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SOLVABILITY OF A BOUNDARY INTEGRAL EQUATION ON A POLYHEDRON

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The boundary integral equation associated with the Dirichlet problem for the Laplace equation on a polyhedral domain is considered. Pointwise estimates for the kernel of the inverse operator are derived. As a consequence, the solvability of the integral equation in the space of continuous functions and in a weighted L_p-space is obtained. Bibliography: 24 *titles.*

1 Introduction

This paper is closely related to our previous works [1]–[3], where integral equations of the harmonic and elastic potential theory on surfaces with conic vertices were considered. In this paper, we investigate the integral equation generated by the Dirichlet problem for the Laplace equation in a 3-dimensional polyhedron that is not necessarily a Lipschitz graph domain.

We use the method proposed by Maz'ya [4]–[7], according to which the analysis of boundary integral equations is reduced to the study of auxiliary boundary value problems. Different applications of the method can be found in [8]–[15].

Based on the estimates for the fundamental solutions of the Dirichlet and Neumann problems [16, 17] (cf. [18] for a detailed exposition), we estimate the kernel of the inverse operator of the integral equation in question, which leads to results on the solvability of this equation in various function spaces and, in particular, in the space C of continuous functions.

The question of the validity of the last result was stated long ago. The solvability of the boundary integral equation in the space C over surfaces of a fairly wide class was established in the multi-dimensional case by Burago, Maz'ya [19] and Kral [20] under the requirement that the essential norm $|T|$ of the double layer potential T is less than 1. This condition can be formulated in geometric terms. However, it does not always hold even for sufficiently simple cones. Angell,

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Kleinman, Kral [21] and Kral, Wendland [22] succeeded in compelling the inequality $|T| < 1$ for certain 3-dimensional polyhedra to hold by replacing the usual norm in C with an equivalent weighted norm. The polyhedral surfaces considered in [21] are constituted by a finite number of rectangles parallel to the coordinate planes.

The solvability in the space C for the above mentioned integral equation on surfaces in $\mathbb{R}^n$ with a finite number of conical points was proved by Grachev and Maz'ya [1]–[3] without any complementary geometric assumptions. Thus, it was shown that the use of the essential norm had been unnecessary and dictated only by the method of proof. We, and,independently Rathsfeld [23], extended this result to arbitrary polyhedra. A direct approach based on the Mellin transform was used in [23]. Some of the results of the present paper were announced in [24] and in Preprint LiTH-MAT-R-91-50.

Now, we briefly describe our results. We assume that Γ is a polyhedron in the three-dimensional Euclidean space. We denote by G^+ the interior of this polyhedron and consider the Dirichlet problem

$$\Delta u = 0 \quad \text{on} \quad G^+, \qquad u = f \quad \text{on} \quad \Gamma. \tag{1.1}$$

Let $O_1, \dots, O_m$ be the vertices of the polyhedron, and let $\mathfrak{M}_1, \dots, \mathfrak{M}_k$ be its edges. We denote by ω_j the opening of the dihedral angle with the edge $\mathfrak{M}_j$ from the side of G^+ and put $\lambda_j = \pi/\omega_j$. We use the notation

$$r_j(x) = \operatorname{dist}(x, \mathfrak{M}_j), \quad \rho_i(x) = \operatorname{dist}(x, O_i),$$
$$r(x) = \min_{1\leqslant j\leqslant k}\{r_j(x)\}, \quad \rho(x) = \min_{1\leqslant i\leqslant m}\{\rho_i(x)\}.$$

Let K_i, $k = 1, 2, \dots, m$, be the cone with vertex O_i which coincides with G^+ near the point O_i. The open set cut by the cone K_i out of the unit sphere S^2 centered at O_i is denoted by Ω_i^+, and the set $S^2\backslash\overline{\Omega_i^+}$ is denoted by Ω_i^-. Let δ_i and ν_i be positive numbers such that $\delta_i(\delta_i+1)$ and $\nu_i(\nu_i+1)$ are the first eigenvalues of the Dirichlet problem on Ω_i^+ and the Neumann problem on Ω_i^- for the Beltrami operator. Further, we denote by $\varkappa_i$ the minimum of δ_i, ν_i, and 1.

Let $W\psi$ denote the classical double layer potential with the density ψ:

$$(W\psi) = \frac{1}{4\pi}\int\limits_\Gamma \frac{\partial}{\partial n_\xi}\Big(\frac{1}{|x-\xi|}\Big)\psi(\xi)\,ds_\xi, \quad x \in G^\pm.$$

We are looking for a solution of Equation (1.1) in the form of a double potential. It is known that the density ψ satisfies the integral equation $(1+T)\psi = 2f$, where T is the operator on Γ defined by the equation $(T\psi)(x) = 2W_0\psi(x) + (1-d(x))\psi(x)$, where $d(x) = 1$ for $x \in \Omega\backslash\overline{\mathfrak{M}_j}$, $d(x) = \omega_j/\pi$ for $x \in \mathfrak{M}_j$, $d(x) = \operatorname{meas}\Omega_i^+/2\pi$ for $x \in O_i$, and $W_0\psi$ is the direct value on Γ of the double layer potential.

The following two theorems present the main results of this paper.

Theorem 1.1. *The operator $1+T : C(\Gamma) \to C(\Gamma)$ performs an isomorphism. The inverse operator admits the representation*

$$(1+T)^{-1}f = (1+L+M)f,$$

where L and M are integral operators on Γ with kernels $L(x,y)$ and $M(x,y)$ admitting the following estimates. If $\mathfrak{M}_j$ is the edge nearest to the point y and O_i is the vertex nearest to y, *then*

$$|M(x,y)| \leqslant c\,\rho(y)^{\varkappa_i-1-\varepsilon}\Big(\frac{r(y)}{\rho(y)}\Big)^{\lambda_j-1-\varepsilon}.$$

If points x and y lie in a neighborhood of a vertex O_i, $i=1,2,\dots,m$, this neighborhood contains no vertices of the polyhedron O_i and, if $\mathfrak{M}_j$, $\mathfrak{M}_l$ are the edges nearest to the points y and x respectively, then

$$|L(x,y)| \leqslant c\,\rho(y)^{-2}\big(r(y)/\rho(y)\big)^{\lambda_j-1-\varepsilon} + c\big(r(y)+|x-y|\big)^{-2}\Big(\frac{r(x)}{r(x)+|x-y|}\Big)^{\lambda_l-\varepsilon}\Big(\frac{r(y)}{r(y)+|x-y|}\Big)^{\lambda_j-1-\varepsilon}$$

for $\rho(x)/2<\rho(y)<2\rho(x)$ and

$$|L(x,y)| \leqslant c\,\rho(y)^{-1}\big(\rho(x)+\rho(y)\big)^{-1}\Big(\frac{\min\{\rho(x),\rho(y)\}}{\rho(x)+\rho(y)}\Big)^{\varkappa_i-\varepsilon}\Big(\frac{r(y)}{\rho(y)}\Big)^{\lambda_j-1-\varepsilon}$$

in the opposite case. Here, ε is an arbitrary positive number.

The following theorem concerns the operator defined by $T\psi = 2W_0\psi$ almost everywhere on Γ as an operator in the weighted L_p-space $L^p_{\beta,\gamma}(\Gamma)$ endowed with the norm

$$\|u\|_{L^p_{\beta,\gamma}(\Gamma)} = \|\rho^\beta\, r^\gamma\, u\|_{L_p(\Gamma)}.$$

Theorem 1.2. *Let $\varkappa=\min\{\varkappa_i\}$, $\lambda=\min\{\lambda_j\}$. If $1\leqslant p<\infty$, $0<\beta+\gamma+2/p<1+\varkappa$, $0<\gamma+1/p<\lambda$ or $p=\infty$, $0\leqslant\beta+\gamma<1+\varkappa$, $0\leqslant\gamma<\lambda$, then the operator $1+T : L^p_{\beta,\gamma}(\Gamma)\to L^p_{\beta,\gamma}(\Gamma)$ performs an isomorphism.*

In Section 2, we collect some preliminary information on boundary value problems and find a representation for the inverse operator of the integral equation in question stated in terms of the inverse operators of boundary value problems. The estimates for $L(x,y)$ and $M(x,y)$ in Theorem 1.1 are obtained in Section 3. Finally, in Section 4, we prove theorems on the unique solvability of the integral equation in the spaces C and $L^p_{\beta,\gamma}$.

2 Representation for the Inverse Operator of the Boundary Integral Equation

2.1 Preliminary information

We use the notation from Section 1. We also denote $G^- = \mathbb{R}^3\backslash\overline{G^+}$ and $B(r,x)=\{y\in\mathbb{R}^3 : |x-y|<r\}$.

We define some weighted Hölder spaces. For the sake of simplicity we introduce the same weight r^γ for all edges and the same weight ρ^β for all vertices. We denote by $N^{l,\alpha}_{\beta,\gamma}(G^+)$ the space of functions on G^+ with the finite norm

$$\|u\|_{N^{l,\alpha}_{\beta,\gamma}(G^+)} = \sup_{x\in F^+}\rho(x)^\beta r(x)^\gamma [u]^{l+\alpha}_{B(r/2,x)\cap G^+} + \sup_{x\in G^+}\rho(x)^\beta r(x)\gamma - l - \alpha|u(x)|. \tag{2.1}$$

Here, β, γ are real numbers, $\alpha\in(0,1)$, l is an integer, $l\geqslant 0$, and

$$[u]^\rho_E = \sup_{x,y\in E}\sum_{|\sigma|=[\rho]}|x-y|^{[\rho]-\rho}|\partial^\sigma_x u(x)-\partial^\sigma_y u(y)|,$$

where E is a subset of $\mathbb{R}^3$, ρ is a positive noninteger, and $[\rho]$ is the integer part of ρ.

