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Approximate Hermite quasi-interpolation

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In this paper we derive approximate quasi-interpolants when the values of a function u and of some of its derivatives are prescribed at the points of a uniform grid. As a byproduct of these formulas we obtain very simple approximants which provide high order approximations for solutions to elliptic differential equations with constant coefficients.

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1. Introduction

The method of approximate approximations is mainly directed to the numerical solution of partial integro-differential equations. The method provides simple formulas for quasi-interpolants, which approximate functions up to a prescribed precision very accurately, but in general the approximants do not converge. The lack of convergence, which is not perceptible in numerical computations, is offset by a greater flexibility in the choice of approximating functions. So it is possible to construct multivariate approximation formulas, which are easy to implement and have additionally the property that pseudodifferential operations can be effectively performed. This allows to create effective numerical algorithms for solving boundary value problems for differential and integral equations.

Approximate quasi-interpolants on a uniform grid are of the form

$$Mu(x) = \mathcal{D}^{-n/2} \sum_{j\in\mathbb{Z}^n} \mathcal{H}\left(\frac{x-hj}{h\sqrt{\mathcal{D}}}\right) u(hj) \tag{1}$$

with positive parameters, "small" h and "large" $\mathcal{D}$, and the generating function $\mathcal{H}$ is sufficiently smooth and of rapid decay. Their properties have been studied in a series of papers [4–7] (cf. also the monograph [11]). It has been shown that the

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quasi-interpolant approximates smooth functions with

$$|Mu(x) - u(x)| \le \varepsilon \sum_{|\beta|=0}^{N-1} (h\sqrt{\mathcal{D}})^{|\beta|} |\partial^\beta u(x)| + c(h\sqrt{\mathcal{D}})^N \|\nabla_N u\|_{L_\infty}, \qquad (2)$$

as long as $\mathcal{H}$ is subject to the moment condition

$$\int\limits_{R^n} x^\alpha \mathcal{H}(x)\, dx = \delta_{|\alpha|0}, \quad 0 \le |\alpha| < N. \qquad (3)$$

In (2) the constant c depends only on $\mathcal{H}$ and ε can be made arbitrarily small if the parameter $\mathcal{D}$ is sufficiently large, so that one can fix $\mathcal{D}$ such that in numerical computations Mu approximates with the order $O(h^N)$. The construction of simple generating functions satisfying (3) for arbitrary N has been addressed in [9], whereas [3, 10] extend the results to the case of nonuniform grids.

In this paper we study more general quasi-interpolation formulas with values of a function u and of some of its derivatives prescribed at the points of a uniform grid. More precisely, we consider approximants of the form

$$Mu(x) = \mathcal{D}^{-n/2} \sum_{j \in \mathrm{Z}^n} \mathcal{H}\left(\frac{x - hj}{h\sqrt{\mathcal{D}}}\right) \mathcal{Q}((-h\sqrt{\mathcal{D}})\partial) u(hj), \qquad (4)$$

where $\mathcal{Q}(t)$, $t \in \mathrm{R}^n$, is a polynomial with $\deg \mathcal{Q} < N$.

We establish estimates of the type (2) if the generating function $\mathcal{H}$ and the coefficients of $\mathcal{Q}$ are connected by suitable conditions (see (14)). It is shown in particular, that for an arbitrary polynomial $\mathcal{Q}$ there exists $\mathcal{H}$ such that the approximate Hermite quasi-interpolant (4) satisfies the estimate (2). On the other hand, the same is true if $\mathcal{H}$ is the Gaussian or a related function and $\mathcal{Q}$ is chosen suitably. As a byproduct we obtain very simple approximants which provide high order approximations to solutions of the Laplace equation or other elliptic differential equations.

Estimate (2) is proved in Section 2 for (4) with $\mathcal{H}$ and $\mathcal{Q}$ connected by conditions (14). In Section 3 we describe a simple method for the construction of $\mathcal{H}$ satisfying the conditions (14) for a given polynomial $\mathcal{Q}$. This result is applied in Section 4 when we consider some examples of Hermite quasi-interpolants. In the particular case $\mathcal{H}(x) = \pi^{-n/2}(\det B)^{-1/2} \mathrm{e}^{-<B^{-1}x,x>}$, with $B = \{b_{ij}\}$ symmetric and positive definite real matrix, we obtain the following quasi-interpolant of order $\mathcal{O}((h\sqrt{\mathcal{D}})^{2M})$

$$Mu(x) = \frac{(\det B)^{-1/2}}{(\pi \mathcal{D})^{n/2}} \sum_{m \in \mathrm{Z}^n} \sum_{s=0}^{M-1} \frac{(-h^2 \mathcal{D})^s}{s! 4^s} \mathcal{B}^s u(hm)\, \mathrm{e}^{-\langle B^{-1}(x-hm), x-hm\rangle/(h^2\mathcal{D})}$$

where $\mathcal{B}$ is the second order partial differential operator $\mathcal{B}u = \sum_{i,j=1}^n b_{ij} \partial_i \partial_j u$. In Section 5 we use Mu for the approximation of solutions of the equation $\mathcal{B}u = 0$. If u satisfies the equation in R^n then Mu has the simple form of the quasi-interpolant of order $\mathcal{O}((h\sqrt{\mathcal{D}})^2)$ but gives an approximation of exponential order plus a small saturation error. The same approximation property holds for functions which satisfy the equation $\mathcal{B}u = 0$ in a domain $\Omega \subset \mathrm{R}^n$. If the function u is extended by zero outside Ω then Mu simplifies to

$$Mu(x) = \frac{(\det B)^{-1/2}}{(\pi\mathcal{D})^{n/2}} \sum_{hm\in\Omega} u(hm)\, \mathrm{e}^{-\langle B^{-1}(x-hm),\, x-hm\rangle/(h^2\mathcal{D})}.$$

We obtain that, for any $\varepsilon > 0$, $M\,u$ approximates pointwise u in a subdomain $\Omega' \subsetneq \Omega$ with

$$|Mu(x) - u(x)| \le \varepsilon \sum_{|\beta|=0}^{2M-1} (h\sqrt{\mathcal{D}})^{|\beta|} |\partial^\beta u(x)| + C_B (h\sqrt{\mathcal{D}})^{2M} c_u,$$

if we choose $\mathcal{D}$ and h appropriately; the constant C_B is independent of $u, h, \mathcal{D}, M$ and c_u depends on u.

In Section 6 we show that the Hermite quasi-interpolant (4) gives the simultaneous approximation of the derivatives of u. If $\partial^\beta \mathcal{H}$ exists and is of rapid decay, for any function $u \in W^L_\infty(\mathrm{R}^n)$ with $L \ge N + |\beta|$, the difference $\partial^\beta Mu(x) - \partial^\beta u(x)$ can be estimated by

$$|\partial^\beta Mu(x) - \partial^\beta u(x)| \le \varepsilon \sum_{|\gamma|=0}^{L-1} (h\sqrt{\mathcal{D}})^{|\gamma|-|\beta|} |\partial^\gamma u(x)| + c_1\, (h\sqrt{\mathcal{D}})^N \|\nabla_{N+|\beta|} u\|_{L_\infty},$$

with c_1 independent of u, h and $\mathcal{D}$.

2. Quasi-interpolants with derivatives

In this section we study the approximation of a smooth function $u(x)$, $x \in \mathrm{R}^n$, by the Hermite quasi-interpolation operator

$$Mu(x) = \mathcal{D}^{-n/2} \sum_{m\in\mathrm{Z}^n} \mathcal{H}\left(\frac{x - hm}{h\sqrt{\mathcal{D}}}\right) \mathcal{Q}\left(-h\sqrt{\mathcal{D}}\,\partial\right) u(hm) \tag{5}$$

where $\mathcal{Q}(t)$, $t \in \mathrm{R}^n$, is a polynomial of degree at most $N-1$

$$\mathcal{Q}(t) = \sum_{|\gamma|=0}^{N-1} a_\gamma t^\gamma,\ t \in \mathrm{R}^n, \quad \text{with } a_\gamma \in \mathrm{R} \ \text{ and } a_0 = 1\,, \tag{6}$$

$\partial = \partial_1 \dots \partial_n$, and $\mathcal{H}$ is a sufficiently smooth, rapidly decaying function. Then

$$Mu(x) = \mathcal{D}^{-n/2} \sum_{m\in\mathrm{Z}^n} \left(\sum_{|\gamma|=0}^{N-1} (-h\sqrt{\mathcal{D}})^{|\gamma|} a_\gamma \partial^\gamma u(hm) \right) \mathcal{H}\left(\frac{x - hm}{h\sqrt{\mathcal{D}}}\right). \tag{7}$$

Our aim is to give conditions on the generating function $\mathcal{H}$ such that (7) is an approximation formula of order $\mathcal{O}((h\sqrt{\mathcal{D}})^N)$ plus terms, which can be made sufficiently small.

Suppose that $u \in C^N(\mathrm{R}^n) \cap L^\infty(\mathrm{R}^n)$. Taking the Taylor expansion of $\partial^\gamma u$ at each node hm leads to

$$\partial^\gamma u(hm) = \sum_{|\alpha|=0}^{N-1-|\gamma|} \frac{(hm-x)^\alpha}{\alpha!} \partial^{\alpha+\gamma} u(x) + \sum_{|\alpha|=N-|\gamma|} \frac{(hm-x)^\alpha}{\alpha!} U_{\alpha+\gamma}(x, hm), \quad 0 \le |\gamma| < N, \tag{8}$$

with

$$U_\alpha(x,y) = N \int_0^1 s^{N-1} \partial^\alpha u(sx + (1-s)y)\, ds\,, \; |\alpha| = N. \tag{9}$$

We write Mu in the form

$$Mu(x) = \sum_{|\beta|=0}^{N-1} (-h\sqrt{\mathcal{D}})^{|\beta|} \partial^\beta u(x) \sum_{\alpha \le \beta} \frac{a_{\beta-\alpha}}{\alpha!} \sigma_\alpha(\frac{x}{h}, \mathcal{D}, \mathcal{H}) + \mathcal{R}_{h,N}(x)$$

with the periodic functions

$$\sigma_\alpha(\xi, \mathcal{D}, \mathcal{H}) = \mathcal{D}^{-n/2} \sum_{m \in \mathbb{Z}^n} \left(\frac{\xi - m}{\sqrt{\mathcal{D}}} \right)^\alpha \mathcal{H}\left(\frac{\xi - m}{\sqrt{\mathcal{D}}} \right), \quad 0 \le |\alpha| < N\,,$$

and the remainder term

$$\mathcal{R}_{h,N}(x) = (-h\sqrt{\mathcal{D}})^N \mathcal{D}^{-n/2} \sum_{|\beta|=N} \sum_{0<\alpha\le\beta} \frac{a_{\beta-\alpha}}{\alpha!} \sum_{m\in\mathbb{Z}^n} U_\beta(x,hm) \left(\frac{x-hm}{h\sqrt{\mathcal{D}}} \right)^\alpha \mathcal{H}\left(\frac{x-hm}{h\sqrt{\mathcal{D}}} \right).$$

Therefore by using the definition of σ_α,

$$Mu(x) - u(x) = u(x) \left(\sigma_0(\frac{x}{h}, \mathcal{D}, \mathcal{H}) - 1 \right) + \sum_{|\beta|=1}^{N-1} (-h\sqrt{\mathcal{D}})^{|\beta|} \partial^\beta u(x) \sum_{\alpha\le\beta} \frac{a_{\beta-\alpha}}{\alpha!} \sigma_\alpha(\frac{x}{h}, \mathcal{D}, \mathcal{H}) + \mathcal{R}_{h,N}(x). \tag{10}$$

