4.0003	0.240E-11	6.1310
	320	0.544E-04	1.9999	0.548E-08	4.0001	0.365E-12	2.7174
ext 3	10	0.261E-01		0.803E-02		0.441E-04	
	20	0.119E-01	1.1338	0.560E-03	3.8421	0.690E-06	5.9997
	40	0.335E-02	1.8275	0.268E-04	4.3875	0.108E-07	6.0000
	80	0.862E-03	1.9590	0.148E-05	4.1767	0.168E-09	6.0000
	160	0.217E-03	1.9899	0.890E-07	4.0553	0.263E-11	6.0000
	320	0.544E-04	1.9975	0.550E-08	4.0151	0.410E-13	6.0030

Table 2: Absolute errors and approximation rates for $\mathcal{K}_\lambda f(0.5, 0.5, 0.5)$ using $\mathcal{K}^{(M)}_{\lambda,h} f(0.5, 0.5, 0.5)$ with the density f given in (3.5) with $u(x) = (x^2-1)^3$ and different extensions, $M = 1, 2, 3$, $\lambda^2 = 1$ and $\lambda^2 = 1 + i$.

$\lambda^2 = 1$:

$\widetilde{f}(\mathbf{x})$	h^{-1}	$M=1$ error	$M=1$ rate	$M=2$ error	$M=2$ rate	$M=3$ error	$M=3$ rate
$f(\mathbf{x})$	10	0.935E-01		0.166E-02		0.222E-15	
	20	0.241E-01	1.9564	0.104E-03	3.9984	0.777E-15	
	40	0.607E-02	1.9883	0.647E-05	3.9999	0.111E-15	
	80	0.152E-02	1.9970	0.405E-06	4.0000	0.555E-16	
	160	0.380E-03	1.9993	0.253E-07	4.0000	0.555E-16	
	320	0.951E-04	1.9998	0.158E-08	4.0000	0.222E-15	
ext 1	10	0.941E-01		0.166E-02		0.779E-10	
	20	0.241E-01	1.9632	0.104E-03	3.9984	0.336E-10	1.2133
	40	0.607E-02	1.9903	0.647E-05	3.9999	0.143E-10	1.2318
	80	0.152E-02	1.9975	0.405E-06	4.0000	0.628E-11	1.1873
	160	0.380E-03	1.9994	0.253E-07	4.0000	0.160E-12	5.2966
	320	0.951E-04	1.9998	0.158E-08	4.0000	0.268E-12	-0.7471
ext 2	10	0.946E-01		0.166E-02		0.201E-11	
	20	0.242E-01	1.9684	0.104E-03	3.9984	0.133E-11	0.5895
	40	0.607E-02	1.9920	0.647E-05	3.9999	0.139E-11	-0.0557
	80	0.152E-02	1.9980	0.405E-06	4.0000	0.641E-13	4.4336
	160	0.380E-03	1.9995	0.253E-07	4.0000	0.532E-12	-3.0524
	320	0.951E-04	1.9999	0.158E-08	4.0000	0.404E-12	0.3980
ext 3	10	0.983E-01		0.166E-02		0.222E-15	
	20	0.245E-01	2.0041	0.104E-03	3.9984	0.722E-15	
	40	0.610E-02	2.0066	0.647E-05	3.9999	0.111E-15	
	80	0.152E-02	2.0022	0.405E-06	4.0000	0.555E-16	
	160	0.380E-03	2.0006	0.253E-07	4.0000	0.555E-16	
	320	0.951E-04	2.0002	0.158E-08	4.0000	0.111E-15	

$\lambda^2 = 1+i$:

$\widetilde{f}(\mathbf{x})$	h^{-1}	$M=1$ error	$M=1$ rate	$M=2$ error	$M=2$ rate	$M=3$ error	$M=3$ rate
$f(\mathbf{x})$	10	0.869E-01		0.168E-02		0.220E-14	
	20	0.224E-01	1.9541	0.105E-03	3.9983	0.729E-15	
	40	0.565E-02	1.9878	0.655E-05	3.9999	0.397E-15	
	80	0.142E-02	1.9969	0.410E-06	4.0000	0.555E-16	
	160	0.354E-03	1.9992	0.256E-07	4.0000	0.128E-15	
	320	0.886E-04	1.9998	0.160E-08	4.0000	0.906E-16	
ext 1	10	0.875E-01		0.168E-02		0.695E-10	
	20	0.225E-01	1.9617	0.105E-03	3.9983	0.301E-10	1.2058
	40	0.566E-02	1.9899	0.655E-05	3.9999	0.128E-10	1.2359
	80	0.142E-02	1.9974	0.410E-06	4.0000	0.563E-11	1.1851
	160	0.354E-03	1.9994	0.256E-07	4.0000	0.144E-12	5.2918
	320	0.886E-04	1.9998	0.160E-08	4.0000	0.240E-12	-0.7425
ext 2	10	0.880E-01		0.168E-02		0.179E-11	
	20	0.225E-01	1.9675	0.105E-03	3.9983	0.119E-11	0.5930
	40	0.566E-02	1.9917	0.655E-05	3.9999	0.124E-11	-0.0642
	80	0.142E-02	1.9979	0.410E-06	4.0000	0.577E-13	4.4276
	160	0.354E-03	1.9995	0.256E-07	4.0000	0.476E-12	-3.0451
	320	0.886E-04	1.9999	0.160E-08	4.0000	0.361E-12	0.3971
ext 3	10	0.918E-01		0.168E-02		0.215E-14	
	20	0.228E-01	2.0074	0.105E-03	3.9983	0.625E-15	
	40	0.568E-02	2.0074	0.655E-05	3.9999	0.296E-15	
	80	0.142E-02	2.0024	0.410E-06	4.0000	0.706E-17	
	160	0.354E-03	2.0006	0.256E-07	4.0000	0.794E-16	
	320	0.886E-04	2.0002	0.160E-08	4.0000	0.119E-15	

Table 3: Absolute errors and approximation rates for $\mathcal{K}_\lambda f(0.4, 0.5, 0)$ using $\mathcal{K}^{(M)}_{\lambda,h} f(0.4, 0.5, 0)$ with the density f given in (3.5) with $u(x) = (1-x^2)^2$ and different extensions, with $M = 1, 2, 3$, $\lambda^2 = 1$ and $\lambda^2 = 1+i$.

$\widetilde{f}(\mathbf{x})$	n	10		10^2		10^3		10^4	
	h^{-1}	error	rate	error	rate	error	rate	error	rate
	10	0.338E-03		0.459E-02		0.487E-01		0.703E+00	
	20	0.605E-05	5.8020	0.732E-04	5.9727	0.746E-03	6.0282	0.751E-02	6.5491
	40	0.976E-07	5.9541	0.115E-05	5.9966	0.117E-04	5.9991	0.117E-03	6.0070
$f(\mathbf{x})$	80	0.154E-08	5.9887	0.179E-07	5.9994	0.182E-06	5.9999	0.183E-05	6.0000
	160	0.241E-10	5.9971	0.280E-09	6.0013	0.285E-08	6.0000	0.285E-07	5.9999
	320	0.376E-12	5.9982	0.513E-11	5.7677	0.445E-10	6.0005	0.446E-09	5.9985

$\widetilde{f}(\mathbf{x})$	n	10^5		10^6		10^7		10^8	
	h^{-1}	error	rate	error	rate	error	rate	error	rate
	20	0.794E-01		0.145E+01					
	40	0.117E-02	6.0852	0.118E-01	6.9443	0.129E+00		0.348E+01	
$f(\mathbf{x})$	80	0.183E-04	6.0012	0.183E-03	6.0133	0.183E-02	6.1364	0.185E-01	7.5527
	160	0.285E-06	5.9992	0.286E-05	5.9975	0.286E-04	5.9985	0.286E-03	6.0174
	320	0.451E-08	5.9842	0.478E-07	5.9030	0.510E-06	5.8096	0.517E-05	5.7889

$\widetilde{f}(\mathbf{x})$	n	10		10^2		10^3		10^4	
	h^{-1}	error	rate	error	rate	error	rate	error	rate
	10	0.352E-03		0.459E-02		0.487E-01		0.703E+00	
	20	0.611E-05	5.8476	0.732E-04	5.9726	0.746E-03	6.0282	0.751E-02	6.5491
ext	40	0.978E-07	5.9652	0.115E-05	5.9966	0.117E-04	5.9991	0.117E-03	6.0070
1	80	0.154E-08	5.9892	0.179E-07	5.9994	0.182E-06	5.9999	0.183E-05	6.0000
	160	0.230E-10	6.0635	0.280E-09	6.0013	0.285E-08	6.0000	0.285E-07	5.9999
	320	0.650E-12	5.1472	0.513E-11	5.7677	0.445E-10	6.0005	0.446E-09	5.9985