We also introduce the space $C^{l,\alpha}_{\beta,\gamma}(G^+)$ $(0<\gamma<l+\alpha,\ l+\alpha-\gamma$ is not integer) of functions u in G^+ with the finite norm

$$\|u\|_{C^{l,\alpha}_{\beta,\gamma}(K)} = \sup_{x\in G^+} \rho(x)^\beta r(x)^\gamma \big[u\big]^{l+\alpha}_{G^+\cap B(r/2,x)} + \sup_{x\in G^+} \rho(x)^\beta \big[u\big]^{l+\alpha-\gamma}_{G^+\cap B(\rho/2,x)} + \sup_{x\in G^+} \rho(x)^{\beta+\gamma-l-\alpha}|u(x)|. \tag{2.2}$$

For the domain G^- we define similar spaces $N^{l,\alpha}_{\beta,\gamma}(G^-)$ and $C^{l,\alpha}_{\beta,\gamma}(G^-)$. Suppose that the ball $B(R,0)$ contains $\overline{G^+}$. We denote by χ a function from the space $C^\infty(\mathbb{R}^3)$ equal to one on $B(R,0)$ and to zero on $\mathbb{R}^3\backslash B(R+1,0)$. A function u in G^- belongs to $N^{l,\alpha}_{\beta,\gamma}(G^-)$ and to $C^{l,\alpha}_{\beta,\gamma}(G^-)$ respectively if and only if the norms (2.1) and (2.2) respectively of $u\chi$ and the norm

$$\sup_{x\in G^-} |x|^{l+\alpha+1}\big[v\big]^{l+\alpha}_{B(|x|/2,x)} + \sup_{x\in G^-} |x|\,|v(x)|$$

of the function $v=(1-\chi)u$ are finite.

Let Γ_i denote a face of the polyhedron Γ. We denote by $N^{l,\alpha}_{\beta,\gamma}(\Gamma_i)$ the space of traces on Γ_i of functions from $N^{l,\alpha}_{\beta,\gamma}(G^+)$ or from $N^{l,\alpha}_{\beta,\gamma}(G^-)$. We say that u belongs to $N^{l,\alpha}_{\beta,\gamma}(\Gamma)$ if the restriction u_i on each Γ_i belongs to $N^{l,\alpha}_{\beta,\gamma}(\Gamma_i)$ and introduce the norm

$$\|u\|_{N^{l,\alpha}_{\beta,\gamma}(\Gamma)} = \sum_i \|u\|_{N^{l,\alpha}_{\beta,\gamma}(\Gamma_i)}.$$

The space of traces on Γ of functions from $C^{l,\alpha}_{\beta,\gamma}(G^+)$ or from $C^{l,\alpha}_{\beta,\gamma}(G^-)$ is denoted by $C^{l,\alpha}_{\beta,\gamma}(\Gamma)$.

Consider the interior Dirichlet problem and the exterior Neumann problem for the Laplace equation

$$\Delta u = 0 \quad \text{on} \quad G^+, \qquad u = f \quad \text{on} \quad \Gamma, \tag{2.3}$$

$$\Delta v = 0 \quad \text{on} \quad G^-, \qquad \partial v/\partial n = g \quad \text{on} \quad \Gamma\backslash\mathfrak{M}, \tag{2.4}$$

where $\partial/\partial n$ stands for the derivative in the direction of the outward normal to $\Gamma\backslash\mathfrak{M} = \bigcup\limits_{1\leqslant i\leqslant k} \overline{\mathfrak{M}_i}$.

Now, we formulate estimates for the fundamental solutions of the problems (2.3) and (2.4). Let K_i, $i=1,2,\ldots,m$, be the cone with the vertex O_i which coincides with G^+ near the point O_i. The open set that the cone K_i cuts from the unit sphere S^2 centered at O_i is denoted by Ω^+, and the set $S^2\backslash\overline{\Omega^+}$ is denoted by Ω^-. Let δ_i and ν_i be positive numbers such that $\delta_i(\delta_i+1)$ and $\nu_i(\nu_i+1)$ are the first positive eigenvalues of the Dirichlet problem in Ω^+ and the Neumann problem in Ω^- for the Laplace–Beltrami operator on S^2. The result formulated here is contained in [16].

Theorem 2.1. *Suppose that* $\delta^+ = \min_{1\leqslant j\leqslant} \delta_i$, $\lambda^+ = \min_{1\leqslant i\leqslant k} \pi/\omega_i$, *and* l *is a positive integer. If* $-\delta^+ < \beta+\gamma-\alpha < 1+\delta^+$ *and* $0<\alpha-\gamma<\min\{1,\lambda^+\}$, *then for any* $f\in C^{l,\alpha}_{\beta,\gamma+l}(\Gamma)$ *there exists a unique solution* $u\in C^{l,\alpha}_{\beta,\gamma+l}(G^+)$ *of the Dirichlet problem* (2.3) *and the solution admits the representation*

$$u(x) = \int_\Gamma \mathcal{P}^+(x,\xi)\, f(\xi)\, ds_\xi \tag{2.5}$$

Suppose that points x and ξ lie in a neighborhood of a vertex O_i, $i = 1, 2, \dots, m,$. If either $2\,\rho(\xi) < \rho(x)$ or $\rho(\xi) > 2\,\rho(x)$, then

$$|\partial_x^\sigma \partial_\xi^\tau \mathcal{P}^+(x,\xi)| \leqslant c_{\sigma,\tau}\,\rho(x)^{-|\sigma|}\rho(\xi)^{-1-|\tau|}\big(\rho(x)+\rho(\xi)\big)^{-1} \times \Big(\frac{\min\{\rho(x),\rho(\xi)\}}{\rho(x)+\rho(\xi)}\Big)^{\delta^+-\varepsilon}\Big(\frac{r(x)}{\rho(x)}\Big)^{\lambda^+-|\sigma|-\varepsilon}\Big(\frac{r(\xi)}{\rho(\xi)}\Big)^{\lambda^+-|\tau|-1-\varepsilon}.$$

In the zone $\rho(\xi) < 2\,\rho(x) < 4\,\rho(\xi)$, the estimates have the form

$$|\partial_x^\sigma \partial_\xi^\tau \mathcal{P}^+(x,\xi)| \leqslant c_{\sigma,\tau}\,|x-\xi|^{-2-|\sigma|-|\tau|} \times \Big(\frac{r(x)}{r(x)+|x-\xi|}\Big)^{\lambda^+-|\sigma|-\varepsilon}\Big(\frac{r(\xi)}{r(\xi)+|x-\xi|}\Big)^{\lambda^+-1-|\tau|-\varepsilon}.$$

In the case $x \in U_i$, $\xi \in U_q$, where U_i and U_q are small neighborhoods of the vertices O_i and O_q with $i \neq q$, the estimates take the form

$$|\partial_x^\sigma \partial_\xi^\tau \mathcal{P}^+(x,\xi)| \leqslant c_{\sigma,\tau}\,\rho(x)^{\delta^+-|\sigma|-\varepsilon}\rho(\xi)^{\delta^+-|\tau|-1-\varepsilon}\Big(\frac{r(x)}{\rho(x)}\Big)^{\lambda^+-|\sigma|-\varepsilon}\Big(\frac{r(\xi)}{\rho(\xi)}\Big)^{\lambda^+-|\tau|-1-\varepsilon},$$

where σ, τ are arbitrary multi-indices and ε is a sufficiently small positive number.

The following result is essentially proved in [17].

Theorem 2.2. *Suppose that $\nu^- = \min_{1\leqslant i\leqslant m}\nu_i$, $\lambda^- = \min_{1\leqslant j\leqslant k}\{\pi/(2\pi-\omega_j)\}$, and l is a positive integer. If $0 < \beta+\gamma-\alpha < 1$ AND $0 < \alpha-\gamma < \min\{1,\lambda^-\}$, then for any $g \in N^{l,\alpha}_{\beta,\gamma+l}(\Gamma)$ there exists a unique solution $v \in C^{l,\alpha}_{\beta,\gamma+l}(G^-)$ of the Neumann problem (2.4) and*

$$v(x) = \int_\Gamma Q^-(x,\xi)\,g(\xi)\,ds_\xi. \tag{2.6}$$

Suppose that points x and ξ lie in a neighborhoods of the vertex O_i, $i = 1, 2, \dots, m,$. If either $2\,\rho(x) < \rho(\xi)$ or $\rho(x) > 2\,\rho(\xi)$, then

$$Q^-(x,\xi) = Q^-(0,\xi) + R^-(x,\xi) \quad \text{for} \quad 2\rho(x) < \rho(\xi), \tag{2.7}$$

$$Q^-(x,\xi) = Q^-(x,0) + R^-(\xi,x) \quad \text{for} \quad 2\rho(\xi) < \rho(x), \tag{2.8}$$

where

$$Q^-(0,\xi) = Q^-(\xi,0) = a_i^-/\rho(\xi) + b_i^- + d_i^-(\xi) \tag{2.9}$$

and $a_i^- = 1/\operatorname{meas}(\Omega_i^-)$, $b_i^- = \text{const}$ For $R^-(x,\xi)$ and $d_i^-(\xi)$ the following estimates hold:

$$|\partial_\xi^\sigma\, d_i^-(\xi)| \leqslant c_\sigma\,\rho(x)^{\nu^- -|\alpha|-\varepsilon}\Big(\frac{r(\xi)}{\rho(\xi)}\Big)^{\lambda^-_{\sigma\varepsilon}},$$

$$|\partial_x^\sigma \partial_\xi^\tau \mathcal{R}^-(x,\xi)| \leqslant c_{\sigma,\tau}\,\rho(x)^{\nu^- -|\sigma|-\varepsilon}\rho(\xi)^{-1-\nu^- -|\tau|+\varepsilon}\Big(\frac{r(x)}{\rho(x)}\Big)^{\lambda^-_{\sigma\varepsilon}}\Big(\frac{r(\xi)}{\rho(\xi)}\Big)^{\lambda^-_{\sigma\varepsilon}}.$$

In the intermediate zone $\rho(x) < 2\,\rho(\xi) < 4\,\rho(x)$, the estimate takes the form

$$|\partial_x^\sigma \partial_\xi^\tau Q^-(x,\xi)| \leqslant \frac{c_{\sigma\tau}}{|x-\xi|^{1+|\sigma|+|\tau|}}\Big(\frac{r(x)}{r(x)+|x-\xi|}\Big)^{\lambda^-_{\sigma\varepsilon}}\Big(\frac{r(\xi)}{r(\xi)+|x-\xi|}\Big)^{\lambda^-_{\tau\varepsilon}}.$$

In the case $x \in U_i$, $\xi \in U_q$, where U_i and U_q are small neighborhoods of the vertices O_i and O_q with $i \neq q$,

$$|\partial_x^\sigma \partial_\xi^\tau Q^-(x,\xi)| \leqslant c_{\sigma,\tau}\, \rho(x)^{\nu_{\sigma\varepsilon}^-} \rho(\xi)^{\nu_{\tau\varepsilon}^-} \Big(\frac{r(x)}{\rho(x)}\Big)^{\lambda_{\sigma\varepsilon}^-} \Big(\frac{r(\xi)}{\rho(\xi)}\Big)^{\lambda_{\tau\varepsilon}^-}.$$

Here, we use the notation

$$\lambda_{\sigma\varepsilon}^- = \min\{0, \lambda^- - |\sigma| - \varepsilon\}, \quad \lambda_{\tau\varepsilon}^- = \min\{0, \lambda^- - |\tau| - \varepsilon\},$$
$$\nu_{\sigma\varepsilon}^- = \min\{0, \nu^- - |\sigma| - \varepsilon\}, \quad \nu_{\tau\varepsilon}^q = \min\{0, \nu^- - |\tau| - \varepsilon\}.$$

In what follows, we need estimates for the fundamental solutions of the Dirichlet and Neumann problems in a dihedral angle. Let D^+ be the interior of the angle with opening ω, and let $D^- = \mathbb{R}^3 \backslash \overline{D^+}$. We denote by F^+ and F^- the sides of D^+, by $\mathfrak{M}$ the edge and by F the boundary, i.e. $F = F^+ \cup F^- \cup \mathfrak{M}$.

