Brezis-Gallouet-Wainger type inequality for irregular domains

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Dedicated to Viktor Burenkov on the occasion of his 70th birthday

Abstract. A Brezis-Gallouet-Wainger logarithmic interpolation-embedding inequality is proved for various classes of irregular domains, in particular, for power cusps and λ-John domains.

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1 Introduction

Brezis and Gallouet [BG] established the unique solvability of the initial-boundary value problem for the nonlinear Schrödinger evolution equations with zero Dirichlet data on the boundary of a bounded domain in \( \mathbb{R}^2 \) or its complementary domain. They used crucially the following interpolation-embedding inequality

\[ \|u\|_{L^\infty(\Omega)} \leq C \left( 1 + \left( \log(1 + \|u\|_{W^{2,2}(\Omega)}) \right)^{1/2} \right) \]  

(1.1)

for every \( u \in W^{2,2}(\Omega) \) with \( \|u\|_{W^{1,2}(\Omega)} = 1 \). Applications of this inequality to the Euler equation can be found in Chapter 13 of M.E. Taylor’s book [Ta].

Brezis and Wainger [BW] extended (1.1) to Sobolev spaces of higher order on \( \mathbb{R}^n \) in the form

\[ \|u\|_{L^\infty(\mathbb{R}^n)} \leq C \left( 1 + \left( \log(1 + \|u\|_{W^{l,q}(\mathbb{R}^n)}) \right)^{(n-k)/n} \right) \]  

(1.2)

for every function \( u \) in \( W^{l,q}(\mathbb{R}^n) \) normalized by

\[ \|u\|_{W^{k,n/k}(\mathbb{R}^n)} = 1, \]
where \( k \) and \( l \) are integers, \( 1 \leq k < l, \) \( ql > n, \) and \( k \leq n. \)

Inequalities (1.1) and (1.2) were studied in different directions in [En], [Go], [Sa].

We prove some inequalities of Brezis-Gallouet-Wainger type for various classes of irregular domains. In particular, classes \( J_\alpha \) introduced in [Ma1], power cusps and \( \lambda \)-John domains are considered.

## 2 Preliminaries

Let \( \Omega \) be an open domain in \( \mathbb{R}^n \) such that \( m_n(\Omega) < \infty, \) where \( m_n \) denotes the \( n \)-dimensional Lebesgue measure on \( \Omega. \) Given \( p \in [1, \infty) \) and sets \( E \subset G \subset \Omega, \) the capacity \( C_p(E, G) \) of the condenser \( (E, G) \) is defined as

\[
C_p(E, G) = \inf \left\{ \int_{\Omega} |\nabla u|^p dx : u \in L^{1,p}(\Omega), \ u > 1 \text{ in } E \text{ and } u \leq 0 \text{ in } \Omega \setminus G \right\}. \tag{2.1}
\]

Here \( L^{1,p}(\Omega) \) denotes the Sobolev space defined as

\[
L^{1,p}(\Omega) = \{ u \in L^p(\Omega, \text{loc}) : u \text{ is weakly differentiable in } \Omega \text{ and } |\nabla u| \in L^p(\Omega) \},
\]

endowed with the seminorm

\[
\|u\|_{L^{1,p}(\Omega)} = \|\nabla u\|_{L^p(\Omega)}.
\]

We need two isocapacitary functions. The first of them

\[
\nu_p : [0, m_n(\Omega)/2] \to [0, \infty]
\]

is defined by

\[
\nu_p(s) = \inf \{ C_p(E, G) : E \text{ and } G \text{ are measurable subsets of } \Omega \text{ such that } E \subset G \text{ and } s \leq m_n(E), m_n(G) \leq m_n(\Omega)/2 \} \quad \text{for } s \in [0, m_n(\Omega)/2]. \tag{2.2}
\]

Clearly, the function \( \nu_p \) is non-decreasing. The isocapacitary inequality on \( \Omega \) is a straightforward consequence of definition(2.2) and tells us that

\[
\nu_p(m_n(E)) \leq C_p(E, G) \tag{2.3}
\]

for every measurable sets \( E \subset G \subset \Omega \) with \( m_n(G) < m_n(\Omega)/2. \)