Let us introduce the functions

$$\mathcal{E}_\alpha(\xi, \mathcal{D}, \mathcal{H}) = \sigma_\alpha(\xi, \mathcal{D}, \mathcal{H}) - \int_{\mathbb{R}^n} x^\alpha \mathcal{H}(x)\, dx\,, \qquad 0 \le |\alpha| \le N-1.$$

Then we have

$$\sigma_0(\xi, \mathcal{D}, \mathcal{H}) - 1 = \mathcal{E}_0(\xi, \mathcal{D}, \mathcal{H}) + \Big(\int_{\mathbb{R}^n} \mathcal{H}(x) dx - 1 \Big) \tag{11}$$

and for $|\beta| = 1, \ldots, N-1$

$$\sum_{\alpha \le \beta} \frac{a_{\beta-\alpha}}{\alpha!} \sigma_\alpha(\xi, \mathcal{D}, \mathcal{H}) = \sum_{\alpha \le \beta} \frac{a_{\beta-\alpha}}{\alpha!} \mathcal{E}_\alpha(\xi, \mathcal{D}, \mathcal{H}) + \sum_{\alpha \le \beta} \frac{a_{\beta-\alpha}}{\alpha!} \int\limits_{\mathrm{R}^n} x^\alpha \mathcal{H}(x) dx. \qquad (12)$$

Theorem 2.1: *Suppose that $\mathcal{H}$ is differentiable up to the order of the smallest integer $n_0 > n/2$, satisfies the decay condition: there exist $K > N + n$ and $C_\beta > 0$ such that*

$$|\partial^\beta \mathcal{H}(x)| \le C_\beta \, (1 + |x|)^{-K}, \quad 0 \le |\beta| \le n_0 \qquad (13)$$

and the conditions

$$\begin{cases} \displaystyle\int\limits_{\mathrm{R}^n} \mathcal{H}(x) dx & = 1 \\ \displaystyle\sum_{\alpha \le \beta} \frac{a_{\beta-\alpha}}{\alpha!} \int\limits_{\mathrm{R}^n} x^\alpha \mathcal{H}(x) dx & = 0\,, \quad |\beta| = 1, \ldots, N-1. \end{cases} \qquad (14)$$

Then for any $\varepsilon > 0$ there exists $\mathcal{D} > 0$ such that, for all $u \in W_\infty^N(\mathrm{R}^n) \cap C^N(\mathrm{R}^n)$ the approximation error of the quasi-interpolant (7) *can be pointwise estimated by*

$$|Mu(x) - u(x)| \le \varepsilon \sum_{|\beta|=0}^{N-1} (h\sqrt{\mathcal{D}})^{|\beta|} |\partial^\beta u(x)| + c(h\sqrt{\mathcal{D}})^N \|\nabla_N u\|_{L_\infty} \qquad (15)$$

with the constant c not depending on h, u and $\mathcal{D}$.

Proof: Under conditions (14) the relations (11) and (12) simplify to

$$\sigma_0(\xi, \mathcal{D}, \mathcal{H}) - 1 = \mathcal{E}_0(\xi, \mathcal{D}, \mathcal{H}),$$

$$\sum_{\alpha \le \beta} \frac{a_{\beta-\alpha}}{\alpha!} \sigma_\alpha(\xi, \mathcal{D}, \mathcal{H}) = \sum_{\alpha \le \beta} \frac{a_{\beta-\alpha}}{\alpha!} \mathcal{E}_\alpha(\xi, \mathcal{D}, \mathcal{H}).$$

Moreover, the assumptions (13) on $\mathcal{H}$ ensure that (see [10])

$$\{\partial^\alpha \mathcal{F}\mathcal{H}(\sqrt{\mathcal{D}}\nu)\} \in l_1(\mathrm{Z}^n), \qquad 0 \le |\alpha| < K - n,$$

$$|\mathcal{E}_\alpha(\xi, \mathcal{D}, \mathcal{H})| = \Big|\sigma_\alpha(\xi, \mathcal{D}, \mathcal{H}) - \int_{\mathrm{R}^n} x^\alpha \mathcal{H}(x)\, dx\Big| \le (2\pi)^{-|\alpha|} \sum_{\nu \in \mathrm{Z}^n \setminus 0} \big|\partial^\alpha \mathcal{F}\mathcal{H}(\sqrt{\mathcal{D}}\nu)\big|,$$

and

$$\sum_{\nu \in \mathrm{Z}^n \setminus 0} |\partial^\alpha \mathcal{F}\mathcal{H}(\sqrt{\mathcal{D}}\nu)| \longrightarrow 0 \ \text{ as } \ \mathcal{D} \to \infty.$$

Hence for any $\varepsilon > 0$ there exists $\mathcal{D}$ such that the following estimates are valid

$$|\sigma_0(\xi, \mathcal{D}, \mathcal{H}) - 1| = |\mathcal{E}_0(\xi, \mathcal{D}, \mathcal{H})| < \varepsilon, \qquad (16)$$

$$\Big|\sum_{\alpha\le\beta}\frac{a_{\beta-\alpha}}{\alpha!}\sigma_\alpha(\xi,\mathcal{D},\mathcal{H})\Big| \le \sum_{\alpha\le\beta}\frac{|a_{\beta-\alpha}|}{\alpha!}|\mathcal{E}_\alpha(\xi,\mathcal{D},\mathcal{H})| < \varepsilon, \quad |\beta| = 1,\dots,N-1. \quad (17)$$

The next step is to estimate the remainder term $\mathcal{R}_{h,N}$. Since

$$|U_\alpha(x,y)| = N\left|\int_0^1 s^{N-1}\partial^\alpha u(sx+(1-s)y)ds\right| \le ||\partial^\alpha u||_{L_\infty}$$

we obtain

$$|\mathcal{R}_{h,N}(x)| \le (h\sqrt{\mathcal{D}})^N\sum_{|\beta|=N}\left\|\partial^\beta u\right\|_{L^\infty}\sum_{0<\alpha\le\beta}\frac{|a_{\beta-\alpha}|}{\alpha!}\|\rho_\alpha(\cdot,\mathcal{D},\mathcal{H})\|_{L_\infty}$$

where

$$\rho_\alpha(\xi,\mathcal{D},\mathcal{H}) = \mathcal{D}^{-n/2}\sum_{m\in\mathbb{Z}^n}\Big|\Big(\frac{\xi-m}{\sqrt{\mathcal{D}}}\Big)^\alpha\mathcal{H}\Big(\frac{\xi-m}{\sqrt{\mathcal{D}}}\Big)\Big|.$$

In view of the decay condition there exist constants c_α such that, for $\mathcal{D}$ sufficiently large ([10])

$$\|\rho_\alpha(\cdot,\mathcal{D},\mathcal{H})\|_{L_\infty} \le c_\alpha, \quad 0\le|\alpha|\le N.$$

Hence we obtain

$$|\mathcal{R}_{h,N}(x)| \le c\,(h\sqrt{\mathcal{D}})^N\sum_{|\beta|=N}\left\|\partial^\beta u\right\|_{L^\infty} \quad (18)$$

with a constant c not depending on h, $\mathcal{D}$ and u. By (10), this leads together with (16) and (17) to the estimate (15). □

3. Construction of generating functions for arbitrary N

Here we describe a simple method for the construction of generating functions $\mathcal{H}$ satisfying the conditions (14) for arbitrary given a_γ with $a_0 = 1$. Let us denote by $\mathcal{A} = \{\mathcal{A}_{\alpha\beta}\}$ the triangular matrix with the elements

$$\mathcal{A}_{\alpha\beta} = \begin{cases} a_{\beta-\alpha} & \alpha\le\beta, \\ 0 & \text{otherwise} \end{cases} \quad |\beta|,|\alpha| = 0,\dots,N-1.$$

The dimension of $\mathcal{A}$ is $(N+n-1)!/((N-1)!\,n!)$. Since $\det\mathcal{A} = 1$ there exists the inverse matrix $\mathcal{A}^{-1} = \{\mathcal{A}^{(-1)}_{\alpha\beta}\}$ and (14) leads to the following conditions

$$\int_{\mathbb{R}^n} x^\alpha\mathcal{H}(x)\,dx = \alpha!\,\mathcal{A}^{(-1)}_{0\alpha}, \quad |\alpha| = 0,\dots,N-1. \quad (19)$$

These conditions can be rewritten as

$$\partial^\alpha(\mathcal{F}\mathcal{H})(0) = \alpha!(-2\pi i)^{|\alpha|}\mathcal{A}^{(-1)}_{0\alpha}, \quad |\alpha| = 0,\dots,N-1.$$

Let us assume (see [9])

$$\mathcal{H}(x) = \mathcal{P}_N\Big(\frac{1}{2\pi i}\frac{\partial}{\partial x}\Big)\,\eta(x),$$

where $\mathcal{P}_N(t)$ is a polynomial of degree less than or equal to $N-1$ and η is a smooth function rapidly decaying as $|x| \to \infty$ with $\mathcal{F}\eta(0) \neq 0$. Conditions (19) give

$$\partial^\alpha\big(\mathcal{P}_N(\lambda)\mathcal{F}\eta(\lambda)\big)(0) = (-2\pi i)^{|\alpha|}\alpha!\,\mathcal{A}^{(-1)}_{0\alpha}, \quad |\alpha| = 0, \dots, N-1.$$

We choose $\mathcal{P}_N$ as the Taylor polynomial of order $(N-1)$ of the function

$$Q(\lambda) = \sum_{|\alpha|=0}^{N-1} \frac{\mathcal{A}^{(-1)}_{0\alpha}(-2\pi i\lambda)^\alpha}{\mathcal{F}\eta(\lambda)}, \quad \lambda \in \mathrm{R}^n.$$

Since

$$\partial^\beta Q(0) = \sum_{\alpha\le\beta} \mathcal{A}^{(-1)}_{0\alpha}(-2\pi i)^{|\alpha|}\frac{\beta!}{(\beta-\alpha)!}\partial^{\beta-\alpha}(\mathcal{F}\eta)^{-1}(0),$$

where we use the notation

$$\partial^{\beta-\alpha}(\mathcal{F}\eta)^{-1}(0) = \partial^{\beta-\alpha}\Big(\frac{1}{\mathcal{F}\eta(\lambda)}\Big)(0),$$

we obtain

$$\mathcal{P}_N(\lambda) = \sum_{|\beta|=0}^{N-1} \frac{\lambda^\beta}{\beta!}\sum_{\alpha\le\beta} \mathcal{A}^{(-1)}_{0\alpha}(-2\pi i)^{|\alpha|}\frac{\beta!}{(\beta-\alpha)!}\partial^{\beta-\alpha}(\mathcal{F}\eta)^{-1}(0)\,.$$

Therefore the equations

$$\begin{aligned}\partial^\alpha\big(\mathcal{P}_N(\lambda)\mathcal{F}\eta(\lambda)\big)(0) &= \partial^\alpha(Q(\lambda)\mathcal{F}\eta(\lambda))(0)\\ &= \partial^\alpha\Big(\sum_{|\gamma|=0}^{N-1}\mathcal{A}^{(-1)}_{0\gamma}(-2\pi i\lambda)^\gamma\Big)(0) = (-2\pi i)^{|\alpha|}\alpha!\,\mathcal{A}^{(-1)}_{0\alpha}\end{aligned}$$

are valid for all $\alpha : 0 \le |\alpha| \le N-1$. We have thus proved the following

Theorem 3.1: *Suppose that* $\eta \in C^{N-1}(\mathrm{R}^n)$ *satisfies*

$$|\eta(x)| \le A\,(1+|x|)^{-K}, \quad x \in \mathrm{R}^n, \quad K > N+n,$$
$$\int_{\mathrm{R}^n} |x|^{N-1}|\partial^\alpha\eta(x)|dx < \infty, \quad 0 \le |\alpha| \le N-1,$$

and $\mathcal{F}\eta(0) \neq 0$. *Then the function*

$$\mathcal{H}(x) = \sum_{|\beta|=0}^{N-1}\sum_{\alpha\le\beta}\mathcal{A}^{(-1)}_{0\alpha}(-1)^{|\alpha|}\frac{\partial^{\beta-\alpha}(\mathcal{F}\eta)^{-1}(0)}{(\beta-\alpha)!(2\pi i)^{|\beta-\alpha|}}\partial^\beta\eta(x)$$

satisfies the conditions (14).