We introduce the space $N_\gamma^{l,\alpha}(D^+)$ equipped with the norm

$$\|u\|_{N_\gamma^{l,\alpha}(D^+)} = \sup_{x\in D^+} r(x)^\gamma \big[u\big]^{l+\alpha}_{D^+\cap B(r/2,x)} + \sup_{x\in D^+} r(x)^{\gamma-l-\alpha}|u(x)|.$$

and the space $C_\gamma^{l,\alpha}(D^+)$, $l+\alpha-\gamma>0$, equipped with the norm

$$\|u\|_{C_\gamma^{l,\alpha}(D^+)} = \sup_{x\in D^+} r(x)^\gamma \big[u\big]^{l+\alpha}_{D^+\cap B(r/2,x)} + \|u\|_{C^{l+\alpha-\gamma}(\overline{D^+})},$$

where $C^s(\overline{D^+})$ is the Hölder space of order s and $r(x) = \operatorname{dist}(x, \mathfrak{M})$.

We denote by $N_\gamma^{l,\alpha}(F^\pm)$ the space of traces on $F^\pm$ of functions from $N_\gamma^{l,\alpha}(D^+)$ or from $N_{\beta,\gamma}^{l,\alpha}(G^-)$. We say that u belongs to $N_\gamma^{l,\alpha}(F)$ if the restriction $u^\pm$ to $F^\pm$ belongs to $N_\gamma^{l,\alpha}(F^\pm)$ and introduce the norm

$$\|u\|_{N_\gamma^{l,\alpha}(F)} = \sum_\pm \|u\|_{N_\gamma^{l,\alpha}(F^\pm)}.$$

The space of traces on F of functions from $C_\gamma^{l,\alpha}(D^+)$ is denoted by $C_\gamma^{l,\alpha}(F)$. Similarly, one defines spaces of functions on D^-.

Consider two boundary value problems

$$\Delta u = 0 \ \text{ in } \ D^+, \quad u = f \ \text{ on } \ F, \tag{2.10}$$

$$\Delta v = 0 \ \text{ in } \ D^-, \quad \partial v/\partial n = g \ \text{ on } \ F\backslash\mathfrak{M}. \tag{2.11}$$

The following theorem was proved in [16].

Theorem 2.3. *Let $0 < \alpha - \gamma < \min\{1, \pi/\omega\}$, and let l be a positive integer. Then for any $f \in C_{\beta,\gamma+l}^{l,\alpha}(F)$ there exists a unique solution $u \in C_{\beta,\gamma+l}^{l,\alpha}(D^+)$ of the Dirichlet problem (2.10). It admits the representation*

$$u(x) = \int_F P^+(x,\xi)\, f(\xi)\, ds_\xi, \tag{2.12}$$

where

$$|\partial_x^\sigma \partial_\xi^\tau P^+(x,\xi)| \leqslant c_{\sigma,\tau}\, |x-\xi|^{-2-|\sigma|-|\tau|} \Big(\frac{r(x)}{r(x)+|x-\xi|}\Big)^{\pi/\omega-|\sigma|-\varepsilon} \Big(\frac{r(\xi)}{r(\xi)+|x-\xi|}\Big)^{\pi/\omega-1-|\tau|-\varepsilon}.$$

Now, we formulate an analogous result for the Neumann problem obtained in [17].

Theorem 2.4. *Let* $0 < \alpha - \gamma < \lambda^-$, $\lambda^- = \min\{1, \pi/(2\pi - \omega)\}$. *and let* l *be a positive integer. Then for any* $g \in C^{l,\alpha}_{\beta,\gamma+l}(F)$ *there exists a unique solution* $v \in C^{l,\alpha}_{\beta,\gamma+l}(D^-)$ *of the Dirichlet problem* (2.11). *It admits the representation*

$$v(x) = \int\limits_F Q^-(x,\xi)\, g(\xi)\, ds_\xi, \tag{2.13}$$

where

$$|\partial_x^\sigma \partial_\xi^\tau \big(Q^-(x,\xi) - a/|x-\xi|\big)| \leqslant c_{\sigma,\tau}\, |x-\xi|^{-1-|\sigma|-|\tau|} \Big(\frac{r(x)}{r(x)+|x-\xi|}\Big)^{\lambda^- -|\sigma|-\varepsilon} \times \Big(\frac{r(\xi)}{r(\xi)+|x-\xi|}\Big)^{\lambda^- -|\tau|-\varepsilon},$$

where $a = 1/\operatorname{meas}(S^2 \cap D^-)$ *and* S^2 *is the unit sphere with center at* $x \in \mathfrak{M}$.

2.2 Representations for the inverse operators

We denote by $V\psi$ and $W\psi$ the single and double layer potentials:

$$(V\psi) = \frac{1}{4\pi} \int\limits_\Gamma \frac{1}{|x-\xi|} \varphi(\xi)\, ds_\xi, \quad x \in \mathbb{R}^3,$$

$$(W\psi) = \frac{1}{4\pi} \int\limits_\Gamma \frac{\partial}{\partial n_\xi} \Big(\frac{1}{|x-\xi|}\Big) \psi(\xi)\, ds_\xi, \quad x \in G^\pm.$$

In what follows, we denote by $(\cdot)^+$ and $(\cdot)^-$ the interior and exterior limit values with respect to G^+. By $W_0\psi$ we mean the direct values of the double layer potential $W\psi$ on Γ. Let the operator T be defined by the equality $(T\psi)(x) = 2W_0\psi(x) + (1-d(x))\,\psi(x)$, where $d(x) = \lim\limits_{\delta\to 0^+} (\operatorname{meas}(G^+ \cap B(\delta,x))/\operatorname{meas} B(\delta,x)$.

Lemma 2.1. *Suppose that* $0 < \beta + \gamma - \alpha < 2$, $0 < \alpha - \gamma < 1$, *and* l *is a positive integer. If* $\psi \in C^{l,\alpha}_{\beta,\gamma+l}(\Gamma)$, *then* $W_0\psi \in C^{l,\alpha}_{\beta,\gamma+l}(\Gamma)$ *and*

$$(W\psi)^\pm = W_0\psi \pm \psi/2, \quad \Big(\frac{\partial(W\psi)}{\partial n}\Big)^+ = \Big(\frac{\partial(W\psi)}{\partial n}\Big)^- \quad \textit{on} \ \ \Gamma\backslash\mathcal{M}. \tag{2.14}$$

Proof. Let δ be so small that the ball $B(2\delta, O_i)$ contains no vertices except O_i. One verifies directly the estimate

$$\sup_{x\in B(\delta,O_i)} \rho(x)^{\beta+\gamma-\alpha}\, |(W\psi)(x)| \leqslant c \sup_{x\in\Gamma} \rho(x)^{\beta+\gamma-\alpha}\, |\psi(x)|. \tag{2.15}$$

We consider the transmission problem

$$\begin{aligned} &\Delta u = 0 \ \text{ in } \ G^+ \cup G^-, \quad u^+ - u^- = \psi \ \text{ on } \ \Gamma, \\ &\Big(\frac{\partial u}{\partial n}\Big)^+ - \Big(\frac{\partial u}{\partial n}\Big)^- = 0 \ \text{ on } \ \Gamma\backslash\mathfrak{M} \end{aligned} \tag{2.16}$$

which is satisfied by $W\psi \in C^{l,\alpha}_{\mathrm{loc}}(\overline{G^{\pm}}\backslash\mathfrak{M})$. We introduce the sets $U_k = \{\xi : 1/2 < 2^k|\xi| < 2\}$ and $V_k = \{\xi : 1/4 < 2^k|\xi| < 4\}$ for $k = 1, 2, \ldots$.

The well-known local Schauder estimate for solutions of (2.16) leads to the inequality

$$\begin{aligned}
&2^{-k(l+\beta)} \sup_{U_k\cap G^{\pm}} r(x)^{\gamma}[u]^{l+\alpha}_{B(r/2,x)\cap G^{\pm}} + 2^{-k(l+\beta)}[u]^{l+\alpha-\gamma}_{U_k\cap G^{\pm}}\\
&\leqslant c\big(2^{-k(l+\beta)} \sup_{V_k\cap\Gamma} r(x)^{\gamma}[\psi]^{l+\alpha}_{B(r/2,x)\cap\Gamma} + 2^{-k(l+\beta)}[\psi]^{l+\alpha-\gamma}_{V_k\cap\Gamma}\\
&+ 2^{-k(\beta+\gamma-\alpha)} \sup_{V_k\cap\Gamma} |\psi(x)| + 2^{-k(\beta+\gamma-\alpha)} \sup_{x\in V_k\cap\Gamma} |u(x)|\big).
\end{aligned}$$

From this inequality and (2.15) we conclude that $W\psi \in C^{l,\alpha}_{\beta,\gamma+l}(G^{\pm})$.

The relations (2.14) follow from similar relations for domains with smooth boundaries. □

Lemma 2.2. *Suppose that* $0 < \beta + \gamma - \alpha < 1$, $0 < \alpha - \gamma < 1$, *and* l *is a positive integer. If* $\varphi \in N^{l-1,\alpha}_{\beta,\gamma+l}(\Gamma)$, *then* $V\varphi \in N^{l,\alpha}_{\beta,\gamma+l}(\Gamma^{\pm})$ *and*

$$\Big(\frac{\partial(V\varphi)}{\partial n}\Big)^{\pm} = -W_0^*\varphi \pm \varphi/2, \quad (V\varphi)^+ = (V\varphi)^-$$

on $\Gamma\backslash\mathcal{M}$. *Here,* W_0^* *is the operator formally adjoint of* W_0.