$\widetilde{f}(\mathbf{x})$	n	10^5		10^6		10^7		10^8	
	h^{-1}	error	rate	error	rate	error	rate	error	rate
	20	0.794E-01		0.145E+01					
ext	40	0.117E-02	6.0852	0.118E-01	6.9443	0.129E+00		0.348E+01	
1	80	0.183E-04	6.0012	0.183E-03	6.0133	0.183E-02	6.1364	0.185E-01	7.5527
	160	0.285E-06	5.9992	0.286E-05	5.9975	0.286E-04	5.9985	0.286E-03	6.0174
	320	0.451E-08	5.9842	0.478E-07	5.9030	0.510E-06	5.8096	0.517E-05	5.7889

$\widetilde{f}(\mathbf{x})$	n	10		10^2		10^3		10^4	
	h^{-1}	error	rate	error	rate	error	rate	error	rate
	10	0.415E-03		0.459E-02		0.487E-01		0.703E+00	
	20	0.632E-05	6.0374	0.732E-04	5.9727	0.746E-03	6.0282	0.751E-02	6.5491
ext	40	0.985E-07	6.0037	0.115E-05	5.9966	0.117E-04	5.9991	0.117E-03	6.0070
2	80	0.154E-08	5.9994	0.179E-07	5.9994	0.182E-06	5.9999	0.183E-05	6.0000
	160	0.241E-10	5.9999	0.280E-09	6.0013	0.285E-08	6.0000	0.285E-07	5.9999
	320	0.408E-12	5.8832	0.513E-11	5.7677	0.445E-10	6.0005	0.446E-09	5.9985

$\widetilde{f}(\mathbf{x})$	n	10^5		10^6		10^7		10^8	
	h^{-1}	error	rate	error	rate	error	rate	error	rate
	20	0.794E-01		0.145E+01					
ext	40	0.117E-02	6.0852	0.118E-01	6.9443	0.129E+00		0.348E+01	
2	80	0.183E-04	6.0012	0.183E-03	6.0133	0.183E-02	6.1364	0.185E-01	7.5527
	160	0.285E-06	5.9992	0.286E-05	5.9975	0.286E-04	5.9985	0.286E-03	6.0174
	320	0.451E-08	5.9842	0.478E-07	5.9030	0.510E-06	5.8096	0.517E-05	5.7889

Table 4: Absolute errors and approximation rates for $\mathcal{K}_\lambda f(0.5, 0..., 0)$ using $\mathcal{K}^{(3)}_{\lambda,h} f(0.5, 0, ..., 0)$ with the density f given in (3.5) with $u(x) = 1 - \sin(\pi x^2/2)$ and different extensions , $n = 10^i$, $i = 1, ..., 8$, $\lambda^2 = 1$.

$\widetilde{f}(\mathbf{x})$	n	10		10^2		10^3		10^4	
	h^{-1}	error	rate	error	rate	error	rate	error	rate
	10	0.699E-03		0.596E-02		0.595E-01		0.759E+00	
	20	0.106E-04	6.0400	0.902E-04	6.0453	0.880E-03	6.0792	0.881E-02	6.4288
	40	0.165E-06	6.0100	0.140E-05	6.0105	0.136E-04	6.0111	0.136E-03	6.0162
$f(\mathbf{x})$	80	0.257E-08	6.0025	0.218E-07	6.0026	0.213E-06	6.0026	0.212E-05	6.0027
	160	0.402E-10	6.0005	0.341E-09	6.0017	0.332E-08	6.0006	0.332E-07	6.0005
	320	0.632E-12	5.9909	0.491E-11	6.1156	0.585E-10	5.9998	0.519E-09	5.9973