Suppose that $\eta(x)$ is radial, that is $\eta(x) = \psi(r)$, $r = |x|$. Then $\partial^\alpha \mathcal{F}\eta(0) = 0$ for any $\alpha = (\alpha_1, \ldots, \alpha_n)$ containing at least one odd α_i and we obtain the formula

$$\mathcal{H}(x) = \sum_{|\beta|=0}^{N-1} (-\partial)^\beta \eta(x) \sum_{2\gamma \le \beta} \mathcal{A}^{(-1)}_{0,\beta-2\gamma} \frac{\partial^{2\gamma}(\mathcal{F}\eta)^{-1}(0)}{(2\gamma)!(-4\pi^2)^{|\gamma|}} .$$

Let us consider the special case $\eta(x) = \pi^{-n/2} \mathrm{e}^{-|x|^2}$ with $\mathcal{F}\eta(\lambda) = \mathrm{e}^{-\pi^2|\lambda|^2}$. Denoting by H_β the Hermite polynomial of n variables defined by

$$H_\beta(t) = \mathrm{e}^{|t|^2} (-\partial)^\beta \mathrm{e}^{-|t|^2}$$

we derive $\partial^\gamma (\mathcal{F}\eta)^{-1}(0) = (-\pi i)^{|\gamma|} H_\gamma(0)$. Note that $H_\gamma(0) = 0$ if γ has odd components, otherwise $H_{2\gamma}(0) = (-1)^\gamma (2\gamma)!/\gamma!$. Hence

$$\mathcal{H}(x) = \pi^{-n/2} \sum_{|\beta|=0}^{N-1} H_\beta(x) \mathrm{e}^{-|x|^2} \sum_{2\gamma \le \beta} \frac{(-1)^{|\gamma|}}{\gamma! 4^{|\gamma|}} \mathcal{A}^{(-1)}_{0\,,\beta-2\gamma}. \tag{20}$$

Assuming $a_\gamma = \delta_{|\gamma|0}$ and $N = 2M$ we find out

$$\begin{aligned} \mathcal{H}(x) &= \pi^{-n/2} \sum_{j=0}^{M-1} \frac{(-1)^j}{4^j} \sum_{|\beta|=j} \frac{H_{2\beta}(x)}{\beta!} \mathrm{e}^{-|x|^2} \\ &= \pi^{-n/2} \sum_{j=0}^{M-1} \frac{(-1)^j}{4^j j!} \Delta^j \mathrm{e}^{-|x|^2} = \pi^{-n/2} L^{(n/2)}_{M-1}(|x|^2) \mathrm{e}^{-|x|^2} , \end{aligned}$$

where $L^{(\gamma)}_k$ are the generalized Laguerre polynomials, which are defined by

$$L^{(\gamma)}_k(y) = \frac{\mathrm{e}^{\,y} y^{-\gamma}}{k!} \left(\frac{d}{dy}\right)^k \left(\mathrm{e}^{-y} y^{k+\gamma}\right), \quad \gamma > -1 .$$

In this case we obtain the classical generating function $\eta(x) = L^{(n/2)}_{M-1}(|x|^2)\mathrm{e}^{-|x|^2}$ (see [9, 13]).

4. Examples

Example 4.1 If $N = 2$, then formula (20) gives

$$\mathcal{H}(x) = \pi^{-n/2} \Big(1 - 2 \sum_{|\beta|=1} a_\beta x^\beta \Big) \mathrm{e}^{-|x|^2}. \tag{21}$$

For the one-dimensional case the corresponding quasi-interpolant is

$$M_a u(x) = (\pi \mathcal{D})^{-1/2} \sum_{m \in \mathbb{Z}} (u(hm) - h\sqrt{\mathcal{D}}\, a\, u'(hm))(1 - 2a \frac{x - hm}{h\sqrt{\mathcal{D}}}) \mathrm{e}^{-(x-hm)^2/(h^2\mathcal{D})} .$$

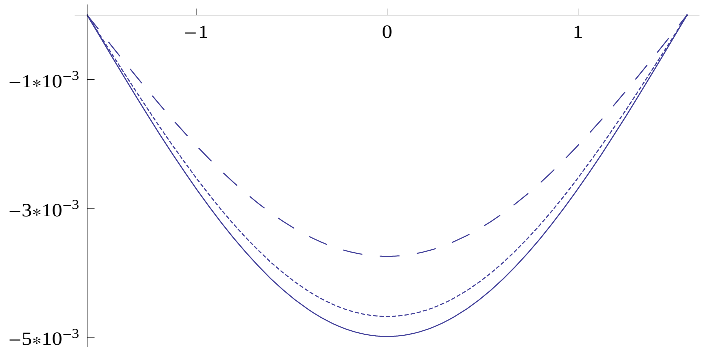

Figure 1. The graphs of $(M_a - I)\cos x$ when $a = 0$ (solid line), $a = 1/8$ (dotted line) and $a = 1/4$ (dashed line).

For $u(x) = \cos x$, $x \in \mathrm{R}$, the difference between $u(x)$ and $M_a u(x)$ is plotted in Figure 1 by taking $h = 0.1$, $\mathcal{D} = 2$ for different values of a.

Example 4.2 Now we are looking for quasi-interpolants of order $\mathcal{O}(h^4)$. In the one-dimensional case

$$\mathcal{A}_{01}^{(-1)} = -a_1,\ \mathcal{A}_{02}^{(-1)} = a_1^2 - a_2,\ \mathcal{A}_{03}^{(-1)} = -a_1^3 + 2a_1a_2 - a_3$$

and formula (20) gives

$$\begin{aligned}\mathcal{H}(x) =&\pi^{-1/2}\mathrm{e}^{-x^2}[(3/2 - \mathcal{A}_{02}^{(-1)}) + (5\mathcal{A}_{01}^{-1} - 12\mathcal{A}_{03}^{(-1)})x \\ &+ (4\mathcal{A}_{02}^{(-1)} - 1)x^2 + (8\mathcal{A}_{03}^{(-1)} - 2\mathcal{A}_{01}^{(-1)})x^3].\end{aligned}$$

If $a_1 = a_2 = a_3 = 0$, then we get the classical generating function $\eta(x) = \pi^{-1/2}\mathrm{e}^{-x^2}(3/2 - x^2)$. Assuming $a_3 = a_2 = 0$, $a_1 = \pm 1/2$ we obtain

$$\mathcal{H}(x) = \pi^{-1/2}(1 \pm x)\mathrm{e}^{-x^2}$$

and the quasi-interpolant of order $\mathcal{O}(h^4)$:

$$M_1 u(x) = (\pi\mathcal{D})^{-1/2} \sum_{m\in \mathrm{Z}} (u(hm) \pm \frac{h\sqrt{\mathcal{D}}}{2} u'(hm))(1 \pm \frac{x - hm}{h\sqrt{\mathcal{D}}})\, \mathrm{e}^{-(x-hm)^2/(h^2\mathcal{D})}\,.$$

With the choice $a_1 = a_3 = 0$, $a_2 = -1/4$ we obtain $\mathcal{H}(x) = \pi^{-1/2}\mathrm{e}^{-x^2}$ and the quasi-interpolant of order $\mathcal{O}(h^4)$:

$$M_2 u(x) = (\pi\mathcal{D})^{-1/2} \sum_{m\in \mathrm{Z}} (u(hm) - \frac{h^2\mathcal{D}}{4} u''(hm))\, \mathrm{e}^{-(x-hm)^2/(h^2\mathcal{D})}\,.$$

In the Figures 2, 3, 4 we show the error graphs for the approximation of $u = \cos(x)$ with the quasi-interpolants Mu, M_1u, M_2u, respectively. We have taken $\mathcal{D} = 2$, $h = 0.05$, $h = 0.1$ and $h = 0.2$. The case of smaller h gives different pictures: it is clearly visible that the error oscillates very fast with Mu and M_1u.

If $n > 1$, by taking $a_{2\beta} = a$ and $a_\beta = 0$ otherwise, we obtain the radial generating

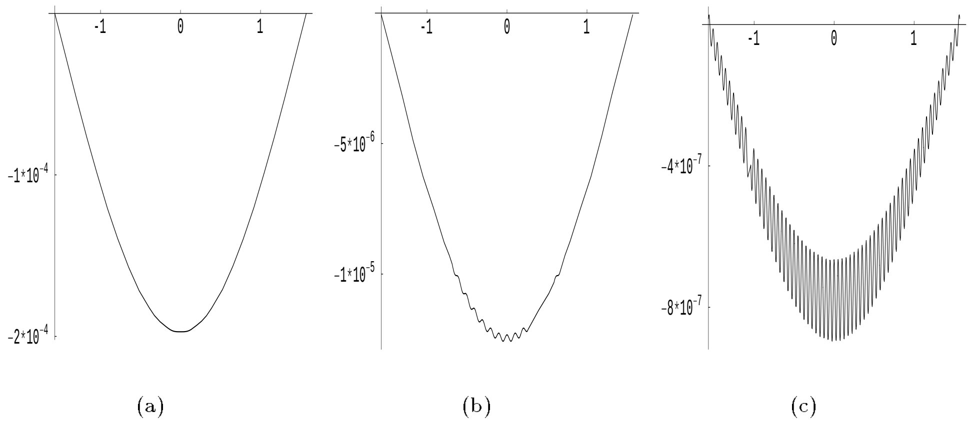

Figure 2. The graphs of $(M - I)\cos(x)$ by assuming $h = 0.2$ (a), $h = 0.1$ (b) and $h = 0.05$ (c).