Proof. One verifies directly the estimates

$$\sup_{x\in B(\delta,O_i)} \rho(x)^{\beta+\gamma-\alpha}|(V\varphi)(x)| \leqslant c \sup_{x\in\Gamma} \rho(x)^{\beta} r(x)^{\gamma-\alpha+1}|\varphi(x)|, \quad i = 1, 2, \ldots, m,$$

where δ is the same as in (2.15). To get the result, it suffices to apply the same argument as in the proof of Lemma 2.1 to the transmission problem

$$\begin{aligned}
&\Delta v = 0 \ \text{ on } \ G^+ \cup G^-, \quad v^+ - v^- = 0 \ \text{ on } \ \Gamma,\\
&\Big(\frac{\partial v}{\partial n}\Big)^+ - \Big(\frac{\partial v}{\partial n}\Big)^- = \varphi \ \text{ on } \ \Gamma\backslash\mathcal{M}. \square
\end{aligned}$$

Lemma 2.3. *Suppose that* $0 < \beta + \gamma - \alpha < 1$, $0 < \alpha - \gamma < 1$, *and* l *is a positive integer. Then the following representation holds*:

$$u = V\Big(\Big(\frac{\partial u}{\partial n}\Big)^+ - \Big(\frac{\partial u}{\partial n}\Big)^-\Big) + W(u^+ - u^-)$$

on $G^+ \cup G^-$ *for all* $u \in C^{l,\alpha}_{\beta,\gamma+l}(G^+ \cup G^-)$ *satisfying* $\Delta u = 0$ *on* $G^+ \cup G^-$. *Here,* $C^{l,\alpha}_{\beta,\gamma}(G^+ \cup G^-)$ *is the space of functions* u *in* $G^+ \cup G^-$ *whose restrictions to* $G^{\pm}$ *belong to* $C^{l,\alpha}_{\beta,\gamma}(G^{\pm})$.

Proof. We use the following classical relations:

$$\begin{aligned}
u(x) &= (V(\partial u/\partial n)^+)(x) + (Wu^+)(x), \quad x \in G^+,\\
0 &= (V(\partial u/\partial n)^+)(x) + (Wu^+)(x), \quad x \in G^-,\\
0 &= -(V(\partial u/\partial n)^-)(x) - (Wu^-)(x), \quad x \in G^+,\\
u(x) &= -(V(\partial u/\partial n)^-)(x) - (Wu^-)(x), \quad x \in G^-,
\end{aligned}$$

for all functions u such that $u \in C^{\infty}(\overline{G^{\pm}})$, $u = O(|x|^{-1})$ as $x \to \infty$, $\Delta u = 0$ on $G^+ \cup G^-$. Using Lemmas 2.1 and 2.2, one can show that these relations extend to all harmonic functions $u \in C^{l,\alpha}_{\beta,\gamma+l}(G^+ \cup G^-)$ on $G^+ \cup G^-$. □

Theorem 2.5. *Suppose that* $0 < \alpha - \gamma < \min\{\lambda^+, \lambda^-\}$, $0 < \beta + \gamma - \alpha < \min\{\delta^+, \nu^-, 1\}$, *and* l *is a positive integer. If* $f \in C^{l,\alpha}_{\beta,\gamma+l}(\Gamma)$, *then there exists a unique solution* $\varphi \in C^{l,\alpha}_{\beta,\gamma+l}(\Gamma)$ *of the integral equation* $(1+T)\varphi = f$ *and this solution can be represented in the form*

$$(1+T)^{-1} f = \frac{1}{2}\Big(1 - Q^- \frac{\partial}{\partial n} P^+\Big) f. \tag{2.17}$$

Here, P^+ *and* Q^- *are the inverse operators of the boundary value problems* (2.3) *and* (2.4) (*cf. Theorems* 2.1 *and* 2.2).

Proof. By Theorems 2.1 and 2.2, the function

$$\varphi = \frac{1}{2}\Big(1 - Q^- \frac{\partial}{\partial n} P^+\Big) f$$

belongs to the space $C^{l,\alpha}_{\beta,\gamma+l}(\Gamma)$. We prove that φ is a solution of the equation $(1+T)\varphi = f$. We introduce the function $u \in C^{l,\alpha}_{\beta,\gamma+l}(G^+ \cup G^-)$ which is a solution of the boundary value problem

$$\Delta u = 0 \ \text{ on } \ G^+ \cup G^-, \quad u^+ = f \ \text{ on } \ \Gamma,$$
$$\Big(\frac{\partial u}{\partial n}\Big)^+ - \Big(\frac{\partial u}{\partial n}\Big)^- = 0 \ \text{ on } \ \Gamma\backslash\mathfrak{M}.$$

It is clear that $u^- = Q^- \dfrac{\partial}{\partial n} P^+ f$. Hence $\varphi = (u^+ - u^-)/2$. By this and Lemmas 2.1, 2.3, we arrive at the chain of equalities

$$\big((1+T)\varphi\big)(x) = 2(W\varphi)^+(x) = (W(u^+ - u^-))^+(x) = u^+(x) = f(x) \tag{2.18}$$

for $x \in \Gamma\backslash\mathfrak{M}$. Since $(W_0 1)(x)$ is the solid angle under which the surface Γ is seen from x, we conclude that $T\varphi \in C(\Gamma)$ for $\varphi \in C(\Gamma)$ and the relations (2.18) hold for all $x \in \Gamma$.

It remains to verify the uniqueness of the solution. Let $\varphi_0 \in C^{l,\alpha}_{\beta,\gamma+l}(\Gamma)$ satisfy $(1+T)\varphi_0 = 0$. Consider the function $u = W\varphi_0$. By Lemma 2.1, u is a solution of (2.3) with $f = 0$. In view of the uniqueness of the solution of (2.3), we conclude that $W\varphi_0 = 0$ on G^+. Since

$$\Big(\frac{\partial(W\varphi_0)}{\partial n}\Big)^- = \Big(\frac{\partial(W\varphi_0)}{\partial n}\Big)^+ = 0 \ \text{ on } \ \Gamma\backslash\mathfrak{M}$$

and (2.4) is uniquely solvable, we conclude that $W\varphi_0 = 0$ in G^-. Thus,

$$\varphi_0 = (W\varphi_0)^+ - (W\varphi_0)^- = 0,$$

which completes the proof. □

3 Estimates for the Kernel of the Inverse Operator

In what follows, we use the notation $\varkappa = \min\{\delta^+, \nu^-, 1\}$ and $\lambda = \min\{\lambda^+, \lambda^-\}$. The goal of this section is to prove the following assertion.

Theorem 3.1. *Suppose that* $0 < \alpha - \gamma < \lambda$, $0 < \beta + \gamma - \alpha < 1$, *and* l *is a positive integer. Then*

$$(1+T)^{-1} f = (1 + L + M) f, \quad f \in C^{l,\alpha}_{\beta,\gamma+l}(\Gamma), \tag{3.1}$$

where L and M are integral operators on Γ. The kernel $M(x,y)$ of the operator M admits the estimate

$$|M(x,y)| \leqslant c\,\rho(y)^{\varkappa-1-\varepsilon}\Big(\frac{r(y)}{\rho(y)}\Big)^{\lambda-1-\varepsilon}.$$

The kernel $L(x,y)$ of L vanishes if $\operatorname{dist}(x,y) \geqslant \delta$, *where δ is a sufficiently small positive number.*

If points x and y lie in a neighborhood of a vertex O_i, $i = 1,2,\dots,m$, and this neighborhood contains no vertices of the polyhedron other than O_i, then the kernel $\mathcal{L}(x,y)$ satisfies

$$|L(x,y)| \leqslant c\,\rho(y)^{-2}\Big(\frac{r(y)}{\rho(y)}\Big)^{\lambda-1-\varepsilon} + c\big(r(y)+|x-y|\big)^{-2}\Big(\frac{r(x)}{r(x)+|x-y|}\Big)^{\lambda-\varepsilon}\Big(\frac{r(y)}{r(y)+|x-y|}\Big)^{\lambda-1-\varepsilon}$$

provided that $\rho(x)/2 < \rho(y) < 2\rho(x)$ and

$$|L(x,y)| \leqslant c\,\rho(y)^{-1}\big(\rho(x)+\rho(y)\big)^{-1}\Big(\frac{\min\{\rho(x),\rho(y)\}}{\rho(x)+\rho(y)}\Big)^{\varkappa-\varepsilon}\Big(\frac{r(y)}{\rho(y)}\Big)^{\lambda-1-\varepsilon}$$

in the opposite case. Here, ε is an arbitrary positive number.

In what follows, by $\{\chi_k\}_{k=1}^3$, η_1, and, η_2 we mean functions in $C^\infty([0,\infty))$ such that
(1) $\sum_{1\leqslant k\leqslant 3}\chi_k = 1$, $\operatorname{supp}\chi_1 \subset [0,5/8)$, $\operatorname{supp}\chi_2 \subset (1/2,2)$, $\chi_3 \subset (8/5,\infty)$,
(2) $\eta_1(t) = 1$ for $t < 1/8$ and $\eta_1(t) = 0$ for $t \geqslant 1/4$,
(3) $\eta_2(t) = 1$ for $t < 5/6$ and $\eta_2(t) = 0$ for $t \geqslant 6/7$.

We assume that the points x and y lie in a neighborhood U_i of the vertex O_i, $i = 1,2,\dots,m$ and U_i contains no other vertices than O_i. Let the origin coincide with O_i.

3.1 Estimates for the kernels $L(x,y)$ and $M(x,y)$ for $|y| < 5|x|/8$

Given $x \in \Gamma\backslash\mathfrak{M}$, consider the Dirichlet problem

$$\begin{aligned} &\Delta_y R^+(y,x) = 0, \quad y \in G^+,\\ &R^+(y,x) = \eta_2(|y|/|x|)R^-(y,x), \quad y \in \Gamma. \end{aligned} \tag{3.2}$$

Lemma 3.1. *Suppose that $0 < -\alpha-\gamma < \lambda$, $-\varkappa < \beta+\gamma-\alpha < 1+\varkappa$, and l is a positive integer. Then there exists a unique solution $R^+(\cdot,x) \in C^{l,\alpha}_{\beta,\gamma+l}(G^+)$ of the problem* (3.2) *for all $x \in U_i \cap (\Gamma\backslash O_i)$ and*

$$|\partial_y^\tau R^+(y,x)| \leqslant c_\tau\,|x|^{-1}|y|^{-|\tau|}\Big(\frac{|y|}{|x|}\Big)^{\varkappa-\varepsilon}\Big(\frac{r(y)}{|y|}\Big)^{\lambda_{\tau\varepsilon}}, \tag{3.3}$$

where $\lambda_{\tau\varepsilon} = \min\{0, \lambda-|\tau|-\varepsilon\}$ and $y \in U_i \cap G^+$.