The second isocapacitory function \( \pi_p : [0, m_n(\Omega)/2] \to [0, \infty] \) is given by

\[
\pi_p(s) = \inf \{ C_p(E, G) : E \text{ is a point and } G \text{ is a measurable subset of } \Omega \text{ such that } E \in G \text{ and } m_n(G) \leq s \} \quad \text{for } s \in [0, m_n(\Omega)/2]. \tag{2.4}
\]

The function \( \pi_p \) is clearly non-increasing and the corresponding isocapacitary inequality on \( \Omega \) is

\[
\pi_p(m_n(G)) \leq C_p(E, G). \tag{2.5}
\]

Variants of these isocapacitary functions were introduced in Chaptrs 3 and 5 of [Ma2] and employed to provide necessary and sufficient conditions for embeddings in the Sobolev space of functions with gradient in \( L^p. \)
We need another function of purely geometric nature associated with $\Omega$. It is called the isoperimetric function of $\Omega$ and will be denoted by $\lambda$. The function $\lambda : [0, m_n(\Omega)/2] \to [0, \infty]$ is given by

$$\lambda(s) = \inf \{ P(E) : s \leq m_n(E) \leq m_n(\Omega)/2 \}. \quad (2.6)$$

Here, $P(E)$ is the De Giorgi perimeter of $E$ which can be defined as

$$P(E) = \mathcal{H}^{n-1}(\partial^* E),$$

where $\partial^* E$ stands for the essential boundary of $E$ in the sense of geometric measure theory and $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure. Recall that $\partial^* E$ agrees with the topological boundary $\partial E$ of $E$ when $E$ is regular enough, e.g. is an open subset of $\Omega$ with a smooth boundary.

The very definition of $\lambda$ leads to the isoperimetric inequality on $\Omega$ which reads

$$\lambda(m_n(E)) \leq P(E) \quad (2.7)$$

for every measurable set $E \subset \Omega$ with $m_n(E) \leq m_n(\Omega)/2$. The isoperimetric function of an open subset of $\mathbb{R}^n$ was introduced in [Ma1] (see also [Ma2]) to characterize Sobolev embeddings for functions with gradient in $L^1$. In more recent years, isoperimetric inequalities and corresponding isoperimetric functions have been intensively investigated on Riemannian manifolds as well, see e.g. [BC, CF, CGL, GP, Gr, MHH, Kle, MJ, Pi, Ri].

The functions $\nu_p$ and $\pi_p$ on one hand, and $\lambda$ on the other hand, are related by the inequalities

$$\nu_p(s) \geq \left( \int_s^{m_n(\Omega)/2} \frac{dr}{\lambda(r)^{p'}} \right)^{1-p} \quad (2.8)$$

and

$$\pi_p(s) \geq \left( \int_0^s \frac{dr}{\lambda(r)^{p'}} \right)^{1-p} \quad (2.9)$$

for $s \in (0, m_n(\Omega)/2)$ with $p' = p/(p-1)$, which follows along the same lines as in [Ma2], Proposition 4.3.4.1. It is shown in [Ma2], Lemma 3.2.4, that $\lambda(s) > 0$ for $s > 0$. Owing to inequalities (2.8) and (2.9), we have $\nu_p(s) > 0$ and $\pi_p(s) > 0$ for $s \in (0, m_n(\Omega)/2)$ as well.