$\widetilde{f}(\mathbf{x})$	n	10^5		10^6		10^7		10^8	
	h^{-1}	error	rate	error	rate	error	rate	error	rate
	20	0.913E-01		0.134E+01					
	40	0.136E-02	6.0671	0.137E-01	6.6101	0.145E+00		0.267E+01	
$f(\mathbf{x})$	80	0.212E-04	6.0035	0.212E-03	6.0113	0.212E-02	6.0906	0.214E-01	6.9639
	160	0.332E-06	5.9994	0.332E-05	5.9966	0.333E-04	5.9966	0.333E-03	6.0087
	320	0.526E-08	5.9779	0.572E-07	5.8594	0.632E-06	5.7186	0.646E-05	5.6865

$\widetilde{f}(\mathbf{x})$	n	10		10^2		10^3		10^4	
	h^{-1}	error	rate	error	rate	error	rate	error	rate
	10	0.690E-03		0.596E-02		0.595E-01		0.759E+00	
	20	0.106E-04	6.0254	0.902E-04	6.0453	0.880E-03	6.0792	0.881E-02	6.4288
ext	40	0.165E-06	6.0068	0.140E-05	6.0105	0.136E-04	6.0111	0.136E-03	6.0162
2	80	0.257E-08	6.0019	0.218E-07	6.0026	0.213E-06	6.0026	0.212E-05	6.0027
	160	0.401E-10	6.0046	0.341E-09	6.0017	0.332E-08	6.0006	0.332E-07	6.0005
	320	0.676E-12	5.8884	0.491E-11	6.1156	0.519E-10	5.9998	0.519E-09	5.9973

$\widetilde{f}(\mathbf{x})$	n	10^5		10^6		10^7		10^8	
	h^{-1}	error	rate	error	rate	error	rate	error	rate
	20	0.913E-01		0.134E+01					
ext	40	0.136E-02	6.0671	0.137E-01	6.6101	0.145E+00		0.267E+01	
2	80	0.212E-04	6.0035	0.212E-03	6.0113	0.212E-02	6.0906	0.214E-01	6.9639
	160	0.332E-06	5.9994	0.332E-05	5.9966	0.333E-04	5.9966	0.333E-03	6.0087
	320	0.526E-08	5.9779	0.572E-07	5.8594	0.632E-06	5.7186	0.646E-05	5.6865

$\widetilde{f}(\mathbf{x})$	n	10		10^2		10^3		10^4	
	h^{-1}	error	rate	error	rate	error	rate	error	rate
	10	0.156E-01		0.590E-02		0.595E-01		0.759E+00	
	20	0.165E-04	9.8811	0.901E-04	6.0349	0.880E-03	6.0791	0.881E-02	6.4288
ext	40	0.943E-07	7.4538	0.140E-05	6.0091	0.136E-04	6.0111	0.136E-03	6.0162
3	80	0.110E-08	6.4188	0.218E-07	6.0021	0.213E-06	6.0026	0.212E-05	6.0027
	160	0.333E-10	5.0496	0.340E-09	6.0016	0.332E-08	6.0006	0.332E-07	6.0005
	320	0.602E-12	5.7901	0.491E-11	6.1156	0.519E-10	5.9998	0.519E-09	5.9973

$\widetilde{f}(\mathbf{x})$	n	10^5		10^6		10^7		10^8	
	h^{-1}	error	rate	error	rate	error	rate	error	rate
	20	0.913E-01		0.134E+01					
ext	40	0.136E-02	6.0671	0.137E-01	6.6101	0.145E+00		0.267E+01	
3	80	0.212E-04	6.0035	0.212E-03	6.0113	0.212E-02	6.0906	0.214E-01	6.9639
	160	0.332E-06	5.9994	0.332E-05	5.9966	0.333E-04	5.9966	0.333E-03	6.0087
	320	0.526E-08	5.9779	0.572E-07	5.8599	0.632E-06	5.7186	0.646E-05	5.6865

Table 5: Absolute errors and approximation rates for $\mathcal{K}_\lambda f(0.4, 0.4, 0, ..., 0)$ using $\mathcal{K}^{(3)}_{\lambda,h} f(0.4, 0.4, 0, ..., 0)$ with the density f given in (3.5) with $u(x) = \mathrm{e}^x(1-x^2)^2$ and different extensions , $n = 10^i$, $i = 1, ..., 8$, $\lambda^2 = 1$.