Proof. We set $x = |x|X$ and $y = |x|Y$. Let $G_{|x|}$ and $\Gamma_{|x|}$ be the images of the sets G^+ and Γ under the mapping $y \to Y$. The problem (3.2) can be written in the form

$$\begin{aligned} &\Delta_Y R_{|x|}(Y,X) = 0, \quad Y \in G_{|x|},\\ &R_{|x|}(Y,X) = H_x(y), \quad Y \in \Gamma_{|x|}, \end{aligned} \tag{3.4}$$

where $R_{|x|}(Y,X) = |x|R^+(|x|Y,|x|X)$, $|X| = 1$, and, in view of the inequalities for $\partial_x^\sigma\partial_\xi^\tau R^-(x,\xi)$ from Theorem 2.2, $\|H_x\|_{C^{l,\alpha}_{\beta,\gamma+l}(G^+_{|x|})} \leqslant c$.

Applying Theorem 2.1 to the solution of the problem (3.4), we get

$$\|R_{|x|}(\cdot, X)\|_{C^{l,\alpha}_{\beta,l+\gamma}(G^+_{|x|})} \leqslant c.$$

Setting $\gamma = \alpha - \lambda + \varepsilon$ and $\beta = -\varkappa - \gamma + \alpha + \varepsilon$, we find

$$|\partial_Y^\tau R_{|x|}(Y, X)| \leqslant c_\tau \, |Y|^{\varkappa - |\tau| - \varepsilon} \Big(\frac{r(Y)}{|Y|}\Big)^{\lambda_{\tau\varepsilon}}.$$

Returning back to the function $R^+(y, x)$, we arrive at (3.3). □

Lemma 3.2. *Suppose that* $0 < \alpha - \gamma < \lambda$, $0 < \beta + \gamma - \alpha < 1$, *and* l *is a positive integer. For any* $\varphi \in C^{l,\alpha}_{\beta,l+\gamma}(\Gamma)$

$$\int\limits_\Gamma Q^-(x,\xi)\frac{\partial}{\partial n_\xi}\int\limits_\Gamma P^+(\xi,y)\chi_1\Big(\frac{|y|}{|x|}\Big)\varphi(y)ds_y ds_\xi = \int\limits_\Gamma L(x,y)\chi_1\Big(\frac{|y|}{|x|}\Big)\varphi(y)ds_y \qquad (3.5)$$

for $x \in \Gamma \cap U_i$, *where*

$$|L(x,y)| \leqslant c\,|x|^{-1}|y|^{-1}\Big(\frac{|y|}{|x|}\Big)^{\varkappa-\varepsilon}\Big(\frac{r(y)}{|y|}\Big)^{\lambda-1-\varepsilon}. \qquad (3.6)$$

Proof. Setting

$$v(\xi) = \int\limits_\Gamma P^+(\xi,y)\chi_1\Big(\frac{|y|}{|x|}\Big)\varphi(y)ds_y$$

and using (2.8), we write the left-hand side of (3.5) in the form

$$\int\limits_\Gamma (\eta_2\Big(\frac{|\xi|}{|x|}\Big)Q^-(0,x)\frac{\partial}{\partial n_\xi}v(\xi)ds_\xi + \int\limits_\Gamma R^+(\xi,x)\frac{\partial}{\partial n_\xi}v(\xi)ds_\xi + \int\limits_\Gamma (1-\eta_2\Big(\frac{|\xi|}{|x|}\Big)Q^-(x,\xi)\frac{\partial}{\partial n_\xi}v(\xi)ds_\xi,$$

where R^+ is the solution of (3.2)

Applying the Green formula to the first and second integrals, we obtain (3.5) with

$$L(x,y) = \sum_{1\leqslant k\leqslant 3} L_k(x,y), \qquad (3.7)$$

where

$$L_1(x,y) = \frac{\partial}{\partial n_y}R^+(y,x),$$

$$L_2(x,y) = -\int\limits_{G^+}\Big(\Delta_\xi\eta_2\Big(\frac{|\xi|}{|x|}\Big)Q^-(0,x)\Big)P^+(\xi,y)ds_\xi,$$

$$L_3(x,y) = -\int\limits_\Gamma\Big(1-\eta_2\Big(\frac{|\xi|}{|x|}\Big)\Big)Q^-(x,\xi)\frac{\partial}{\partial n_\xi}P^+(\xi,y)ds_\xi.$$

We estimate each term in (3.7). The inequality (3.6) for $\mathcal{L}_1(x,y)$ follows directly from (3.3). Let us estimate $\mathcal{L}_2(x,y)$. It is clear that $6|\xi| > 5|x|$ on the support of the function

$\xi \to \Delta_\xi \eta_2(|\xi|/|x|)$. From this and the inequality $8|y| < 5|x|$ we conclude that $3|\xi| > 4|y|$. By Theorems 2.1 and 2.2,

$$|L_2(x,y)| \leqslant c \int\limits_{\xi\in G^+:6|\xi|>5|x|} |\xi|^{-3}|x|^{-1}|y|^{-1}\Big(\frac{|y|}{|\xi|}\Big)^{\delta^+-\varepsilon}\Big(\frac{r(y)}{|y|}\Big)^{\lambda^+-1-\varepsilon} d\xi$$
$$\leqslant c|x|^{-1}|y|^{-1}\Big(\frac{|y|}{|x|}\Big)^{\delta^+-\varepsilon}\Big(\frac{r(y)}{|y|}\Big)^{\lambda^+-1-\varepsilon}.$$

Finally, to obtain the required estimate for $\mathcal{L}_3(x,y)$, we write it as the sum of two integrals over the sets $\Gamma_1 = \{\xi \in \Gamma : |\xi| < 2\,|x|\}$ and $\Gamma_2 = \{\xi \in \Gamma : |\xi| > 2\,|x|\}$. By Theorems 2.1 and 2.2,

$$|L_3(x,y)| \leqslant c \int\limits_{\substack{\xi\in\Gamma:6|\xi|>5|x|\\ |x-\xi|<3|x|}} |x-\xi|^{-1}|\xi|^{-2}K(y,\xi)ds_\xi + c\int\limits_{\xi\in\Gamma:|\xi|>2|x|}\big(|\xi|^{-3}+\rho(\xi)^{\delta^+-1-\varepsilon}\big)K(y,\xi)ds_\xi$$
$$\leqslant c\,|x|^{-1}|y|^{-1}\Big(\frac{|y|}{|x|}\Big)^{\delta^+-\varepsilon}\Big(\frac{r(y)}{|y|}\Big)^{\lambda-1-\varepsilon}.$$

Here, we used the notation

$$K(y,\xi) = |y|^{-1}(|y|/|\xi|)^{\delta^+-\varepsilon}\Big(\frac{r(y)}{|y|}\Big)^{\lambda^+-1-\varepsilon}\Big(\frac{r(\xi)}{\rho(\xi)}\Big)^{\lambda^+-1-\varepsilon}.$$ □

3.2 Estimates for the kernels $L(x,y)$ and $M(x,y)$ for $5|y| > 8|x|$

Let $x \in \Gamma \cap U_i$. Consider the boundary value problems

$$\Delta_y R^+(y,x) = 0, \quad y\in G^+, \quad R^+(y,x) = \eta_2\Big(\frac{|x|}{|y|}\Big)R^-(y,x), \quad y\in\Gamma, \tag{3.8}$$
$$\Delta d^+(y) = 0, \quad y\in G^+, \quad d^+(y) = d^-(y), \quad y\in\Gamma. \tag{3.9}$$

Lemma 3.3. *Suppose that* $0 < \alpha-\gamma < \lambda$, $-\varkappa < \beta+\gamma-\alpha < 1+\varkappa$, *and* l *is a positive integer. Then the problems* (3.8) *and* (3.9) *have unique solutions* $R^+(\cdot,x) \in C^{l,\alpha}_{\beta,\gamma+l}(G^+)$, *respectively,* $d^+ \in C^{l,\alpha}_{\beta,\gamma+l}(G^+)$ *for all* $x \in U_i\cap(\Gamma\backslash O_i)$ *and*

$$|\partial_y^\tau R^+(y,x)| \leqslant c_\tau\,|y|^{-1-|\tau|}\Big(\frac{|x|}{|y|}\Big)^{\varkappa-\varepsilon}\Big(\frac{r(y)}{|y|}\Big)^{\lambda_{\tau\varepsilon}}, \quad y\in G^+\cap O_i, \tag{3.10}$$
$$|\partial_y^\tau d^+(y)| \leqslant c_\tau\,\rho(y)^{\varkappa-|\tau|-\varepsilon}\Big(\frac{r(y)}{\rho(y)}\Big)^{\lambda_{\tau\varepsilon}}, \quad y\in G^+\cap O_i. \tag{3.11}$$

Proof. The inequality (3.11) is a direct consequence of Theorem 2.1, and the inequality (3.11) is proved in a similar manner as Lemma 3.1. □

Lemma 3.4. *Suppose that* $0 < \alpha-\gamma < \lambda$, $0 < \beta+\gamma-\alpha < 1$, *and* l *is a positive integer. For any* $\varphi \in C^{l,\alpha}_{\beta+l}(\Gamma)$

$$\int\limits_\Gamma Q^-(x,\xi)\frac{\partial}{\partial n_\xi}\int\limits_\Gamma P^+(\xi,y)\chi_3\Big(\frac{|y|}{|x|}\Big)\varphi(y)ds_y\Big)ds_\xi$$
$$= \int\limits_\Gamma \big(M(x,y)+L(x,y)\big)\chi_3\Big(\frac{|y|}{|x|}\Big)\varphi(y)ds_y, \quad x\in\Gamma\backslash\mathfrak{M}, \tag{3.12}$$

where $y \in \Gamma \cap O_i$ *and*

$$|M(x,y)| \leqslant c\,|y|^{\varkappa-1-\varepsilon}\Big(\frac{r(y)}{|y|}\Big)^{\lambda-1-\varepsilon}, \tag{3.13}$$

$$|L(x,y)| \leqslant c\,|y|^{-2}(|x|/|y|)^{\varkappa-\varepsilon})\Big(\frac{r(y)}{|y|}\Big)^{\lambda-1-\varepsilon}. \tag{3.14}$$