Clearly,

$$\lambda(s) \leq c_n s^{(n-1)/n}. \quad (2.10)$$

Moreover, if $\Omega$ is bounded and Lipschitz, then

$$\lambda(s) \approx s^{(n-1)/n} \quad \text{near } s = 0. \quad (2.11)$$

Here and in what follows, the notation

$$f \approx g \quad \text{near } 0 \quad (2.12)$$

for functions $f, g : (0, \infty) \to [0, \infty)$ means that there exist positive constants $c_1, c_2$, and $s_0$ such that

$$c_1 g(c_1 s) \leq f(s) \leq c_2 g(c_2 s) \quad \text{if } s \in (0, s_0). \quad (2.13)$$

Furthermore, for any $\Omega$,

$$\nu_p(s) \leq c_{n,p} s^{(n-p)/n} \quad \text{for } n > p \quad (2.14)$$
and
\[ \nu_p(s) \leq c_n \left( \log \frac{1}{s} \right)^{n-1} \quad \text{for } n = p. \]
(2.15)

Both inequalities can be verified by setting appropriate test functions in the above definition of the \( p \)-capacity.

If \( \Omega \) is bounded and Lipschitz, then, by (2.11) and (2.8),
\[ \nu_p(s) \approx \begin{cases} \frac{s^{(n-p)/n}}{(\log \frac{1}{s})^{n-1}} & \text{if } n > p, \\ \frac{1}{s^{n-p}} & \text{if } n = p \end{cases} \]
(2.16)

near \( s = 0 \).

Similarly to (2.14) and (2.15), we obtain the equality
\[ \pi_p(s) = 0 \quad \text{for } n \geq p. \]
(2.17)

Finally, if \( \Omega \) is bounded and Lipschitz, then
\[ \pi_p(s) \approx \frac{s^{(n-p)/n}}{(\log \frac{1}{s})^{n-1}} \quad \text{for } p > n. \]
(2.18)

In what follows we use the notations
\[ G_\sigma = \{ x \in \Omega : u(x) > \sigma \}, \]
\[ L_\sigma = \{ x \in \Omega : u(x) < \sigma \}, \]
\[ E_\sigma = \{ x \in \Omega : u(x) = \sigma \}. \]

3 Main results for arbitrary \( \Omega \)

**Theorem 1** For every \( \varepsilon \in (0, m_n(\Omega)/2) \) and for all \( u \in L^{1,p}(\Omega) \cap L^{1,r}(\Omega) \), \( p \geq 1 \), \( r \geq 1 \),
\[ \text{osc}_\Omega u \leq \pi_p(\varepsilon)^{-1/p} \left\| \nabla u \right\|_{L^p(\Omega)} + \nu_p(\varepsilon)^{-1/r} \left\| \nabla u \right\|_{L^r(\Omega)}, \]
(3.1)

where
\[ \text{osc}_\Omega u = \text{ess sup}_\Omega u - \text{ess inf}_\Omega u. \]

**Proof.** Let the numbers \( T \) and \( t \) be chosen so that
\[ m_n \{ x : u(x) > T \} \leq m_n(\Omega)/2 \leq m_n \{ x : u(x) \geq T \}, \]
\[ m_n \{ x : u(x) > t \} \leq \varepsilon \leq m_n \{ x : u(x) \geq t \}. \]

Furthermore, let
\[ S := \text{ess sup}_\Omega u. \]
Then
\[ S - t \leq \left( c_p(\mathcal{E}_S, \mathcal{G}_t) \right)^{-1/p} \left( \int_{\mathcal{G}_t} |\nabla u|^p dx \right)^{1/p} \]
and
\[ t - T \leq \left( c_r(\mathcal{G}_t, \mathcal{G}_T) \right)^{-1/r} \left( \int_{\mathcal{G}_T \setminus \mathcal{G}_t} |\nabla u|^r dx \right)^{1/r} \]
which implies
\[ \text{ess sup}_\Omega u - T \leq \pi_p(\varepsilon)^{-1/p} \left\| \nabla u \right\|_{L^p(\mathcal{G}_T)} + \nu_p(\varepsilon)^{-1/r} \left\| \nabla u \right\|_{L^r(\mathcal{G}_T)}. \]
An analogous estimate for \( T \, \text{ess inf} \, \Omega \) \( u \) is proved in the same way. Adding both estimates, we arrive at (3.1).

We obtain a direct consequence of Theorem 1 and the above lower estimates for the isocapacitary functions which is formulated in terms of the isoperimetric function \( \lambda \).