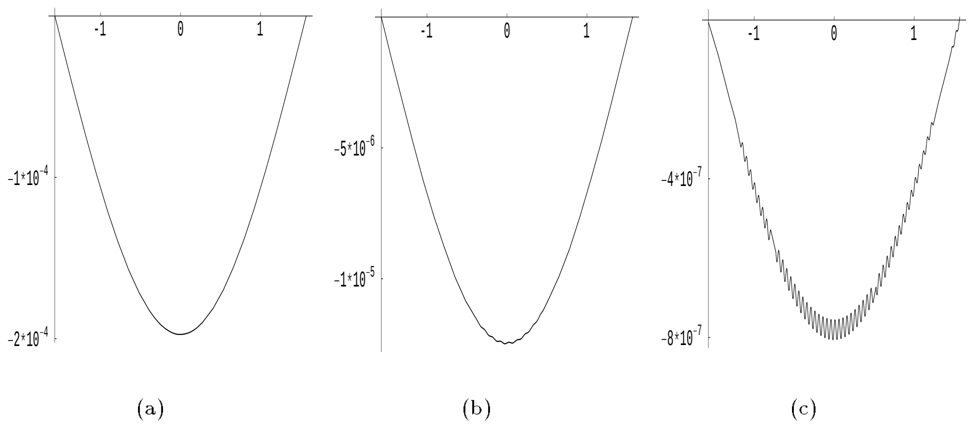

Figure 3. The graphs of $(M_1 - I)\cos(x)$ by assuming $h = 0.2$ (a), $h = 0.1$ (b) and $h = 0.05$ (c).

function

$$\begin{aligned}\mathcal{H}_a(x) =&(1 - \sum_{|\beta|=1} (a_{2\beta} + 1/4)(4x^{2\beta} - 2))\mathrm{e}^{-|x|^2}\pi^{-n/2}\\ =&(1 + n(2a + 1/2) - (1 + 4a)|x|^2)\mathrm{e}^{-|x|^2}\pi^{-n/2}\end{aligned}$$

for the approximate approximation of order $\mathcal{O}(h^4)$

$$M_a u(x) = (\pi\mathcal{D})^{-n/2} \sum_{m\in\mathbb{Z}^n} (u(hm) + h^2\,\mathcal{D}\,a\,\Delta u(hm))\mathcal{H}_a(\frac{x - hm}{h\sqrt{\mathcal{D}}}).$$

Example 4.3 Let $n > 1$ and set in (7) $N = 2M$ and

$$\begin{cases} a_\gamma \;= 0\,, \quad \text{if } \gamma \text{ has odd components,} \\ a_{2\gamma} = \dfrac{(-1)^{|\gamma|}}{\gamma! 4^{|\gamma|}}, \quad 0 \le |\gamma| \le M - 1. \end{cases} \tag{22}$$

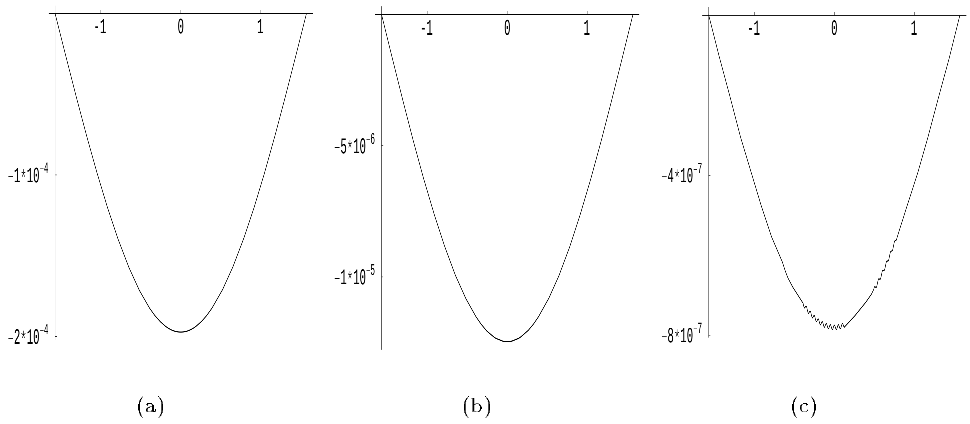

Figure 4. The graphs of $(M_2 - I)\cos(x)$ by assuming $h = 0.2$ (a), $h = 0.1$ (b) and $h = 0.05$ (c).

Keeping in mind that

$$\sum_{|\gamma|=s} \frac{s!}{\gamma!}\,\partial^{2\gamma} u(x) = \Delta^s u(x) \tag{23}$$

we obtain

$$\mathcal{Q}\Big((-h\sqrt{\mathcal{D}})\partial\Big)\, u = \sum_{j=0}^{M-1} \frac{(-1)^j (h\sqrt{\mathcal{D}})^{2j}}{4^j} \sum_{|\gamma|=j} \frac{\partial^{2\gamma}}{\gamma!} u = \sum_{j=0}^{M-1} \frac{(-1)^j (h^2\mathcal{D})^j}{j! 4^j} \Delta^j u.$$

The function $\mathcal{H}(x) = \pi^{-n/2} \mathrm{e}^{-|x|^2}$ satisfies conditions (14). In fact

$$\frac{1}{\pi^{n/2}} \int\limits_{\mathrm{R}^n} x^\alpha \mathrm{e}^{-|x|^2} dx = \frac{(-\pi i)^\alpha}{(2\pi)^\alpha} H_\alpha(0) = \begin{cases} 0, & \text{if } \alpha \text{ has odd components,} \\ \dfrac{(2\gamma)!}{\gamma! 2^{2|\gamma|}}, & \text{if } \alpha = 2\gamma\,. \end{cases}$$

Then the equations (14) are valid for all $\beta : 0 \le |\beta| \le M-1$ because of

$$\sum_{\alpha\le\beta} \frac{a_{2(\beta-\alpha)}}{\alpha! 2^{2|\alpha|}} = \sum_{\alpha\le\beta} \frac{(-1)^{|\beta-\alpha|}}{(\beta-\alpha)! 4^{|\beta-\alpha|}} \frac{1}{\alpha! 4^{|\alpha|}} = \delta_{|\beta|0}, \quad |\beta| = 0, \ldots, M-1.$$

Therefore, a general approximation of order $N = 2M$ is given by

$$M^{(N)} u(x) = (\pi\mathcal{D})^{-n/2} \sum_{m\in\mathrm{Z}^n} \sum_{s=0}^{M-1} (h\sqrt{\mathcal{D}})^{2s} \frac{(-1)^s}{s! 4^s} \Delta^s u(hm)\, \mathrm{e}^{-|x-hm|^2/(h^2\mathcal{D})}. \tag{24}$$

If $M = 2$ then we find the "fourth order formula"

$$M^{(4)} u(x) = (\pi\mathcal{D})^{-n/2} \sum_{m\in\mathrm{Z}^n} \big[u(hm) - \frac{h^2\mathcal{D}}{4} \Delta u(hm)\big] \mathrm{e}^{-|x-hm|^2/(h^2\mathcal{D})}.$$

For $M = 3$ we obtain the "sixth order formula"

$$M^{(6)}u(x) = (\pi\mathcal{D})^{-n/2}\sum_{m\in\mathbb{Z}^n}\big[u(hm) - \frac{h^2\mathcal{D}}{4}\Delta u(hm) + \frac{h^4\mathcal{D}^2}{32}\Delta^2 u(hm)\big]\mathrm{e}^{-|x-hm|^2/(h^2\mathcal{D})}.$$

Note the additive structure of the formula (24)

$$M^{(N+2)}u(x) = M^{(N)}u(x) + \frac{(h\sqrt{\mathcal{D}})^N(-1)^M}{(\pi\mathcal{D})^{n/2}M!4^M}\sum_{m\in\mathbb{Z}^n}\Delta^M u(hm)\,\mathrm{e}^{-|x-hm|^2/(h^2\mathcal{D})}.$$

Example 4.4 We consider the second order partial differential operator

$$\mathcal{B}u = \sum_{i,k=1}^{n} b_{ik}\partial_i\partial_k u\,,$$

where the matrix $B = \{b_{ik}\} \in \mathbb{R}^{n\times n}$ is symmetric and positive definite. Define the quasi-interpolant

$$Mu(x) = \frac{(\det B)^{-1/2}}{(\pi\mathcal{D})^{n/2}}\sum_{m\in\mathbb{Z}^n}\sum_{s=0}^{M-1}\frac{(-h^2\mathcal{D})^s}{s!4^s}\mathcal{B}^s u(hm)\,\mathrm{e}^{-\langle B^{-1}(x-hm),x-hm\rangle/(h^2\mathcal{D})}. \tag{25}$$

Assume C such that $B^{-1} = C^T C$. If we consider the linear transformation $\xi = C\,x$ and introduce $U(\xi) = u(x)$, in the new coordinates we have ([12, p.42])

$$\mathcal{B}u(x) = \Delta U(\xi).$$

Then (25) will take the form

$$Mu(x) = \mathcal{M}U(\xi) = \frac{\det C}{(\pi\mathcal{D})^{n/2}}\sum_{m\in\mathbb{Z}^n}\sum_{s=0}^{M-1}\frac{(-h^2\mathcal{D})^s}{s!4^s}\Delta^s U(h\,C\,m)\,\mathrm{e}^{-|\xi-h\,C\,m|^2/(h^2\mathcal{D})}.$$

Keeping in mind (23) and (8) we rewrite

$$\mathcal{M}U(\xi) = \sum_{|\beta|=0}^{2M-1}(-h\sqrt{\mathcal{D}})^{|\beta|}\partial^\beta U(\xi)\,\mathcal{E}_\beta\big(\frac{\xi}{h},\mathcal{D}\big) + R_h(\xi)$$

where

$$\mathcal{E}_\beta(\xi,\mathcal{D}) = \sum_{2\gamma\le\beta}\frac{(-1)^{|\gamma|}}{\gamma!(\beta-2\gamma)!4^{|\gamma|}}\Sigma_{\beta-2\gamma}(\xi,\mathcal{D})$$

with

$$\Sigma_\alpha(\xi,\mathcal{D}) = \frac{\det C}{(\pi\mathcal{D})^{n/2}}\sum_{m\in\mathbb{Z}^n}\left(\frac{\xi-C\,m}{\sqrt{\mathcal{D}}}\right)^\alpha \mathrm{e}^{-|\frac{\xi-C\,m}{\sqrt{\mathcal{D}}}|^2}\,,$$

and

$$R_h(\xi) = \frac{(h\sqrt{\mathcal{D}})^{2M}\det C}{(\pi\mathcal{D})^{n/2}}$$
$$\times \sum_{|\beta|=2M}\sum_{2\gamma\le\beta}\frac{(-1)^{|\gamma|}}{4^{|\gamma|}\gamma!(\beta-2\gamma)!}\sum_{m\in\mathbb{Z}^n}U_\beta(\xi,hCm)\left(\frac{\xi-hCm}{h\sqrt{\mathcal{D}}}\right)^{\beta-2\gamma}\mathrm{e}^{-|\frac{\xi-hCm}{h\sqrt{\mathcal{D}}}|^2}.$$

Now we use Poisson's summation formula on affine grids (see [11, p.23])

$$\frac{\det C}{\mathcal{D}^{n/2}}\sum_{m\in\mathbb{Z}^n}\Big(\frac{\xi-C\,m}{\sqrt{\mathcal{D}}}\Big)^\delta\eta\Big(\frac{\xi-C\,m}{\sqrt{\mathcal{D}}}\Big) = \Big(\frac{i}{2\pi}\Big)^{|\delta|}\sum_{\nu\in\mathbb{Z}^n}\partial^\delta\mathcal{F}\eta(\sqrt{\mathcal{D}}C^{-T}\nu)\mathrm{e}^{2\pi i\langle\xi,C^{-T}\nu\rangle}, \tag{26}$$

where we denote by $C^{-T}=(C^T)^{-1}$. In our case $\eta(x)=\pi^{-n/2}\mathrm{e}^{-|x|^2}$ and $\mathcal{F}\eta(\lambda)=\mathrm{e}^{-\pi^2|\lambda|^2}$. Since $\partial^\delta\mathcal{F}\eta(\lambda)=(-\pi)^\delta H_\delta(\pi\,\lambda)\mathrm{e}^{-\pi^2|\lambda|^2}$ we obtain

$$\partial^\delta\mathcal{F}\eta(0)=\begin{cases}0\,, & \text{if } \delta \text{ has odd components,}\\[2mm] \pi^{2\gamma}\dfrac{(-1)^{|\gamma|}(2\gamma)!}{\gamma!}, & \text{if } \delta=2\gamma\,.\end{cases}$$