Proof. Setting $v(\xi) = P^+(\xi,y)\chi_3(|y|/|x|)\varphi(y)ds_y$ and using (2.7), (2.9), we write the left-hand side of (3.12) in the form

$$\int\limits_\Gamma (\eta_2(|x|/|\xi|)\Big(\frac{a_i^-}{|\xi|} + b_i^- + d^+(\xi)\Big)\frac{\partial}{\partial n_\xi}v(\xi)ds_\xi$$
$$+\int\limits_\Gamma R^+(\xi,x)\frac{\partial}{\partial n_\xi}v(\xi)ds_\xi + \int\limits_\Gamma \Big(1-\eta_2\Big(\frac{|x|}{|\xi|}\Big)\Big)Q^-(x,\xi)\frac{\partial}{\partial n_\xi}v(\xi)ds_\xi,$$

where $R^+(\xi,x)$ and $d_i^+(\xi)$ are solutions of the problems (3.8) and (3.9). Applying the Green formula, we arrive at (3.12), where

$$L(x,y)+M(x,y) = \frac{\partial}{\partial n_y}R^+(y,x) + \frac{\partial}{\partial n_y}\Big(\eta_2\Big(\frac{|x|}{|y|}\Big)\Big(\frac{a_i^-}{|y|} + b_i^- + d^+(y)\Big)\Big)$$

$$-\int\limits_{G^+}\Delta_\xi\Big(\eta_2\Big(\frac{|x|}{|\xi|}\Big)\Big(\frac{a_i^-}{|\xi|} + b_i^- + d^+(\xi)\Big)\Big)P^+(\xi,y)\,ds_\xi + \int\limits_\Gamma\Big(1-\eta_2\Big(\frac{|x|}{|\xi|}\Big)\Big)Q^-(x,\xi)\frac{\partial}{\partial n_\xi}P^+(\xi,y)\,ds_\xi.$$

To obtain the estimates (3.13) and (3.14), it suffices to use Theorems 2.1, 2.2 and Lemma 3.3 (cf. the proof of Lemma 3.2). □

3.3 Estimates for the kernel $L(x,y)$ for $|y|/2 < |x| < 2|y|$

The goal of this subsection is to prove the following assertion.

Lemma 3.5. *Suppose that* $0 < \alpha - \gamma < \lambda$, $0 < \beta + \gamma - \alpha < 1$, *and* l *is a positive integer. For any* $\varphi \in C^{l,\alpha}_{\beta,l+\gamma}(\Gamma)$

$$\int\limits_\Gamma Q^-(x,\xi)\frac{\partial}{\partial n_\xi}\int\limits_\Gamma \mathcal{P}^+(\xi,y)\chi_2\Big(\frac{|y|}{|x|}\Big)\varphi(y)ds_y\,ds_\xi = -\varphi(x) + \int\limits_\Gamma \mathcal{L}(x,y)\chi_2\Big(\frac{|y|}{|x|}\Big)\varphi(y)ds_y, \tag{3.15}$$

where $|L(x,y)| \leqslant c\,(r(y))^{-2}$ *if* $|x-y| < r(x)/2$, *and*

$$|L(x,y)| \leqslant \frac{c}{|x-y|^2}\Big(\frac{r(x)}{|x-y|}\Big)^{\lambda-\varepsilon}\Big(\frac{r(y)}{|x-y|}\Big)^{\lambda-1-\varepsilon} + \frac{c}{|y|^2}\Big(\frac{r(y)}{|y|}\Big)^{\lambda-1-\varepsilon}$$

otherwise.

First we formulate an auxiliary assertion. Suppose that a point $x \in \Gamma\backslash O_i$ lies in a neighborhood of the edge $\mathfrak{M}_j$ together with the ball $B(\delta|x|,x)$ of radius $\delta|x|$, where δ is a sufficiently small positive number. We denote by D_j^+ and D_j^- the interior and exterior of the dihedral angle which coincides with G^+ near the edge $\mathfrak{M}_j$. In what follows, we omit the subscript j in $D_j^\pm$ and use the notation for the diherdral angle $D^\pm$ introduced in Subsection 1.2.

Lemma 3.6. *The following estimates hold on the set* $\{y \in \Gamma : |x|/2 < |y| < 2|x|\}$:

$$|\partial_x^\sigma \partial_y^\tau (P^+(x,y) - \mathcal{P}^+(x,y)| \leqslant c_{\sigma\tau} |x|^{-2-|\tau|-|\sigma|},$$
$$|\partial_x^\sigma \partial_y^\tau (Q^-(x,y) - \mathcal{Q}^-(x,y)| \leqslant c_{\sigma\tau} |x|^{-1-|\tau|-|\sigma|} \Big(\frac{r(x)}{|x|}\Big)^{\lambda^-_{\sigma\varepsilon}} \Big(\frac{r(y)}{|y|}\Big)^{\lambda^-_{\tau\varepsilon}},$$

where $P^+(x,y)$, $Q^-(x,y)$ *are the kernels of the operators* (2.12), (2.13), *and* $\lambda^-_{\sigma\varepsilon} = \min\{0, \lambda^- - |\sigma| - \varepsilon\}$, $\lambda^-_{\tau\varepsilon} = \min\{0, \lambda^- - |\tau| - \varepsilon\}$.

The proof is similar to that of Lemma 2.6 in [2]. The only difference is that one has to use theorems on the solvability of the Dirichlet and Neumann problems in domains with edges (cf. [16]) instead of similar assertions for smooth boundaries.

Let $x \in \Gamma \backslash O_i$. Consider the problem

$$\Delta \mathcal{R}^+(x,y) = 0, \quad y \in G^+,$$
$$\mathcal{R}^+(x,y) = \chi_2\Big(\frac{|x|}{|y|}\Big)\big(Q^-(x,y) - \mathcal{Q}^-(x,y)\big), \quad y \in \Gamma.$$

It follows essentially from Theorem 2.1 that

$$|\partial_y^\tau \mathcal{R}^+(x,y)| \leqslant c_\tau\, |x|^{-1-|\tau|} \Big(\frac{r(y)}{|y|}\Big)^{\lambda_{\tau\varepsilon}} \tag{3.16}$$

for all y with $|x|/2 < |y| < 2|x|$, where $\lambda_{\tau\varepsilon} = \min\{0, \lambda - |\tau| - \varepsilon\}$.

Proof of Lemma 3.5. We write the left-hand side of (3.16) as

$$\int_\Gamma \chi_2\Big(\frac{|x|}{|y|}\Big) \mathcal{Q}^-(x,\xi) \frac{\partial}{\partial n_\xi} v(\xi)\, ds_\xi + \int_\Gamma \mathcal{R}^+(x,\xi) \frac{\partial}{\partial n_\xi} v(\xi)\, ds_\xi$$
$$+ \int_\Gamma (1 - \chi_2\Big(\frac{|x|}{|\xi|}\Big)) Q^-(x,\xi) \frac{\partial}{\partial n_\xi} v(\xi)\, ds_\xi, \tag{3.17}$$

where

$$v(\xi) = \int_\Gamma P^+(\xi,y) \chi_2\Big(\frac{|y|}{|x|}\Big) \varphi(y)\, ds_y.$$

Replacing $P^+(x,\xi)$ by $\mathcal{P}^+(x,\xi)$ in the first term and applying the Green formula to the second term, we write (3.17) as

$$\int_\Gamma \mathcal{Q}^-(x,\xi) \frac{\partial}{\partial n_\xi} \int_\Gamma \mathcal{P}^+(\xi,y)\, \eta_1\Big(\frac{|y-x|}{\delta|x|}\Big) \varphi(y)\, ds_y\, ds_\xi + \int_\Gamma L'(x,y) \chi_2\Big(\frac{|y|}{|x|}\Big) \varphi(y)\, ds_y, \tag{3.18}$$

where

$$L'(x,y) = \frac{\partial}{\partial n_\xi}\mathcal{R}^+(x,y) - \int\limits_\Gamma \mathcal{Q}^-(x,\xi)\frac{\partial}{\partial n_\xi}\mathcal{P}^+(\xi,y)\left(\eta_1\Big(\frac{|y-\xi|}{\delta|x|}\Big)-1\right)\eta_1\Big(\frac{|y-x|}{\delta|x|}\Big)\,\varphi(y)\,ds_\xi$$

$$+\int\limits_\Gamma \chi_2\Big(\frac{|x|}{|\xi|}\Big)\;\mathcal{Q}^-(x,\xi)\frac{\partial}{\partial n_\xi}P^+(\xi,y)\Big(1-\eta_1\Big(\frac{|y-x|}{\delta|x|}\Big)\Big)ds_\xi$$

$$+\int\limits_\Gamma \mathcal{Q}^-_\chi(x,\xi)\frac{\partial}{\partial n_\xi}P^+(\xi,y)\Big(1-\eta_1\Big(\frac{|y-\xi|}{\delta|x|}\Big)\Big)\eta_1\Big(\frac{|y-x|}{\delta|x|}\Big)\,ds_\xi$$

$$+\int\limits_\Gamma \mathcal{Q}^-_\chi(x,\xi)\frac{\partial}{\partial n_\xi}\big(P^+(\xi,y)-\mathcal{P}^+(\xi,y)\big)\eta_1\Big(\frac{|y-\xi|}{\delta|x|}\Big)\eta_1\Big(\frac{|y-x|}{\delta|x|}\Big)ds_\xi$$

$$+\int\limits_\Gamma \Big(1-\chi_2\Big(\frac{|x|}{|\xi|}\Big)\Big)\mathcal{Q}^-(x,\xi)\frac{\partial}{\partial n_\xi}P^+(\xi,y)\,ds_\xi.$$

Here,

$$\mathcal{Q}^- = \mathcal{Q}^-_\chi(x,\xi) = \chi_2\Big(\frac{|x|}{|\xi|}\Big)\mathcal{Q}^-(x,\xi).$$

Estimating each term with the help of Theorems 2.1, 2.2 and Lemma 3.6, we find

$$|L'(x,y)| \leqslant c\,|x|^{-2}\Big(\frac{r(y)}{|y|}\Big)^{\lambda-1-\varepsilon}.$$

We set $x' = x/|x|$, $\xi' = \xi/|x|$, $y' = y/|x|$. Since the functions $\mathcal{Q}^-(x,y)$ and $\mathcal{P}^+(x,y)$ are homogeneous, the first term in (3.18) takes the form

$$\int\limits_\Gamma \mathcal{Q}^-(x',\xi')\frac{\partial}{\partial n_\xi}\int\limits_\Gamma \mathcal{P}^+(\xi',y')\,\eta_1\Big(\frac{|y'-x'|}{\delta}\Big)\varphi(y)\,ds_{y'}\,ds_{\xi'}.$$

To complete the proof, it suffices to refer the following assertion. □

Lemma 3.7. *Let F be the boundary of the dihedral angle with opening ω, and let $\Lambda = \pi/(\pi+|\pi-\omega|)$. If $\varphi \in C^{l,\alpha}_{\gamma+l}(F)$, $0<\alpha-\gamma<\Lambda$,* supp $\varphi \subset B(1,0)$, *then*

$$\int\limits_\Gamma \mathcal{Q}^-(x,\xi)\frac{\partial}{\partial n_\xi}\mathcal{P}^+(\xi,y)\varphi(y)\,ds_y\,ds_\xi = -\varphi(x) + \int\limits_F \mathcal{L}(x,y)\varphi(y)\,ds_y \tag{3.19}$$

for $x \in F\backslash\mathfrak{M}$. Moreover, $|\mathcal{L}(x,y)| \leqslant c\,(r(y))^{-2}$ for $|x-y|<r(x)/2$ and

$$|\mathcal{L}(x,y)| \leqslant c\,|x-y|^{-2}\Big(\frac{r(x)}{|x-y|}\Big)^{\Lambda-\varepsilon}\Big(\frac{r(y)}{|x-y|}\Big)^{\Lambda-1-\varepsilon}$$

otherwise.