**Corollary 1** Let \( p \geq 1 \), \( r \geq 1 \). Then, for every \( \varepsilon \in (0, m_n(\Omega)/2) \) and for all \( u \in L^{1,p}(\Omega) \cap L^{1,r}(\Omega) \)

\[
\text{osc}_\Omega u \leq \left( \int_0^\varepsilon \frac{d\mu}{\lambda(\mu)^{p'}} \right)^{1/p'} \| \nabla u \|_{L^p(\Omega)} + \left( \int_\varepsilon^{m_n(\Omega)/2} \frac{d\mu}{\lambda(\mu)^{r'}} \right)^{1/r'} \| \nabla u \|_{L^r(\Omega)}. \tag{3.2}
\]

**Proof.** The result follows from the lower estimates (5.6) and (5.7) inserted into (3.1).

**Remark 1.** One can add the inequality \( p > n \) in Theorem 1 since (3.1) has no sense for \( n \geq p \) because of (2.17).

In the case \( r = 1 \) the estimate (3.2) is simplified as

\[
\text{osc}_\Omega u \leq \left( \int_0^\varepsilon \frac{d\mu}{\lambda(\mu)^{p'}} \right)^{1/p'} \| \nabla u \|_{L^p(\Omega)} + \lambda(\varepsilon)^{-1} \| \nabla u \|_{L^1(\Omega)}, \tag{3.3}
\]

where \( p > n \).

4 Domains of the class \( J_\alpha \)

We say that a domain belongs to the class \( J_\alpha, \alpha > 0 \), if there is a constant \( K_\alpha \) such that

\[
\lambda(\mu) \geq K_\alpha \mu^\alpha
\]

for \( \mu \in (0, m_n(\Omega)/2) \). This class was introduced in [Ma1] and studied in detail in [Ma2], [Ma3].

Corollary 1 can be made more visible for \( \Omega \in J_\alpha \).

**Corollary 2** Let \( p > n \), \( r \geq 1 \) and let

\[
\frac{1}{p'} > \alpha > \frac{1}{r'}.
\]

Then, for every \( \varepsilon \in (0, m_n(\Omega)/2) \) and for all \( u \in L^{1,p}(\Omega) \)

\[
\text{osc}_\Omega u \leq K_\alpha^{-1} \left( (1 - \alpha p')^{-1/p'} \varepsilon^{-\alpha + 1/p'} \| \nabla u \|_{L^p(\Omega)} + \alpha r'^{-1} \varepsilon^{-\alpha + 1/r'} \| \nabla u \|_{L^r(\Omega)} \right). \tag{4.1}
\]

Taking the minimum value of the right-hand side in \( \varepsilon \), we arrive at the following alternatives.

**Corollary 3** Let the conditions of Corollary 2 hold.

(i) If

\[
\| \nabla u \|_{L^r(\Omega)} \leq \frac{r'(1 - \alpha p')^{1/p}}{p'(\alpha r' - 1)^{1/r}} \left( \frac{m_n(\Omega)}{2} \right)^{1/r - 1/p} \| \nabla u \|_{L^p(\Omega)},
\]
then
\[ \text{osc}_\Omega u \leq C_{\alpha, p, r} K^{-1}_\alpha \| \nabla u \|_{L^p(\Omega)}^{1 - 1/p'} \| \nabla u \|_{L^r(\Omega)}^{\alpha - 1/p'} \]

with
\[ C_{\alpha, p, r} = (1 - \alpha p')(\alpha r' - 1)^{p - 1} \frac{\alpha - 1/p'}{\alpha + 1/p'} \times \left( (\frac{\alpha - 1/r'}{\alpha + 1/p'})^{\alpha - 1/p'} + (\frac{\alpha + 1/p'}{\alpha - 1/r'})^{\alpha - 1/p'} \right). \]