Formula (26) applied to Σ_α gives

$$\Sigma_\alpha(\xi,\mathcal{D})=\begin{cases}\Big(\dfrac{i}{2\pi}\Big)^{|\alpha|}\displaystyle\sum_{\nu\ne0}\partial^\alpha\mathcal{F}\eta(\sqrt{\mathcal{D}}C^{-T}\nu)\mathrm{e}^{2\pi i\langle\xi,C^{-T}\nu\rangle}, \ \alpha \text{ has odd components,}\\ \dfrac{\alpha!}{2^{|\alpha|}\gamma!}+\Big(\dfrac{i}{2\pi}\Big)^{|\alpha|}\displaystyle\sum_{\nu\ne0}\partial^\alpha\mathcal{F}\eta(\sqrt{\mathcal{D}}C^{-T}\nu)\mathrm{e}^{2\pi i\langle\xi,C^{-T}\nu\rangle}, \ \alpha=2\gamma.\end{cases}$$

We deduce that

$$\mathcal{E}_\beta(\xi,\mathcal{D})=\delta_{|\beta|0}$$
$$+\sum_{2\gamma\le\beta}\frac{(-1)^{|\gamma|}}{\gamma!(\beta-2\gamma)!\,4^{|\gamma|}}\Big(\frac{i}{2\pi}\Big)^{|\beta-2\gamma|}\sum_{\nu\ne0}\partial^{\beta-2\gamma}\mathcal{F}\eta(\sqrt{\mathcal{D}}C^{-T}\nu)\mathrm{e}^{2\pi i\langle\xi,C^{-T}\nu\rangle}$$

and

$$|\mathcal{E}_\beta(\xi,\mathcal{D})-\delta_{|\beta|0}|\le\sum_{2\gamma\le\beta}\frac{\pi^{|2\gamma-\beta|}}{\gamma!(\beta-2\gamma)!\,2^{|\beta|}}\sum_{\nu\ne0}\big|\partial^{\beta-2\gamma}\mathcal{F}\eta(\sqrt{\mathcal{D}}C^{-T}\nu)\big|.$$

By repeating the same arguments used in the proof of Theorem 2.1 we derive that $|\mathcal{E}_\beta(\xi,\mathcal{D})-\delta_{|\beta|0}|<\varepsilon_1$ for prescribed $\varepsilon_1>0$ and sufficiently large $\mathcal{D}$, and the remainder is bounded by

$$|R_h(\xi)|\le c_B(h\sqrt{\mathcal{D}})^{2M}\sum_{|\beta|=2M}\|\partial^\beta U\|_{L^\infty}$$

with the constant c_B independent of $U, h, \mathcal{D}$. Hence

$$
\begin{aligned}
|Mu(x) - u(x)| &= |\mathcal{M}U(\xi) - U(\xi)| \\
&\le \varepsilon_1 \sum_{|\beta|=0}^{2M-1} (h\sqrt{\mathcal{D}})^{|\beta|} |\partial^\beta U(\xi)| + c_B (h\sqrt{\mathcal{D}})^{2M} \sum_{|\beta|=2M} ||\partial^\beta U||_{L^\infty} \\
&\le \varepsilon \sum_{|\beta|=0}^{2M-1} (h\sqrt{\mathcal{D}})^{|\beta|} |\partial^\beta u(x)| + C_B (h\sqrt{\mathcal{D}})^{2M} \sum_{|\beta|=2M} ||\partial^\beta u||_{L^\infty} ,
\end{aligned}
$$

where C_B depends only on the matrix B. Therefore the quasi-interpolant (25) approximates u with the order $\mathcal{O}((h\sqrt{\mathcal{D}})^{2M})$ up to the saturation error.

5. An application of formula (24)

Here we consider the approximation of harmonic functions. Suppose that $\Delta u = 0$ in some domain $\Omega \subseteq \mathrm{R}^n$. Then for any $N = 2M$ and $x \in \Omega$ the Hermite quasi-interpolant (24) has the simple form

$$
M^{(N)}u(x) = Mu(x) = (\pi\mathcal{D})^{-n/2} \sum_{hm\in\Omega} u(hm)\, \mathrm{e}^{-|x-hm|^2/(h^2\mathcal{D})} , \qquad (27)
$$

i.e., it coincides with the well known quasi-interpolation formula of second order. However, Theorem 2.1 indicates higher approximation rates. This will be studied here in more detail.

First we consider the case $\Omega = \mathrm{R}^n$. Then

$$
u(\xi) = \sum_{|\beta|=0}^{\infty} \frac{\partial^\beta u(x)}{\beta!} (\xi - x)^\beta , \quad \xi \in \mathrm{R}^n
$$

and the series converges absolutely in R^n. Moreover, u has the analytic extension

$$
\tilde{u}(\zeta) = \sum_{|\beta|=0}^{\infty} \frac{\partial^\beta u(x)}{\beta!} (\zeta - x)^\beta , \quad \zeta \in \mathrm{C}^n , \qquad (28)
$$

cf. e.g. [1, 14]. Using formula (10) for the quasi-interpolant with the generating function (24) we obtain

$$
Mu(x) - u(x) = \mathcal{E}_{h,2M}(x) + \mathcal{R}_{h,2M}(x) , \qquad (29)
$$

where

$$
\begin{aligned}
\mathcal{E}_{h,2M}(x) &= u(x)\Big((\pi\mathcal{D})^{-n/2} \sum_{m\in\mathrm{Z}^n} \mathrm{e}^{-|x-hm|^2/(h^2\mathcal{D})} - 1\Big) \\
&+ (\pi\mathcal{D})^{-n/2} \sum_{|\beta|=1}^{2M-1} (-h\sqrt{\mathcal{D}})^{|\beta|} \partial^\beta u(x) \sum_{m\in\mathrm{Z}^n} \mathrm{e}^{-|x-hm|^2/(h^2\mathcal{D})} \sum_{\alpha\le\beta} \frac{a_{\beta-\alpha}}{\alpha!} \Big(\frac{x-hm}{h\sqrt{\mathcal{D}}}\Big)^\alpha
\end{aligned}
$$

constitutes the saturation error and the remainder term has the form

$$\mathcal{R}_{h,2M}(x) = (-h\sqrt{\mathcal{D}})^N (\pi\mathcal{D})^{-n/2} \times \sum_{|\beta|=2M} \sum_{m\in\mathbb{Z}^n} U_\beta(x,hm) \mathrm{e}^{-|x-hm|^2/(h^2\mathcal{D})} \sum_{0<\alpha\le\beta} \frac{a_{\beta-\alpha}}{\alpha!} \Big(\frac{x-hm}{h\sqrt{\mathcal{D}}}\Big)^\alpha . \tag{30}$$

From (22) we see that

$$\sum_{\alpha\le\beta} \frac{a_{\beta-\alpha}}{\alpha!}\, x^\alpha = \sum_{2\gamma\le\beta} \frac{a_{2\gamma}}{(\beta-2\gamma)!}\, x^{\beta-2\gamma} = \sum_{2\gamma\le\beta} \frac{(-1)^{|\gamma|}}{\gamma!\,(\beta-2\gamma)!\,2^{2|\gamma|}}\, x^{\beta-2\gamma} ,$$

which by using the representation of Hermite polynomials

$$H_k(\tau) = \sum_{0\le 2j\le k} \frac{(-1)^j k!}{j!\,(k-2j)!}\,(2\tau)^{k-2j} ,$$

shows that

$$\sum_{\alpha\le\beta} \frac{a_{\beta-\alpha}}{\alpha!}\, x^\alpha = \frac{1}{\beta!\,2^{|\beta|}} H_\beta(x) .$$

Hence we obtain

$$\mathcal{E}_{h,2M}(x) = \sum_{|\beta|=0}^{2M-1} \Big(-\frac{h\sqrt{\mathcal{D}}}{2}\Big)^{|\beta|} \frac{\partial^\beta u(x)}{\beta!}\, \sigma_\beta\Big(\frac{x}{h},\mathcal{D}\Big)$$

with the functions

$$\begin{aligned} \sigma_0(x,\mathcal{D}) &= (\pi\mathcal{D})^{-n/2} \sum_{m\in\mathbb{Z}^n} \mathrm{e}^{-|x-m|^2/\mathcal{D}} - 1 , \\ \sigma_\beta(x,\mathcal{D}) &= (\pi\mathcal{D})^{-n/2} \sum_{m\in\mathbb{Z}^n} H_\beta\Big(\frac{x-m}{\sqrt{\mathcal{D}}}\Big)\, \mathrm{e}^{-|x-m|^2/\mathcal{D}} ,\ |\beta| = 1,\dots,2M-1 . \end{aligned} \tag{31}$$

It follows from the definition of H_β and from Poisson's summation formula that

$$\sigma_\beta(x,\mathcal{D}) = (-1)^{|\beta|}\mathcal{D}^{|\beta|/2}\partial^\beta\sigma_0(x,\mathcal{D}) = (-2\pi i)^{|\beta|}\mathcal{D}^{|\beta|/2} \sum_{m\in\mathbb{Z}^n\setminus\{0\}} m^\beta \mathrm{e}^{-\pi^2\mathcal{D}|m|^2} \mathrm{e}^{2\pi i\langle m,x\rangle}.$$

Thus the saturation error can be expressed as

$$\begin{aligned} \mathcal{E}_{h,2M}(x) &= \sum_{|\beta|=0}^{2M-1} \frac{\partial^\beta u(x)}{\beta!} \Big(\frac{h\mathcal{D}}{2}\Big)^\beta \partial^\beta\, \sigma_0\Big(\frac{x}{h},\mathcal{D}\Big) \\ &= \sum_{m\in\mathbb{Z}^n\setminus\{0\}} \tilde{u}_{2M}(x+i\pi h\mathcal{D}m) \mathrm{e}^{-\pi^2\mathcal{D}|m|^2} \mathrm{e}^{2\pi i\langle m,x\rangle/h} , \end{aligned} \tag{32}$$

where

$$\tilde{u}_N(\zeta) = \sum_{|\beta|=0}^{N-1} \frac{\partial^\beta u(x)}{\beta!} (\zeta - x)^\beta$$

is the Taylor polynomial of the analytic extension $\tilde{u}$. Note that (32) is valid for any M.

Theorem 5.1: *Suppose that the harmonic in* R^n *function* u *is such that the series*

$$\sum_{|\beta|=0}^{\infty} \frac{\partial^\beta u(x)}{\sqrt{\beta!}} y^\beta \tag{33}$$

converges absolutely for any $y \in \mathrm{R}^n$. *If* $\sqrt{\mathcal{D}}h < 1$, *then the quasi-interpolant* (27) *approximates* u *with*

$$Mu(x) - u(x) = \lim_{M\to\infty} \mathcal{E}_{h,2M}(x) = \sum_{m\in\mathrm{Z}^n\setminus\{0\}} \tilde{u}(x + \pi i h\mathcal{D}m) \mathrm{e}^{-\pi^2\mathcal{D}|m|^2} \mathrm{e}^{2\pi i\langle m,x\rangle/h},$$

where $\tilde{u}$ *is the analytic extension of* u *onto* C^n.