Now, we formulate an auxiliary assertion. Consider the problem

$$\begin{aligned}&\Delta\mathcal{R}^+(x,y) = 0, \quad y\in D^+,\\ &\mathcal{R}^+(x,y) = \Big(\mathcal{Q}^-(x,y) - \frac{a}{|x-y|}\Big)\Big(1-\eta_1\Big(\frac{|y-x|}{r(x)}\Big)\Big), \quad y\in F,\end{aligned} \tag{3.20}$$

where a, $\mathcal{Q}^-$, and r are the same as in Theorem 2.4.

Lemma 3.8. *Suppose that* $0 < \alpha - \gamma < \Lambda$, $\Lambda = \pi/(\pi + |\pi - \omega|)$, $x \in B(1,0)$. *Then there exists a unique solution of the problem* (3.20) *and*

$$|\partial_y^\sigma \mathcal{R}^+(x,y)| \leqslant c\,|x-y|^{-1-|\sigma|}\Big(\frac{r(x)}{|x-y|}\Big)^{\Lambda-\varepsilon}\Big(\frac{r(y)}{|x-y|}\Big)^{\Lambda-|\sigma|-\varepsilon} \quad \textit{for } |x-y| > r(x).$$

Proof. Since $\mathcal{R}^+(x,y)$ is homogeneous, we can assume that $|x-y| = 1$. We introduce the function $v(y) = r(x)^{-\Lambda+\varepsilon}\mathcal{R}^+(x,y)$. It is clear that v solves the problem

$$\Delta v = 0 \text{ on } D^+, \quad v = \psi \text{ on } F,$$

where $|\partial_y^\sigma \psi| \leqslant c\,r(y)^{\Lambda-|\sigma|-\varepsilon}$. The required estimate follows from Theorem 2.3. □

Proof of Lemma 3.7. For the sake of definiteness, we will assume that x lies on the face F^+ of the polyhedron. We represent the left-hand side of (3.19) as the sum of two terms obtained from the initial expression by replacing φ by φ_1 and by φ_2, where $\varphi_1 = \varphi - \varphi_2$ and $\varphi_2(x,y) = \varphi(x,y)\eta_1(|x-y|/r(x)\delta)$. Here, δ is so small that $F^- \cap B(\delta r(x), x) = \varnothing$.

We write the first term in the form

$$\int\limits_F \eta_1\Big(\frac{4|x-y|}{\delta r(x)}\Big)\mathcal{Q}^-(x,\xi)\frac{\partial}{\partial n_\xi}v(\xi)ds_\xi + \int\limits_F \Big(1-\eta_1\Big(\frac{4|x-y|}{\delta r(x)}\Big)\Big)\mathcal{Q}^-(x,\xi)\frac{\partial}{\partial n_\xi}v(\xi)ds_\xi,$$

where

$$v(\xi) = \int\limits_F \mathcal{P}^+(\xi,y)\Big(1-\eta_1\Big(\frac{|x-y|}{\delta r(x)}\Big)\Big)\varphi(y)ds_y.$$

Applying the Green formula to the second integral and using the solution of (3.20), we find that it is equal to

$$\int\limits_F \mathcal{L}_1^+(x,y)\Big(1-\eta_1\Big(\frac{|x-y|}{\delta r(x)}\Big)\Big)\varphi(y)ds_y,$$

where

$$\mathcal{L}_1^+(x,y) = \frac{\partial}{\partial n_y}\Big(\mathcal{R}^+(x,y) + \frac{a}{|x-y|}\Big) - \int\limits_F \eta_1\Big(\frac{4|x-\xi|}{\delta r(x)}\Big)\mathcal{Q}^-(x,\xi)\frac{\partial}{\partial n_\xi}\mathcal{P}^+(\xi,y)ds_\xi$$
$$+ \int\limits_{D+} \Delta_\xi\Big(\mathcal{R}^+(x,\xi) + \frac{a}{|x-\xi|}\Big)\Big(1-\eta_1\Big(\frac{4|x-\xi|}{\delta r(x)}\Big)\mathcal{P}^+(\xi,y)\,d\xi.$$

Since

$$\Big|\frac{\partial}{\partial n_y}\frac{1}{|x-y|}\Big| \leqslant \frac{r(x)}{|x-y|^3}, \quad |x-y| > r(x),$$

the estimate

$$|\mathcal{L}_1(x,y)| \leqslant c\,|x-y|^{-2}\Big(\frac{r(x)}{|x-y|}\Big)^{\Lambda-\varepsilon}\Big(\frac{r(y)}{|x-y|}\Big)^{\Lambda-1-\varepsilon}$$

follows from Theorems 2.3, 2.4 and Lemma 3.8.

Next, we need to show that

$$\int_F \mathcal{Q}^-(x,\xi)\frac{\partial}{\partial n_\xi}\int_F \mathcal{P}^+(\xi,y)\eta_1\Big(\frac{|x-y|}{\delta r(x)}\Big)\varphi(y)ds_y\,d\xi$$

$$= -\varphi(x) + \int_F \mathcal{L}_2(x,y)\eta_1\Big(\frac{|x-y|}{\delta r(x)}\Big)\varphi(y)ds_y, \quad x \in \Gamma\backslash\mathfrak{M}, \tag{3.21}$$

where

$$|\mathcal{L}_2(x,y)| \leqslant c\,(r(y))^{-2}. \tag{3.22}$$

We write the left-hand side of (3.21) as the sum of two terms by setting $\mathcal{Q}^-(x,\xi) = Q_1(x,\xi) + Q_2(x,\xi)$ and $Q_1(x,\xi) = \mathcal{Q}^-(x,\xi)\,\eta_1(|x-\xi|/(4\delta r(x))$. It is clear that the inequality $|y-\xi| > s\,r(y)$ holds for $|x-\xi| > \delta\,r(x)/2$, $|x-y| < \delta\,r(x)/4$, where s is a certain positive number. Hence, by Theorems 2.3 and 2.4, the second term is the integral operator with a kernel satisfying (3.22).

We denote by $P_0^+(x,y)$ and $Q_0^-(x,y)$ the kernels of the inverse operators of the corresponding boundary value problems in the half-space, that is the problems obtained from (2.10) and (2.11) by replacing $D^\pm$ and F by $\mathbb{R}^3_\pm$ and $\mathbb{R}^2$, where $\mathbb{R}^2$ is the plane containing F^+, $\mathbb{R}^3_+$ is the half-space with boundary $\mathbb{R}^2$ containing points of the polyhedron near the origin and $\mathbb{R}^3_- = \mathbb{R}^3\backslash(\mathbb{R}^3_+ \cup \mathbb{R}^2)$. The following estimates are well known:

$$|\partial_\xi^\sigma \partial_y^\tau (\mathcal{P}^+(\xi,y) - P_0^+(\xi,y))| \leqslant c_{\sigma\tau} r(x)^{-2-|\tau|-|\sigma|},$$
$$|\partial_\xi^\sigma \partial_y^\tau (\mathcal{Q}^-(\xi,y) - Q_0^-(\xi,y))| \leqslant c_{\sigma\tau} r(x)^{-1-|\tau|-|\sigma|}$$

for $\xi, y \in D^\pm \cap B(\delta r(x), x)$. Therefore, to obtain the representation (3.21), it suffices to show that the expression

$$\int_{F^+} \eta_1\Big(\frac{|x-\xi|}{4\delta r(x)}\Big) Q_0^-(x,\xi)\frac{\partial}{\partial n_\xi}\int_{F+} P_0^+\eta_1\Big(\frac{|x-\xi|}{\delta r(x)}\Big)\varphi(y)ds_y\,ds_\xi$$

admits a similar representation. The last assertion follows directly from the relation

$$\int_{\mathbb{R}^2} Q_0^-(x,\xi)\frac{\partial}{\partial n_\xi}\int_{\mathbb{R}^2} P_0^+(\xi,y)\varphi(y)ds_y\,ds_\xi = -\varphi(x). \qquad \square$$

Proof of Theorem 3.1. If the point x is placed near y, then the estimates for the kernels $M(x,y)$, $L(x,y)$ follow from Lemmas 3.2, 3.4, 3.5. In the opposite case, such estimates can be obtained similarly and even simpler. $\square$

Remar 3.1. Using known results on the asymptotic behavior of solutions to the Dirichlet and Neumann problems near boundary singularities, one can improve estimates of the kernels $M(x,y)$ and $L(x,y)$. For example, if the points x, y lie in the neighborhood of a vertex O_i which does not contain other vertices and for a certain edge $\mathfrak{M}_j$ we have the estimates

$$\operatorname{dist}(x,\mathfrak{M}_j) \leqslant c\operatorname{dist}(x,\mathfrak{M}_s), \quad \operatorname{dist}(y,\mathfrak{M}_j) \leqslant c\operatorname{dist}(y,\mathfrak{M}_s)$$

for all s such that $1 \leqslant s \leqslant k$, $s \neq j$, then the numbers $\varkappa$ and λ may be replaced by $\min\{\delta_j, \nu_j\}$ and $\pi/(\pi + |\pi - \omega_j|)$, where ω_j is the opening of the dihedral angle with the edge $\mathfrak{M}_j$.

Remar 3.2. The following representation holds for the inverse operator of the integral equation associated with the exterior Neumann problem:

$$(1+T^*)^{-1}g = (1+L^*+M^*)g, \quad g \in N^{l-1,\alpha}_{\beta,\gamma+l}(\Gamma),$$

where the kernels of the operators L^* and $M*$ obey the estimates which can be obtained from the estimates for the kernels L and M in (3.1) by replacing x by y and vice versa.