(ii) If
\[ \| \nabla u \|_{L^r(\Omega)} > \frac{r'(1 - \alpha p')^{1/p'} (m_n(\Omega))}{2} \big( \frac{1}{r'} - 1 \big) \| \nabla u \|_{L^p(\Omega)}, \]

then
\[ \text{osc}_\Omega u \leq K^{-1}_\alpha (1 - \alpha p')^{-1/p'} \big( \frac{m_n(\Omega)}{2} \big)^{-\alpha + 1/p'} \| \nabla u \|_{L^p(\Omega)} \]
\[ + (\alpha r' - 1)^{-1/\alpha} \big( \frac{m_n(\Omega)}{2} \big)^{-\alpha + 1/r'} \| \nabla u \|_{L^r(\Omega)}. \]

**Remark 2.** Without taking care of constant factors, we can deduce the following inequality of Gagliardo-Nirenberg type (cf. [Ga], [Ni]) from Corollary 3:
\[ \text{osc}_\Omega u \leq c_{\alpha, p, r} K^{-1}_\alpha \| \nabla u \|_{L^p(\Omega)}^{1 - 1/p'} \| \nabla u \|_{L^r(\Omega)} \]

for an arbitrary function \( u \) in \( L^{1,p}(\Omega) \), where \( \Omega \) is a domain of the class \( J_\alpha \) with \( 1/p' > \alpha > 1/r' \), \( p > n, r \geq 1 \), and
\[ \gamma = \frac{\alpha - 1/r'}{1/r' - 1/p} \].

5 **The case \( \Omega \in J_{1/r'} \)**

We turn to the critical case when \( \Omega \) belongs to the class \( J_{1/r'} \).

**Corollary 4** Let \( p > n, p > r \geq 1 \) and let \( \Omega \in J_{1/r'} \). Then, for every \( \varepsilon \in (0, m_n(\Omega)/2) \) and for all \( u \in L^{1,p}(\Omega) \),
\[ \text{osc}_\Omega u \leq K^{-1}_{1/r'} \left( \left( \frac{p - 1}{p - r} \right)^{1/p'} \varepsilon^{(p - r)/p} \| \nabla u \|_{L^p(\Omega)} \right) \]
\[ + \left( \log \frac{m_n(\Omega)}{2\varepsilon} \right)^{1/r} \| \nabla u \|_{L^r(\Omega)} \].

**Corollary 5** Let \( p > n, p > r \geq 1 \) and let \( \Omega \in J_{1/r'} \). Then, for all \( u \in L^{1,p}(\Omega) \),
\[ \text{osc}_\Omega u \leq 2^{1/r} K^{-1}_{1/r'} \min \left\{ \left( \frac{p r}{p - 1} \right)^{1/r'} \| \nabla u \|_{L^r(\Omega)} \left( \log Q \| \nabla u \|_{L^p(\Omega)} \right)^{1/r'}, \right. \]
\[ \left. \left( \frac{p - 1}{p - r} \right)^{1/p'} \left( \frac{m_n(\Omega)}{2} \right)^{p - r} \| \nabla u \|_{L^p(\Omega)} \right\}, \]