Proof: We have to show that $|\mathcal{R}_{h,2M}(x)| \to 0$ as $M \to \infty$. To estimate (30) we rewrite

$$\sum_{0\le 2j<k} \frac{(-1)^j}{j!\,(k-2j)!\,2^{2j}}\, \tau^{k-2j} = \frac{1}{2^k\, k!}\big(H_k(\tau) - H_k(0)\big)$$

$$= \frac{1}{2^{k-1}\,(k-1)!} \int_0^\tau H_{k-1}(t)\, dt\,,$$

which implies for $|\beta| = 2M$

$$\sum_{0<\alpha\le\beta} \frac{a_{\beta-\alpha}}{\alpha!} \Big(\frac{x}{\sqrt{\mathcal{D}}}\Big)^\alpha = \frac{1}{2^{2M}} \prod_{\beta_j>0} \frac{1}{\beta_j!} \Big(H_{\beta_j}\Big(\frac{x_j}{\sqrt{\mathcal{D}}}\Big) - H_{\beta_j}(0)\Big)$$

$$= \frac{1}{2^{2M}} \prod_{\beta_j>0} \frac{2}{(\beta_j-1)!} \int_0^{x_j/\sqrt{\mathcal{D}}} H_{\beta_j-1}(t)\, dt\,.$$

Consequently, the remainder $\mathcal{R}_{h,2M}$ takes the form

$$\Big(\frac{h\sqrt{\mathcal{D}}}{2}\Big)^N (\pi\mathcal{D})^{-n/2} \sum_{|\beta|=2M} \sum_{m\in\mathrm{R}^n} U_\beta(x, hm)$$

$$\times \mathrm{e}^{-|x-hm|^2/(h^2\mathcal{D})} \prod_{\beta_j>0} \frac{2}{(\beta_j-1)!} \int_0^{z_j} H_{\beta_j-1}(t)\, dt\,,$$

where we use the notation $z_j = (x_j - hm_j)/(h\sqrt{\mathcal{D}})$. Then Cramer's inequality for

Hermite polynomials

$$|H_k(x)| \le 2^{k/2}\sqrt{k!}\,\mathrm{e}^{\,x^2/2} \tag{34}$$

(see [2, 15]), leads to the estimate

$$\begin{aligned}
|\mathcal{R}_{h,2M}(x)| &\le \Big(\frac{h\sqrt{\mathcal{D}}}{2}\Big)^N (\pi\mathcal{D})^{-n/2} \\
&\qquad \times \sum_{|\beta|=2M}\sum_{m\in\mathrm{R}^n} |U_\beta(x,hm)|\,\mathrm{e}^{-|x-hm|^2/(h^2\mathcal{D})} \prod_{\beta_j>0} \frac{2^{(\beta_j+1)/2}}{\sqrt{(\beta_j-1)!}} \int\limits_0^{|z_j|} \mathrm{e}^{\,t^2/2}\,dt \\
&\le \Big(\frac{h\sqrt{\mathcal{D}}}{2}\Big)^N (\pi\mathcal{D})^{-n/2} \sum_{|\beta|=2M} C_\beta \sum_{m\in\mathrm{R}^n} |U_\beta(x,hm)|\,S_\beta(x-hm)\,.
\end{aligned} \tag{35}$$

For the last inequality we use the notations

$$S_\beta(x-hm) = \prod_{\beta_j=0} \mathrm{e}^{-(x_j-hm_j)^2/(\mathcal{D}h^2)} \prod_{\beta_j>0} \frac{|x_j-hm_j|}{h\sqrt{2\mathcal{D}}} \mathrm{e}^{\,-(x_j-hm_j)^2/(2h^2\mathcal{D})}\,,$$

$$C_\beta = \prod_{\beta_j>0} \frac{2^{\beta_j+1/2}}{\sqrt{(\beta_j-1)!}} = 2^{2M} \prod_{\beta_j>0} \sqrt{\frac{2}{(\beta_j-1)!}}$$

and the estimate

$$\mathrm{e}^{\,-z_j^2} \int\limits_0^{|z_j|} \mathrm{e}^{\,t^2/2}\,dt \le |z_j|\mathrm{e}^{\,-z_j^2/2}\,.$$

By (9) we obtain for harmonic u

$$\begin{aligned}
|U_\beta(x,hm)| &= \Big| \int\limits_0^1 s^{N-1}\partial^\beta u(x+(1-s)(x-hm))\,ds \Big| \\
&\le \sum_{|\alpha|=0}^\infty \frac{\big|\partial^{\beta+\alpha}u(x)\big|}{\alpha!} \big|(x-hm)^\alpha\big| N \int\limits_0^1 s^{N-1}(1-s)^{|\alpha|}ds\,,
\end{aligned}$$

which shows that

$$\sum_{m\in\mathrm{R}^n} |U_\beta(x,hm)|\,S_\beta(x-hm) \le \sum_{|\alpha|=0}^\infty \frac{|\alpha|!\,N!}{(N+|\alpha|)!} \frac{\big|\partial^{\beta+\alpha}u(x)\big|}{\alpha!} \sum_{m\in\mathrm{R}^n} S_\beta(x-hm)\big|(x-hm)^\alpha\big|.$$

To get an upper bound of the last sum we write

$$\sum_{m\in\mathbb{R}^n} S_\beta(x-m)\big|(x-m)^\alpha\big|$$
$$=\prod_{\beta_j=0}\sum_{m_j\in\mathbb{R}}|x_j-m_j|^{\alpha_j}\mathrm{e}^{-(x_j-m_j)^2/\mathcal{D}}\prod_{\beta_j>0}\sum_{m_j\in\mathbb{R}}\frac{|x_j-m_j|^{\alpha_j+1}}{\sqrt{2\mathcal{D}}}\mathrm{e}^{-(x_j-m_j)^2/(2\mathcal{D})},$$

and note that for $\mathcal{D}>\mathcal{D}_0$

$$\sum_{m_j\in\mathbb{R}}|x_j-m_j|^{\alpha_j}\mathrm{e}^{-(x_j-m_j)^2/\mathcal{D}}\le c\mathcal{D}^{(\alpha_j+1)/2}\int_0^\infty y^{\alpha_j}\mathrm{e}^{-y^2}dy=\frac{c\mathcal{D}^{(\alpha_j+1)/2}}{2}\Gamma\Big(\frac{\alpha_j+1}{2}\Big),$$
$$\sum_{m_j\in\mathbb{R}}\frac{|x_j-m_j|^{\alpha_j+1}}{\sqrt{2\mathcal{D}}}\mathrm{e}^{-(x_j-m_j)^2/(2\mathcal{D})}\le\frac{c\,(2\mathcal{D})^{(\alpha_j+1)/2}}{2}\Gamma\Big(\frac{\alpha_j+2}{2}\Big).$$

Hence we obtain the estimate

$$\sum_{m\in\mathbb{R}^n} S_\beta(x-m)\big|(x-m)^\alpha\big|\le\frac{c\,\mathcal{D}^{(|\alpha|+n)/2}}{2^n}\prod_{\beta_j=0}\Gamma\Big(\frac{\alpha_j+1}{2}\Big)\prod_{\beta_j>0}2^{(\alpha_j+1)/2}\,\Gamma\Big(\frac{\alpha_j+2}{2}\Big)$$

with a constant c independent of α, β, and $\mathcal{D}$. Now we use that for $\alpha_j\ge 2$

$$\Gamma\Big(\frac{\alpha_j+1}{2}\Big)\le\frac{\sqrt[4]{\pi}}{2^{\alpha_j/2}}\sqrt{\alpha_j!}\,,$$

which leads to

$$\sum_{m\in\mathbb{R}^n} S_\beta(x-m)\big|(x-m)^\alpha\big|\le c_1\,\mathcal{D}^{(|\alpha|+n)/2}\prod_{\beta_j=0}\frac{\sqrt{\alpha_j!}}{2^{\alpha_j/2}}\prod_{\beta_j>0}\sqrt{(\alpha_j+1)!}$$

with another constant c_1. Thus we derive from (35)

$$|\mathcal{R}_{h,2M}(x)|$$
$$\le C(h\sqrt{\mathcal{D}})^N\sum_{|\beta|=2M}\sum_{|\alpha|=0}^\infty\frac{|\alpha|!\,|\beta|!\,\mathcal{D}^{(|\alpha|+n)/2}}{|\beta+\alpha|!}\frac{\big|\partial^{\beta+\alpha}u(x)\big|}{\alpha!}\prod_{\beta_j=0}\frac{\sqrt{\alpha_j!}}{2^{\alpha_j/2}}\prod_{\beta_j>0}\frac{\sqrt{2(\alpha_j+1)!}}{\sqrt{(\beta_j-1)!}}$$
$$=C(h\sqrt{\mathcal{D}})^N\sum_{|\beta|=2M}\sum_{|\alpha|=0}^\infty\frac{|\alpha|!\,|\beta|!\,\mathcal{D}^{(|\alpha|+n)/2}}{|\beta+\alpha|!}\frac{\big|\partial^{\beta+\alpha}u(x)\big|}{\sqrt{\alpha!\,\beta!}}\prod_{\beta_j=0}2^{-\alpha_j/2}\prod_{\beta_j>0}\sqrt{2(\alpha_j+1)\beta_j}$$

with a constant C not depending on M, $\mathcal{D}$, h, and u. Since

$$\frac{|\alpha|!\,|\beta|!}{|\beta+\alpha|!}\sqrt{\frac{(\alpha+\beta)!}{\alpha!\,\beta!}}\le\frac{|\alpha|!\,|\beta|!}{|\beta+\alpha|!}\frac{(\alpha+\beta)!}{\alpha!\,\beta!}\le 1\,,$$

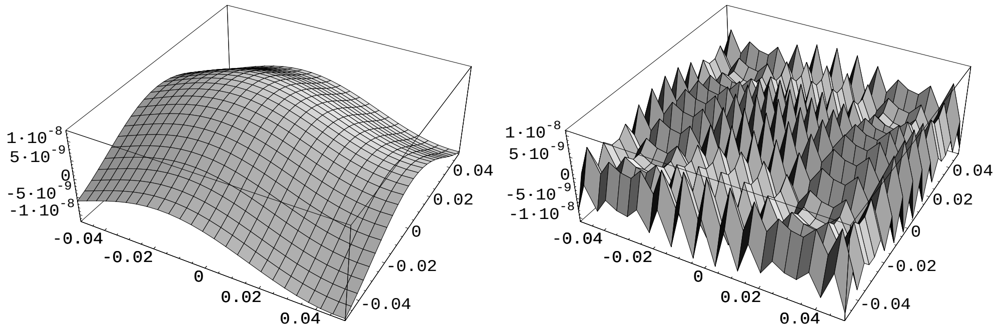

Figure 5. The graphs of $Mu - u$ with $u(x_1, x_2) = \mathrm{e}^{x_1} \cos x_2$, $\mathcal{D} = 2$, $h = 2^{-3}$ (on the left) and $h = 2^{-7}$ (on the right).

the remainder can be estimated by

$$|\mathcal{R}_{h,2M}(x)| \le C(h\sqrt{\mathcal{D}})^N \sum_{|\beta|=2M} \sum_{|\alpha|=0}^{\infty} \frac{\left|\partial^{\beta+\alpha} u(x)\right|}{\sqrt{(\alpha+\beta)!}} \mathcal{D}^{(|\alpha|+n)/2} \prod_{\beta_j > 0} \sqrt{2(\alpha_j+1)\beta_j}\,.$$

Thus, $|\mathcal{R}_{h,2M}(x)| \to 0$ for any fixed $\mathcal{D}$ if (33) holds. □

Remark 1 : *The assertion of Theorem 5.1 is a concrete realization of a general approximation result for analytic functions. Let u be an entire function in C^n of order less than 2. Theorem 7.1 in [7] states that the semi-discrete convolution*

$$u_h(x) = \sum_{m \in \mathrm{Z}^n} u_m \, \mathrm{e}^{-|x-hm|^2/(h^2\mathcal{D})}$$

with coefficients

$$u_m := \int_{\mathrm{R}^n} \mathrm{e}^{-\pi^2\mathcal{D}|y|^2} \, u(hm + i\pi\mathcal{D}hy)\, dy \tag{36}$$

differs from u by

$$u_h(x) - u(x) = \sum_{m \in \mathrm{Z}^n \setminus \{0\}} \tilde{u}(x + i\pi h\mathcal{D}m) \mathrm{e}^{-\pi^2\mathcal{D}|m|^2} \mathrm{e}^{2\pi i \langle m,x\rangle/h}\,,$$

(cf. also [8, Lemma 2.1]). It can be easily seen from (28) and (36) that the coefficients $u_m = (\pi\mathcal{D})^{-n/2} u(hm)$ if the restriction of u to R^n is harmonic.