4 Solvability of the Integral Equation

In this section, we use the above notation. We also denote by $L^p_{\beta,\gamma}(\Gamma)$ the space of functions u equipped with the norm

$$\|u\|_{L^p_{\beta,\gamma}(\Gamma)} = \|\rho^\beta r^\gamma u\|_{L_p(\Gamma)}.$$

Lemma 4.1. *The operators L and M satisfying the estimates in Theorem* 3.1 *are continuous in $L^p_{\beta,\gamma}(\Gamma)$ for $1 \leqslant p < \infty$, $0 < \beta+\gamma+2/p < 1+\varkappa$, $0 < \gamma+1/p < \lambda$ and $p = \infty$, $0 \leqslant \beta+\gamma < 1+\varkappa$, $0 \leqslant \gamma < \lambda$.*

Proof. Let $\varphi \in L^p_{\beta,\gamma}(\Gamma)$. It suffices to show that $L\varphi \in L^p_{\beta,\gamma}(\Gamma \cap U)$, respectively, $M\varphi \in L^p_{\beta,\gamma}(\Gamma \cap U)$, where U is a neighborhood of a vertex O_i. For the sake of convenience, we assume that the point O_i coincides with the origin. We denote by χ a function from $C_0^\infty(\mathbb{R}^3)$ that is equal to 1 on $\overline{U}$. We also assume that supp χ contains no other vertices of the polyhedron except O_i. We verify the following inequality for the function $\psi = \varphi\chi$:

$$\|L\psi\|_{L^p_{\beta,\gamma}(\Gamma\cap U)} \leqslant c\,\|\psi\|_{L^p_{\beta,\gamma}(\Gamma)}. \tag{4.1}$$

The same estimate for the operator M is obvious for $1 \leqslant 0 < \infty$, $\beta+\gamma+2/p > 0$, $\gamma+1/p > 0$ and $p = \infty$, $\beta+\gamma \geqslant 0$, $\gamma \geqslant 0$.

We set

$$L\psi = \sum_{1\leqslant i\leqslant 3} L_i\psi = \sum_{1\leqslant i\leqslant 3} \int_{\Gamma_i} L(x,y)\,\psi(y)\,ds_y,$$

where $\Gamma_1 = \{\xi \in \Gamma : 2|\xi| < |x|\}$, $\Gamma_2 = \{\xi \in \Gamma : |x|/2 < |\xi| < 2|x|\}$, $\Gamma_3 = \{\xi \in \Gamma : |\xi| > 2|x|\}$, and prove (4.1) for each integral $L_i\psi$. We use the Hardy inequality

$$\|\rho^\alpha F\|_{L_p(\mathbb{R}^1_+)} \leqslant c\,\|\rho^{\alpha+1} f\|_{L_p(\mathbb{R}^1_+)}, \tag{4.2}$$

where

$$F(\rho) = \int_0^\rho f(t)dt, \quad \alpha < -1/p, \quad \text{and} \quad F(\rho) = \int_\rho^\infty f(t)dt, \quad \alpha > -1/p.$$

Let $\overline{\psi}$ be the function on $\mathbb{R}^1_+$ defined by

$$\overline{\psi}(\rho) = \int_\ell r(\theta)^{\lambda-1-\varepsilon}|\psi(\rho\theta)|\,d\ell_\theta,$$

where $\ell = \partial K_i \cap S^2$, S^2 is the unit sphere with center at O_i, ∂K_i is the boundary of the cone K_i which coincides with G^+ near the point O_i and a positive ε is so small that $\lambda - \varepsilon > \gamma + 1/p$.

We set

$$F(\rho) = \int\limits_0^\rho \tau^{\varkappa-\varepsilon}\overline{\psi}(\tau)\, d\tau.$$

By the estimates for $L(x,y)$ in Theorem 3.1,

$$\|\rho^{\beta+\gamma} L_1\psi\|_{L_p(\Gamma\cap U)} \leqslant c\, \|\rho^{\beta+\gamma-1-\varkappa+\varepsilon+1/p} F\|_{L_p(\mathbb{R}^1_+)}.$$

Using (4.2) and taking into account that $\lambda-\varepsilon > \gamma+1/p$, we arrive at the desired estimate for $L_1\psi$ for $\beta+\gamma+2/p < 1+\varkappa-\varepsilon$.

The estimate (4.1) for $L_3\psi$ is proved in a similar way. It suffices to consider the function

$$F(\rho) = \int\limits_\rho^\infty \tau^{-1-\varkappa+\varepsilon}\overline{\psi}(\tau)\, d\tau$$

and to apply the inequality (4.2).

To obtain (4.1) for $L_2\psi$, we use the following assertion. □

Lemma 4.2. *Let F be the boundary of the dihedral angle with edge $\mathfrak{M}$, and let $\mathcal{L}$ be the integral operator on F with the kernel $\mathcal{L}(x,y)$ satisfying the estimate in Lemma 3.7. Then the operator $\mathcal{L}$ is continuous in $L_{p,t}(F)$ for $1 \leqslant p \leqslant \infty$, $-\lambda < t+1/p < \lambda$, where $L_{p,t}(F)$ is the function space equipped with the norm $\|u\|_{L_{p,t}(F)} = \|r^t u\|_{L_p(F)}$.*

Proof. Let $1 < p < \infty$. We denote by $\mathcal{L}_i$, $i=1,2$, the operator with the kernel $(\mathcal{L}\zeta_i)(x,y)$, where

$$\zeta_1(x,y) = \Big(1-\eta_1\Big(\frac{|x-y|}{r(x)}\Big)\Big), \quad \zeta_2(x,y) = \eta_1\Big(\frac{|x-y|}{r(x)}\Big).$$

For the operator $\mathcal{L}_1$ we have

$$\|\mathcal{L}_1\varphi\|^p_{L_{p,t}(F)} \leqslant c\int\limits_F r(x)^{p(t+\lambda-\varepsilon)}\Big(\int\limits_F \frac{r(y)^{\lambda-1-\varepsilon}}{|x-y|^{1+2\lambda-2\varepsilon}}\zeta_1(x,y)\varphi(y)ds_y\Big)^p ds_x. \tag{4.3}$$

By the Hölder inequality, the interior integral is majorized by

$$\Big(\int\limits_F \frac{r(y)^{\delta q}}{|x-y|^{2+\alpha q}}\zeta_1 ds_y\Big)^{\frac{1}{q}}\Big(\int\limits_F \frac{r(y)^{p(\lambda-1-\varepsilon-\delta)}}{|x-y|^{2+p(2\lambda-1-2\varepsilon-\alpha)}}\zeta_1|\varphi|^p ds_y\Big)^{\frac{1}{p}},$$

where $q = p/(p-1)$, $p \neq 1$. Setting $\delta > -1/q$, $\alpha-\delta > 0$, we conclude that the first factor in the last expression is estimated by $r(x)^{\delta-\alpha}$. Hence

$$\|\mathcal{L}_1\varphi\|^p_{L_{p,t}(F)} \leqslant c\int\limits_F r(y)^{p(\lambda-1-\varepsilon-\delta)}|\varphi|^p\Big(\int\limits_F \frac{r(x)^{p(t+\lambda-\varepsilon+\delta-\alpha)}}{|x-y|^{2+2(\lambda-1-2\varepsilon-\alpha)p}}\zeta_1(x,y)ds_x\Big)^p ds_y.$$

Suppose that ε, $\alpha-\delta$, $\delta+1-1/p$ are so small that $\lambda+t+1/p-\varepsilon+\delta-\alpha > 0$, $\lambda-1-\varepsilon-\delta-t > 0$. Then the last inequality leads to the estimate

$$\|\mathcal{L}_1\varphi\|_{L_{p,t}(F)} \leqslant c\,\|\varphi\|_{L_{p,t}(F)}. \tag{4.4}$$

In order to establish (4.4) for $p = 1$, it suffices to change the order of integration in the right-hand side of (4.3). In the case $p = \infty$, the estimate (4.4) follows directly from the estimates for the kernel of the operator L_1.

Using the inequalities $c_1 r(y) < r(x) < c_2 r(y)$ and $c_1, c_2 > 9$, for $x, y \in \operatorname{supp} \zeta_2$, we arrive at the estimate (4.4) for the operator $\mathcal{L}_2$. $\square$

Lemma 4.3. *The operator* T *is continuous in the spaces* $C(\Gamma)$ *and* $L^p_{\beta,\gamma}(\Gamma)$ *for all* $1 \leqslant p < \infty$, $0 < \beta + \gamma + 2/p < 2$, $0 < \gamma + 1/p < 1$ *and* $p = \infty$, $0 \leqslant \beta + \gamma < 2$, $0 \leqslant \gamma < 1$.

Proof. Let points x, y be placed in a neighborhood of a vertex O_i. One can verify directly that the kernel $T(x, y)$ of the operator T admits the estimates

$$|T(x,y)| \leqslant c\,\frac{r(x)}{(r(x) + |x-y|)^3} + c\,\frac{1}{\rho(x)^2}$$

if $\rho(x)/2 < \rho(y) < 2\rho(x)$, and

$$|T(x,y)| \leqslant c\,\frac{\rho(x)}{(\rho(x) + \rho(y))^3} + c$$

otherwise.

It is known that $T\varphi \in C(\Gamma)$ for $\varphi \in C(\Gamma)$ (cf. [19, 20]). Hence, by the above estimates for $T(x, y)$ all assertions of this lemma follow from Lemma 4.1. $\square$

Using Theorems 2.5 and Lemmas 4.1, 4.3, we arrive at the following assertion.

Theorem 4.1. *Suppose that* $1 \leqslant p < \infty$, $0 < \beta + \gamma + 2/p < 1 + \varkappa$, $0 < \gamma + 1/p < \lambda$, $p = \infty$, $0 \leqslant \beta + \gamma < 1 + \varkappa$, $0 \leqslant \gamma < \lambda$. *Then the inverse operator of the integral equation associated with the Dirichlet problem is continuous in the spaces* $C(\Gamma)$ *and* $L^p_{\beta,\gamma}(\Gamma)$.

This result along with Lemma 4.3 shows, in particular, that the mappings $1 + T : L_p(\Gamma) \to L_p(\Gamma)$ and $1 + T^* : L_{p/(p-1)}(\Gamma) \to L_{p/(p-1)}(\Gamma)$, where $p > 2/(1 + \varkappa)$ and $p > 1/\lambda$, are isomorphic.

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