where
\[ Q = \left( \frac{m_n(\Omega)}{2} \right)^{p - r} \left( \frac{e(p - r)}{p(r - 1)} \right)^{1/r'} \left( \frac{(p - 1)r}{p - r} \right)^{1/p'}. \]
Proof. Since
\[ t^\varepsilon + 1 \leq 2^{1-\varepsilon}(t + 1)^\varepsilon \]
for \( t \geq 1 \) and \( 0 \leq \varepsilon \leq 1 \), we have
\[ \text{osc}_\Omega u \leq 2^{1/r} K_{1/r}^{-1} \left( A' \varepsilon \frac{p-r}{p-r} + B' \log \frac{m_n(\Omega)}{2\varepsilon} \right)^{1/r'}, \]  
where
\[ A = \left( \frac{(p-1)r}{p-r} \right)^{1/r'} \| \nabla u \|_{L^p(\Omega)} \]  
and
\[ B = \| \nabla u \|_{L^r(\Omega)}. \]  
Hence
\[ \text{osc}_\Omega u \leq 2^{1/r} K_{1/r}^{-1} \left( \min_{[0,m_n(\Omega)/2]} f(\varepsilon) \right)^{1/r'}, \]  
where
\[ f(\varepsilon) = A' \varepsilon \frac{p-r}{p-r} + B' \log \frac{m_n(\Omega)}{2\varepsilon}. \]
The only root of \( f'(\varepsilon) \) is given by
\[ \varepsilon_0 = \left( \frac{p(r-1)}{p-r} \right)^{\frac{p(r-1)}{p-r-1}} \left( \frac{B}{A} \right)^{\frac{r-r'}{p-r}}. \]
If \( \varepsilon_0 < m_n(\Omega)/2 \), then the minimum value of \( f \) on \([0,m_n(\Omega)/2] \) is
\[ f(\varepsilon_0) = B' \frac{p(r-1)}{p-r} \log \left( \left( \frac{m_n(\Omega)}{2} \right)^{\frac{p-r-1}{p-r}} \frac{e(p-r)}{p(r-1)} \left( \frac{A}{B} \right)^{r'} \right). \]
If \( \varepsilon_0 \geq m_n(\Omega)/2 \), then the minimum value of \( f \) on \([0,m_n(\Omega)/2] \) is
\[ f(m_n(\Omega)/2) = A' \left( \frac{m_n(\Omega)}{2} \right)^{\frac{p-r}{p-r-1}}. \]
Combining this with (5.7), we arrive at
\[ \text{osc}_\Omega u \leq 2^{1/r} K_{1/r}^{-1} \min \left\{ \left( \frac{p(r-1)}{p-r} \right)^{1/r'} B' \left( \log \left( Q_0 \frac{A}{B} \right) \right)^{1/r'}, A' \left( \frac{m_n(\Omega)}{2} \right)^{\frac{p-r}{p-r-1}} \right\}, \]  
with the constant \( Q_0 \) given by
\[ Q_0 = \left( \frac{m_n(\Omega)}{2} \right)^{\frac{p-r}{p-r-1}} \left( \frac{e(p-r)}{p(r-1)} \right)^{1/r'}. \]
(Note that \( Q_0 A \geq B \)). In order to obtain (5.2), we put the values of \( A \) and \( B \) defined in (5.5) and (5.6) into (5.10). The result follows. \( \square \)

Remark 3. Without taking into account the values of the constants in (5.1) and (5.2), we can write both inequalities in the form
\[ \text{osc}_\Omega u \leq c_0 \left( \varepsilon^{(n-r)/pr} \| \nabla u \|_{L^p(\Omega)} + \left( \log \frac{m_n(\Omega)}{2\varepsilon} \right)^{1/r'} \| \nabla u \|_{L^r(\Omega)} \right), \]  
(5.11)
where \( \varepsilon \in (0, m_n(\Omega)/2) \) and \( c_0 \) depends only on \( K_{1/r'} \), \( p \), \( r \), and

\[
\text{osc}_\Omega u \leq c_0 (1 + |\log(c_2\|\nabla u\|_{L^p(\Omega)})|)^{1/r'},
\]

(5.12)

provided \( \|\nabla u\|_{L^r(\Omega)} = 1 \). We recall that \( p > n \), \( p > r \geq 1 \), and \( \Omega \in \mathcal{J}_1/r' \). In particular, if \( \Omega \) is a bounded Lipschitz domain in \( \mathbb{R}^n \), then \( r = n \) and, for example, (5.12) becomes

\[
\text{osc}_\Omega u \leq c_1 (1 + |\log(c_2\|\nabla u\|_{L^p(\Omega)})|)^{(n-1)/n},
\]

(5.13)

with \( \|\nabla u\|_{L^n(\Omega)} = 1 \) (cf. [BG]).