We have applied the simple quasi-interpolant (27) to the harmonic function $u(x_1, x_2) = \mathrm{e}^{x_1} \cos x_2$ in R^2 by assuming $\mathcal{D} = 2$ (see Figure 5), $\mathcal{D} = 3$ (Figure 6), $\mathcal{D} = 4$ (Figure 7), $h = 2^{-3}$ and 2^{-7}. The experiments confirm that the quasi-interpolation error $Mu - u$ has reached its saturation bound also for large h because it does not decrease if h becomes smaller.

Let now u be harmonic in some convex domain $\Omega \subset \mathrm{R}^n$ and we consider the approximant

$$Mu(x) = (\pi\mathcal{D})^{-n/2} \sum_{hm \in \Omega} u(hm)\, \mathrm{e}^{-|x-hm|^2/(h^2\mathcal{D})}\,. \tag{37}$$

Theorem 5.2 : *Suppose that the function u is harmonic in a convex domain*

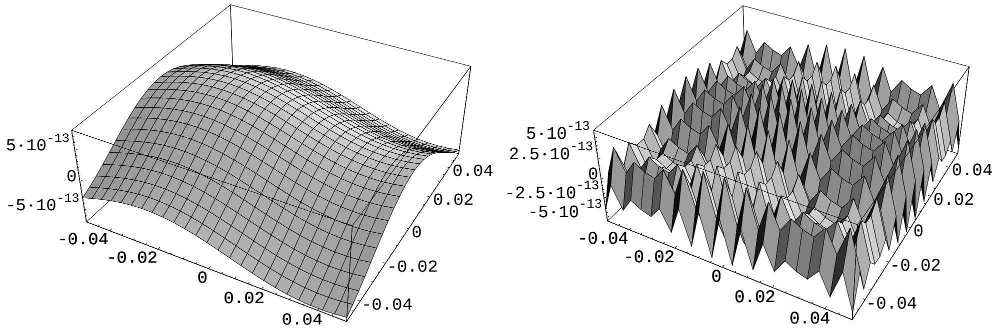

Figure 6. The graphs of $Mu - u$ with $u(x_1, x_2) = \mathrm{e}^{x_1} \cos x_2$, $\mathcal{D} = 3$, $h = 2^{-3}$ (on the left) and $h = 2^{-7}$ (on the right).

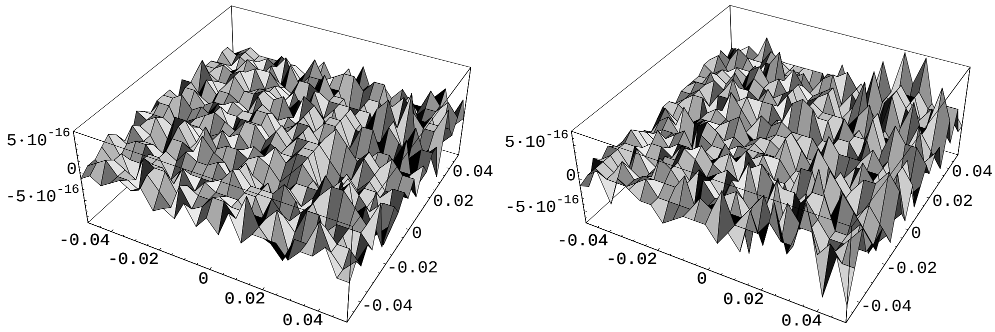

Figure 7. The graphs of $Mu - u$ with $u(x_1, x_2) = \mathrm{e}^{x_1} \cos x_2$, $\mathcal{D} = 4$, $h = 2^{-3}$ (on the left) and $h = 2^{-7}$ (on the right).

$\Omega \subset \mathrm{R}^n$ and satisfies for a given $N = 2M$

$$c_u = \sum_{|\beta|=2M} \|\partial^\beta u\|_{L_\infty(\Omega)} \prod_{\beta_j > 0} \sqrt{\frac{2}{(\beta_j - 1)!}} < \infty \,.$$

Then for any $\varepsilon > 0$ and subdomain $\Omega' \subsetneq \Omega$ there exists $\mathcal{D} > 0$ and $h > 0$ such that the quasi-interpolant (37) provides for all $x \in \Omega'$ the estimate

$$|u(x) - Mu(x)| \le C(h\sqrt{\mathcal{D}})^N c_u + \varepsilon \sum_{|\beta|=0}^{N-1} (h\sqrt{\mathcal{D}})^{|\beta|} |\partial^\beta u(x)| \,, \tag{38}$$

where the constant C depends only on the space dimension.

Proof: Analogously to the case $\Omega = \mathrm{R}^n$ we obtain

$$|u(x) - Mu(x)| \le |\mathcal{R}_{h,2M}(x)| + |\mathcal{E}_{h,2M}(x)|$$

with

$$|\mathcal{E}_{h,2M}(x)| \le \; |u(x)| \Big| (\pi\mathcal{D})^{-n/2} \sum_{hm \in \Omega} \mathrm{e}^{-|x-hm|^2/(h^2\mathcal{D})} - 1 \Big|$$

$$+ (\pi\mathcal{D})^{-n/2} \sum_{|\beta|=1}^{2M-1} \Big(\frac{h\sqrt{\mathcal{D}}}{2} \Big)^{|\beta|} \frac{|\partial^\beta u(x)|}{\beta!} \Big| \sum_{hm \in \Omega} H_\beta \Big(\frac{x - hm}{h\sqrt{\mathcal{D}}} \Big) \, \mathrm{e}^{-|x-hm|^2/(h^2\mathcal{D})} \Big| .$$

and

$$|\mathcal{R}_{h,2M}(x)| \le \Big(\frac{h\sqrt{\mathcal{D}}}{2}\Big)^N (\pi\mathcal{D})^{-n/2} \sum_{|\beta|=2M} C_\beta \sum_{hm\in\Omega} |U_\beta(x,hm)|\, S_\beta(x-hm)\,,$$

see (35). Since

$$|U_\beta(x,hm)| \le \|\partial^\beta u\|_{L_\infty(\Omega)} \quad \text{and} \quad \sum_{hm\in\Omega} S_\beta(x-hm) \le c\,\mathcal{D}^{n/2}$$

with a constant c depending only on n, we get the inequality

$$|\mathcal{R}_{h,2M}(x)| \le C\,(h\sqrt{\mathcal{D}})^N c_u\,.$$

To estimate $|\mathcal{E}_{h,2M}(x)|$ we use the functions σ_β given by (31) and write

$$(\pi\mathcal{D})^{-n/2} \sum_{hm\in\Omega} \mathrm{e}^{-|x-hm|^2/(h^2\mathcal{D})} - 1 = \sigma_0\Big(\frac{x}{h},\mathcal{D}\Big) - (\pi\mathcal{D})^{-n/2} \sum_{hm\notin\Omega} \mathrm{e}^{-|x-hm|^2/(h^2\mathcal{D})}\,,$$

$$\begin{aligned}(\pi\mathcal{D})^{-n/2} \sum_{hm\in\Omega} H_\beta\Big(\frac{x-hm}{h\sqrt{\mathcal{D}}}\Big)\, \mathrm{e}^{-|x-hm|^2/(h^2\mathcal{D})} \\ = \sigma_\beta\Big(\frac{x}{h},\mathcal{D}\Big) - (\pi\mathcal{D})^{-n/2} \sum_{hm\notin\Omega} H_\beta\Big(\frac{x-hm}{h\sqrt{\mathcal{D}}}\Big)\, \mathrm{e}^{-|x-hm|^2/(h^2\mathcal{D})}\,.\end{aligned}$$

Furthermore, for $x\in\Omega$ we derive

$$\Big|(\pi\mathcal{D})^{-n/2} \sum_{hm\notin\Omega} H_\beta\Big(\frac{x-hm}{h\sqrt{\mathcal{D}}}\Big)\, \mathrm{e}^{-|x-hm|^2/(h^2\mathcal{D})}\Big| \le \delta_\beta(h^{-1}\,\mathrm{dist}(x,\partial\Omega),\mathcal{D})\,,$$

where $\delta_\beta(r,\mathcal{D})$, $r\ge 0$, denotes the rapidly decaying function

$$\delta_\beta(r,\mathcal{D}) = \sup_{x\in\mathbb{R}^n} (\pi\mathcal{D})^{-n/2} \sum_{\substack{m\in\mathbb{Z}^n \\ |x-m|>r}} \Big|H_\beta\Big(\frac{x-m}{\sqrt{\mathcal{D}}}\Big)\Big|\, \mathrm{e}^{-|x-m|^2/\mathcal{D}}\,.$$

Thus, for any domain Ω, fixed parameter h and multiindex β we can find a subdomain $\Omega'_{\beta,h} \subsetneq \Omega$ such that

$$\sup_{x\in\Omega'_{\beta,h}} \delta_\beta(h^{-1}\,\mathrm{dist}(x,\partial\Omega),\mathcal{D}) \le \|\sigma_\beta(\cdot,\mathcal{D})\|_{L_\infty}\,, \tag{39}$$

which gives

$$|\mathcal{E}_{h,2M}(x)| \le 2\sum_{|\beta|=0}^{2M-1} \Big(\frac{h\sqrt{\mathcal{D}}}{2}\Big)^{|\beta|} \frac{|\partial^\beta u(x)|}{\beta!} \|\sigma_\beta(\cdot,\mathcal{D})\|_{L_\infty} \quad \text{for all} \quad x\in \bigcap_{|\beta|=0}^{2M-1} \Omega'_{\beta,h}\,.$$

Now we have to choose h such that $\Omega' \subset \bigcap \Omega'_{\beta,h}$, which is possible since $\Omega'_{\beta,h}\to\Omega$ as $h\to 0$. $\square$

Remark 2: *Obviously the assertion of Theorems* 5.1 *and* 5.2 *can be extended to the case that a solution* u *of the second order equation*

$$\sum_{i,k=1}^{n} b_{ik}\partial_i\partial_k u = 0$$

in Ω *is approximated by the quasi-interpolant*

$$Mu(x) = \frac{(\det B)^{-1/2}}{(\pi\mathcal{D})^{n/2}} \sum_{hm\in\Omega} u(hm)\,\mathrm{e}^{-\langle B^{-1}(x-hm),x-hm\rangle/(h^2\mathcal{D})}$$

with the matrix $B = \{b_{ik}\}$.