6 Higher order Sobolev spaces

**Theorem 2** Let \( \Omega \in \mathcal{J}_\alpha \), \( \alpha < 1 \), and let \( u \) denote an arbitrary function in \( W^{l,q}(\Omega) \) with integer \( l \) and \( q \geq 1 \). Further let \( r = 1/(1-\alpha) \) and

\[
l(1-\alpha) < 1/q.
\]

(6.1)

If

\[
\|u\|_{W^{l,r}(\Omega)} = 1,
\]
then

\[
\|u\|_{L^\infty(\Omega)} \leq C_{\alpha,q,l} \left( 1 + \left( \log(1 + \|u\|_{W^{l,q}(\Omega)}) \right)^\alpha \right).
\]

(6.2)

**Proof.** Obviously, \( W^{l-1,q}(\Omega) \subset L^p(\Omega) \) implies \( W^{l,q}(\Omega) \subset W^{1,p}(\Omega) \). If

\[
q(l-1)(1-\alpha) < 1
\]

(6.3)

then, by Corollary 4.9/1 [Ma2], the embedding \( W^{l-1,q}(\Omega) \subset L^p(\Omega) \) holds with

\[
p = \frac{q}{1 - q(l-1)(1-\alpha)}.
\]

(6.4)

Since by (6.4)

\[
\frac{1}{p} = \frac{1}{q} - (l-1)(1-\alpha),
\]

(6.3) obviously holds. (That is the value of \( p \) given by (6.4) can be used in (5.11).) In the case

\[
q(l-1)(1-\alpha) = 1
\]

by Corollary 4.9/1, [Ma2], the role of \( p \) can be played by an arbitrary large number, and for

\[
q(l-1)(1-\alpha) > 1
\]

we can put \( p = \infty \) in view of Theorem 5.6.5/2 in [Ma2]. Therefore, we always have

\[
\|\nabla u\|_{L^p(\Omega)} \leq c \|u\|_{W^{l,q}(\Omega)}.
\]

(6.5)

Putting the estimate (6.5) into the right-hand side of (5.11), we obtain

\[
\|u\|_{L^\infty(\Omega)} \leq c_1 \left( \varepsilon^\sigma \|u\|_{W^{l,q}(\Omega)} + \left( \log \frac{m_n(\Omega)}{2\varepsilon} \right)^{1/r'} \|u\|_{W^{1,r}(\Omega)} \right)
\]

(6.6)

with \( \sigma > 0 \) and \( \varepsilon \in (0, m_n(\Omega)/2) \). Minimizing the right-hand side in \( \varepsilon \) and arguing as in Corollary 5, we complete the proof.
7 Whirlpool domain

**Example 1.** Let \( \Omega \) be the domain
\[
\{ x = (x', x_n) : |x'| < \varphi(x_n), \ 0 < x_n < 1 \},
\]
where \( \varphi \) is a continuously differentiable convex function on \([0, 1]\), \( \varphi(0) = 0 \). The area minimizing function satisfies
\[
c [\varphi(t)]^{n-1} - \lambda (v_{n-1} \int_0^t [\varphi(\tau)]^{n-1} d\tau) \leq [\varphi(t)]^{n-1}
\]
for sufficiently small \( t \). (See [Ma2], p. 175-176). Here \( v_{n-1} \) is the volume of the unit ball in \( \mathbb{R}^{n-1} \). Now the inequality (3.2) implies
\[
\text{osc}_\Omega u \leq c \left( \int_0^s [\varphi(\tau)]^{n-1} d\tau \right)^{1/p'} \| \nabla u \|_{L^p(\Omega)} + \left( \int_0^s [\varphi(\tau)]^{n-1} d\tau \right)^{1/r'} \| \nabla u \|_{L^r(\Omega)}
\]
for sufficiently small \( \delta > 0 \).