6. Approximation of derivatives

Here we study the approximation of derivatives using Hermite quasi-interpolation operator (5). We introduce the continuous convolution (see [11])

$$C_\delta v(x) = \delta^{-n}\int_{\mathrm{R}^n} \mathcal{H}\left(\frac{x-y}{\delta}\right)\mathcal{Q}(-\delta\,\partial)v(y)dy \tag{40}$$

where $\mathcal{Q}(t)$ is the polynomial in (6).

Theorem 6.1: *Suppose that* $\mathcal{H}$ *satisfies the decay condition* (13) *with* $K > L+n$, $L \in \mathrm{N}$, $L \geq N$. *For any* $\varepsilon > 0$ *there exists* $\mathcal{D} > 0$ *such that for any function* $u \in W^L_\infty(\mathrm{R}^n)$

$$|Mu(x) - C_{h\sqrt{\mathcal{D}}}\,u(x)| \leq \varepsilon \sum_{|\gamma|=0}^{L-1} (h\sqrt{\mathcal{D}})^{|\gamma|}|\partial^\gamma u(x)| + c_1\,(h\sqrt{\mathcal{D}})^L \sum_{|\gamma|=L} ||\partial^\gamma u||_{L_\infty}\,, \tag{41}$$

where the constant c_1 *does not depend on* u, h *and* $\sqrt{\mathcal{D}}$.

Proof: Suppose that the function $u \in W^L_\infty(\mathrm{R}^n)$. The Taylor expansion (8) with N replaced by L gives the following form of the quasi-interpolant Mu in (7)

$$\begin{aligned} Mu(x) &= \sum_{|\gamma|=0}^{N-1} a_\gamma \sum_{|\alpha|=0}^{L-1-|\gamma|} \frac{(-h\sqrt{\mathcal{D}})^{|\alpha+\gamma|}}{\alpha!}\partial^{\alpha+\gamma}u(x)\sigma_\alpha(\frac{x}{h},\mathcal{D},\mathcal{H}) \\ &+ (-h\sqrt{\mathcal{D}})^L \sum_{|\gamma|=0}^{N-1} a_\gamma \sum_{|\alpha|=L-|\gamma|} \frac{\mathcal{D}^{-n/2}}{\alpha!} \sum_{m\in\mathrm{Z}^n} U_{\alpha+\gamma}(x,hm)\left(\frac{x-hm}{h\sqrt{\mathcal{D}}}\right)^\alpha \mathcal{H}\left(\frac{x-hm}{h\sqrt{\mathcal{D}}}\right). \end{aligned}$$

Similarly the Taylor expansion of u around y leads to

$$\begin{aligned} C_\delta u(x) &= \sum_{|\gamma|=0}^{N-1} a_\gamma \sum_{|\alpha|=0}^{L-1-|\gamma|} (-\delta)^{|\alpha+\gamma|}\partial^{\alpha+\gamma}u(x)\frac{1}{\alpha!}\int_{\mathrm{R}^n}\tau^\alpha\mathcal{H}(\tau)d\tau \\ &+ (-\delta)^L \sum_{|\gamma|=0}^{N-1} a_\gamma \sum_{|\alpha|=L-|\gamma|} \frac{1}{\alpha!}\int_{\mathrm{R}^n}\tau^\alpha\mathcal{H}(\tau)U_{\alpha+\gamma}(x,x-\tau\delta)d\tau. \end{aligned} \tag{42}$$

Setting $\delta = h\sqrt{\mathcal{D}}$, we obtain

$$Mu(x) - C_{h\sqrt{\mathcal{D}}}u(x) = \sum_{|\gamma|=0}^{N-1} a_\gamma \sum_{|\alpha|=0}^{L-1-|\gamma|} \frac{(-h\sqrt{\mathcal{D}})^{|\alpha+\gamma|}}{\alpha!} \partial^{\alpha+\gamma}u(x)\mathcal{E}_\alpha(\frac{x}{h}, \mathcal{D}, \mathcal{H})$$

$$+ (-h\sqrt{\mathcal{D}})^L \sum_{|\gamma|=0}^{N-1} a_\gamma \sum_{|\alpha|=L-|\gamma|} \frac{1}{\alpha!}[\mathcal{D}^{-n/2} \sum_{m\in \mathrm{Z}^n} U_{\alpha+\gamma}(x, hm) \left(\frac{x-hm}{h\sqrt{\mathcal{D}}}\right)^\alpha \mathcal{H}\left(\frac{x-hm}{h\sqrt{\mathcal{D}}}\right)$$

$$- \int\limits_{\mathrm{R}^n} \tau^\alpha \mathcal{H}(\tau) U_{\alpha+\gamma}(x, x-\tau\delta) d\tau].$$

Then

$$|Mu(x) - C_{h\sqrt{\mathcal{D}}}u(x)| \le \sum_{|\gamma|=0}^{N-1} |a_\gamma| \sum_{|\alpha|=0}^{L-1-|\gamma|} \frac{(h\sqrt{\mathcal{D}})^{|\alpha+\gamma|}}{\alpha!} |\partial^{\alpha+\gamma}u(x)|\, |\mathcal{E}_\alpha(\frac{x}{h}, \mathcal{D}, \mathcal{H})|$$

$$+ (h\sqrt{\mathcal{D}})^L \sum_{|\gamma|=0}^{N-1} |a_\gamma| \sum_{|\alpha|=L-|\gamma|} \frac{||\partial^{\alpha+\gamma}u||_{L_\infty}}{\alpha!} \left(||\rho_\alpha(\cdot, \mathcal{D}, \mathcal{H})||_{L_\infty} + \int_{\mathrm{R}^n} |\tau^\alpha \mathcal{H}(\tau)| d\tau \right).$$

Proceeding as in the proof of Theorem 2.1 we deduce (41). □

Theorem 6.2: *If $\mathcal{H}$ satisfies the conditions* (13) *with $K > N+n$ and* (14)*, then for any $v \in W_\infty^N(\mathrm{R}^n)$*

$$|C_\delta v(x) - v(x)| \le c_2 \delta^N \sum_{|\alpha|=N} \|\partial^\alpha v\|_{L_\infty}. \tag{43}$$

Proof: The representation (42) with $L = N$ gives

$$C_\delta v(x) = \sum_{|\alpha|=0}^{N-1} (-\delta)^{|\alpha|} \partial^\alpha v(x) \sum_{\gamma\le\alpha} \frac{a_{\alpha-\gamma}}{\gamma!} \int_{\mathrm{R}^n} \tau^\gamma \mathcal{H}(\tau) d\tau + R_{\delta,\mathcal{H}}(x) \tag{44}$$

where the remainder $R_{\delta,\mathcal{H}}$ satisfies

$$|R_{\delta,\mathcal{H}}(x)| \le \delta^N \sum_{|\alpha|=N} \|\partial^\alpha v\|_{L_\infty} \sum_{\gamma\le\alpha} \frac{|a_{\alpha-\gamma}|}{\gamma!} \int_{\mathrm{R}^n} |\tau^\gamma \mathcal{H}(\tau)| d\tau \le c_2\delta^N \sum_{|\alpha|=N} \|\partial^\alpha v\|_{L_\infty}. \tag{45}$$

If condition (14) holds, in view of (44), we obtain (43). □

Theorem 6.2 leads immediately to the next corollary.

Corollary 6.3: *Suppose that $\mathcal{H}$ satisfies conditions* (13) *with $K > N+n$ and* (14)*. Then for u such that $\partial^\beta u \in W_\infty^N(\mathrm{R}^n)$,*

$$|C_\delta \partial^\beta u(x) - \partial^\beta u(x)| \le c_2 \delta^N \sum_{|\alpha|=N} \|\partial^{\alpha+\beta} u\|_{L_\infty}. \tag{46}$$

If the derivative $\partial^\beta \mathcal{H}$ exists and satisfies the decay condition (13) then the convolution satisfies

$$\partial^\beta C_\delta\, u(x) = (h\sqrt{\mathcal{D}})^{-|\beta|} C_{\delta,\partial^\beta\mathcal{H}}\, u(x) = C_\delta \partial^\beta u(x) \tag{47}$$

where

$$C_{\delta,\partial^\beta\mathcal{H}} u(x) = \delta^{-n} \int_{\mathrm{R}^n} \partial^\beta \mathcal{H}\left(\frac{x-y}{\delta}\right) \mathcal{Q}(-\delta\,\partial) u(y) dy$$

and the quasi-interpolant Mu in (7) satisfies the equation

$$\partial^\beta M u = (h\sqrt{\mathcal{D}})^{-|\beta|} M_{\partial^\beta\mathcal{H}} u \tag{48}$$

with

$$M_{\partial^\beta\mathcal{H}} u = \mathcal{D}^{-n/2} \sum_{m\in\mathrm{Z}^n} \partial^\beta\mathcal{H}\left(\frac{x-hm}{h\sqrt{\mathcal{D}}}\right) \mathcal{Q}\left(-h\sqrt{\mathcal{D}}\,\partial\right) u(hm).$$

Hence keeping in mind (47) and (48) we write the difference

$$\begin{aligned}
&(h\sqrt{\mathcal{D}})^{|\beta|}[\partial^\beta M u(x) - \partial^\beta u(x)] \\
&\quad = (h\sqrt{\mathcal{D}})^{|\beta|}[\partial^\beta M u(x) - \partial^\beta C_\delta u(x)] + (h\sqrt{\mathcal{D}})^{|\beta|}[\partial^\beta C_\delta u(x) - \partial^\beta u(x)] \\
&\quad = [M_{\partial^\beta\mathcal{H}} u(x) - C_{\delta,\partial^\beta\mathcal{H}} u(x)] + (h\sqrt{\mathcal{D}})^{|\beta|}[C_\delta \partial^\beta u(x) - \partial^\beta u(x)].
\end{aligned}$$

From (46) and (41) we obtain the following result.

Theorem 6.4: *Suppose that $\mathcal{H}$ satisfies the conditions* (13) *with $K > N + n$ and* (14). *Moreover suppose that the derivative $\partial^\beta\mathcal{H}$ exists and satisfies the conditions* (13) *with $K > L + n$. The for any $\varepsilon > 0$ there exits $\mathcal{D} > 0$ such that, for any $u \in W^L_\infty(\mathrm{R}^n)$ with $L \geq N + |\beta|$,*

$$\begin{aligned}
|\partial^\beta M u(x) - \partial^\beta u(x)| &\leq \varepsilon \sum_{|\gamma|=0}^{L-1} (h\sqrt{\mathcal{D}})^{|\gamma|-|\beta|} |\partial^\gamma u(x)| \\
&\quad + c_1\,(h\sqrt{\mathcal{D}})^{L-|\beta|} \sum_{|\gamma|=L} ||\partial^\gamma u||_{L_\infty} + c_2 (h\sqrt{\mathcal{D}})^N \sum_{|\alpha|=N} ||\partial^{\alpha+\beta} u||_{L_\infty}.
\end{aligned}$$

We deduce that formula (5) gives the simultaneous approximation of the derivatives $\partial^\beta u$ with the saturation term $\varepsilon h^{-|\beta|}$.

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