For the power \( \beta \)-cusp
\[
\Omega = \left\{ x : \sum_{i=1}^{n-1} x_i^2 < x_n^{2\beta}, \ 0 < x_n < 1 \right\}, \quad \beta > 1,
\]
one has by (7.2)
\[
c_1 s^\alpha \leq \lambda(s) \leq c_2 s^\alpha, \quad \alpha = \frac{\beta(n-1)}{\beta(n-1) + 1}.
\]
For this particular case (7.3) takes the form
\[
\text{osc}_\Omega u \leq c \left( \delta^{1+\frac{\beta(n-1)}{p-1} \frac{p-1}{p}} [\nabla u]_{L^p(\Omega)} + \delta^{1+\frac{\beta(n-1)}{r-1} \frac{r-1}{r}} [\nabla u]_{L^r(\Omega)} \right),
\]
where \( p-1+\beta(n-1) > 0 \) and \( r-1+\beta(n-1) < 0 \). In the critical case \( r-1+\beta(n-1) = 0 \) one has for small \( \delta > 0 \)
\[
\text{osc}_\Omega u \leq c \left( \delta^{1+\frac{\beta(n-1)}{p-1} \frac{p-1}{p}} [\nabla u]_{L^p(\Omega)} + (\log \delta)^{1/r'} [\nabla u]_{L^{r'}(\Omega)} \right).
\]
Minimizing the right-hand side in (7.5) and (7.6), we arrive at inequalities independent of $\delta$ of Gagliardo-Nirenberg and Brezis-Wainger type for the $\beta$-cusp. For instance, the result of Theorem 2 for the $\beta$-cusp runs as follows.

**Theorem 3** Let $\Omega$ be the $\beta$-cusp (7.4) and let $u$ denote an arbitrary function in $W^{l;q}(\Omega)$, where $l$ is integer, $q \geq 1$, and

$$ql > \beta(n-1).$$

If

$$\|u\|_{W^{1,1+\beta(n-1)}(\Omega)} = 1,$$

then

$$\|u\|_{L^\infty(\Omega)} \leq C_{\beta,q,l} \left( 1 + \left( \log(1 + \|u\|_{W^{l;q}(\Omega)}) \right)^\alpha \right)$$

with $\alpha = \beta(n-1)/[1 + \beta(n-1)]$.

**Remark 4.** Let us show that the power $\alpha$ of the logarithm in (7.7) is the best possible. We choose

$$u(x) = \frac{\log 1}{\left( \log \frac{1}{\delta} \right)^{1/[1+\beta(n-1)]}}$$

with a small $\delta > 0$. Then

$$\|u\|_{L^\infty(\Omega)} \approx \left( \log \frac{1}{\delta} \right)^{\beta(n-1)/[1+\beta(n-1)]}$$

and

$$\|\nabla u\|_{L^{1+\beta(n-1)}(\Omega)} \approx \left( \log \frac{1}{\delta} \right)^{-1/[1+\beta(n-1)]} \left( \int_0^1 \frac{x_n^{\beta(n-1)}}{(x_n + \delta)^{1+\beta(n-1)}} dx_n \right)^{1/[1+\beta(n-1)]} \approx 1.$$

Similarly,

$$\|u\|_{W^{l;q}(\Omega)} \approx \delta^{(1+\beta(n-1)-lq)/q}.$$ Using this information in the inequality (7.7), we see that $\alpha$ on its right-hand side cannot be diminished.

**8 \lambda-John domains**

We recall that a bounded domain $\Omega \subset \mathbb{R}^n$ is $\lambda$-John, $\lambda \geq 1$, if there is a constant $C > 0$ and a distinguished point $x_0 \in \Omega$ such that every $x \in \Omega$ can be joined to $x_0$ by a rectifiable arc $\gamma \subset \Omega$ along which

$$\text{dist}(y, \partial\Omega) \geq C |\gamma(x,y)|^\lambda, \quad y \in \gamma,$$

where $|\gamma(x,y)|$ is the length of the portion of $\gamma$ joining $x$ to $y$. Clearly, the class of $\lambda$-John domains increases with $\lambda$. By Kilpeläinen and Malý [KM], every $\lambda$-John domain belongs to the class $J_{\lambda(n-1)/n}$. This fact together with Theorem 2 implies inequality (6.2) with $\alpha = \lambda(n-1)/n$ for every $\lambda$-John domain.
References


