Brezis-Gallouet-Wainger type inequality for irregular domains

Vladimir Maz'ya

Department of Mathematical Sciences
University of Liverpool, M&O Building
Liverpool, L69 3BX, UK

Department of Mathematics
Linköping University
SE-58183 Linköping, Sweden

Tatyana Shaposhnikova

Department of Mathematics
Linköping University
SE-58183 Linköping, Sweden

Dedicated to Viktor Burenkov on the occasion of his 70th birthday

Abstract. A Brezis-Gallouet-Wainger logarithmic interpolation-embedding inequality is proved for various classes of irregular domains, in particular, for power cusps and λ-John domains.

2010 MSC. Primary: 46E35; Secondary: 32U20

Keywords: logarithmic interpolation inequalities, Sobolev embeddings, capacities

1 Introduction

Brezis and Gallouet [BG] established the unique solvability of the initial-boundary value problem for the nonlinear Schrödinger evolution equations with zero Dirichlet data on the boundary of a bounded domain in $\mathbb{R}^2$ or its complementary domain. They used crucially the following interpolation-embedding inequality

$$\|u\|_{L^\infty(\Omega)} \le C\Big(1 + \big(\log(1 + \|u\|_{W^{2,2}(\Omega)})\big)^{1/2}\Big) \tag{1.1}$$

for every $u \in W^{2,2}(\Omega)$ with $\|u\|_{W^{1,2}(\Omega)} = 1$. Applications of this inequality to the Euler equation can be found in Chapter 13 of M.E. Taylor's book [Ta].

Brezis and Wainger [BW] extended (1.1) to Sobolev spaces of higher order on $\mathbb{R}^n$ in the form

$$\|u\|_{L^\infty(\mathbb{R}^n)} \le C\Big(1 + \big(\log(1 + \|u\|_{W^{l,q}(\mathbb{R}^n)})\big)^{(n-k)/n}\Big) \tag{1.2}$$

for every function u in $W^{l,q}(\mathbb{R}^n)$ normalized by

$$\|u\|_{W^{k,n/k}(\mathbb{R}^n)} = 1,$$

where k and l are integers, $1 \le k < l$, $ql > n$, and $k \le n$.

Inequalities (1.1) and (1.2) were studied in different directions in [En], [Go], [Sa].

We prove some inequalities of Brezis-Gallouet-Wainger type for various classes of irregular domains. In particular, classes $\mathcal{J}_\alpha$ introduced in [Ma1], power cusps and λ-John domains are considered.

2 Preliminaries

Let Ω be an open domain in $\mathbb{R}^n$ such that $m_n(\Omega) < \infty$, where m_n denotes the n-dimensional Lebesgue measure on Ω. Given $p \in [1, \infty)$ and sets $E \subset G \subset \Omega$, the capacity $C_p(E, G)$ of the condenser (E, G) is defined as

$$C_p(E,G) = \inf\Big\{ \int_\Omega |\nabla u|^p dx : \, u \in L^{1,p}(\Omega), \ u > 1 \text{ in } E \text{ and}$$
$$u \le 0 \text{ in } \Omega \backslash G \text{ (up to a set of standard } p\text{−capacity zero)}\Big\}. \tag{2.1}$$

Here $L^{1,p}(\Omega)$ denotes the Sobolev space defined as

$$L^{1,p}(\Omega) = \big\{ u \in L^p(\Omega, loc) : \, u \text{ is weakly differentiable in } \Omega \text{ and } |\nabla u| \in L^p(\Omega) \big\},$$

endowed with the seminorm

$$\|u\|_{L^{1,p}(\Omega)} = \|\nabla u\|_{L^p(\Omega)}.$$

We need two isocapacitary functions. The first of them

$$\nu_p : [0, m_n(\Omega)/2] \to [0, \infty]$$

is defined by

$$\nu_p(s) = \inf\big\{ C_p(E,G) \colon E \text{ and } G \text{ are measurable subsets of } \Omega \text{ such that}$$
$$E \subset G \text{ and } s \le m_n(E), \, m_n(G) \le m_n(\Omega)/2 \big\} \ \text{ for } s \in [0, m_n(\Omega)/2]. \tag{2.2}$$

Clearly, the function ν_p is non-decreasing. The isocapacitary inequality on Ω is a straightforward consequence of definition(2.2) and tells us that

$$\nu_p(m_n(E)) \le C_p(E, G) \tag{2.3}$$

for every measurable sets $E \subset G \subset \Omega$ with $m_n(G) < m_n(\Omega)/2$.

The second isocapacitary function $\pi_p : [0, m_n(\Omega)/2] \to [0, \infty]$ is given by

$$\pi_p(s) = \inf\big\{ C_p(E,G) \colon E \text{ is a point and } G \text{ is a measurable subset of } \Omega$$
$$\text{such that } E \in G \text{ and } m_n(G) \le s \big\} \text{ for } s \in [0, m_n(\Omega)/2]. \tag{2.4}$$

The function π_p is clearly non-increasing and the corresponding isocapacitary inequality on Ω is

$$\pi_p(m_n(G)) \le C_p(E, G). \tag{2.5}$$

Variants of these isocapacitary functions were introduced in Chaptrs 3 and 5 of [Ma2] and employed to provide necessary and sufficient conditions for embeddings in the Sobolev space of functions with gradient in L^p.

We need another function of purely geometric nature associated with Ω. It is called the isoperimetric function of Ω and will be denoted by λ. The function $\lambda : [0, m_n(\Omega)/2] \to [0, \infty]$ is given by

$$\lambda(s) = \inf\big\{P(E) : s \le m_n(E) \le m_n(\Omega)/2\big\}. \tag{2.6}$$

Here, $P(E)$ is the De Giorgi perimeter of E which can be defined as

$$P(E) = \mathcal{H}^{n-1}(\partial^* E),$$

where $\partial^* E$ stands for the essential boundary of E in the sense of geometric measure theory and $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure. Recall that $\partial^* E$ agrees with the topological boundary ∂E of E when E is regular enough, e.g. is an open subset of Ω with a smooth boundary.

The very definition of λ leads to the isoperimetric inequality on Ω which reads

$$\lambda(m_n(E)) \le P(E) \tag{2.7}$$

for every measurable set $E \subset \Omega$ with $m_n(E) \le m_n(\Omega)/2$. The isoperimetric function of an open subset of $\mathbb{R}^n$ was introduced in [Ma1] (see also [Ma2]) to characterize Sobolev embeddings for functions with gradient in L^1. In more recent years, isoperimetric inequalities and corresponding isopertimetric functions have been intensively investigated on Riemannian manifolds as well, see e.g. [BC, CF, CGL, GP, Gr, MHH, Kle, MJ, Pi, Ri].

The functions ν_p and π_p on one hand, and λ on the other hand, are related by the inequalities

$$\nu_p(s) \ge \Big(\int_s^{m_n(\Omega)/2} \frac{dr}{\lambda(r)^{p'}}\Big)^{1-p} \tag{2.8}$$

and

$$\pi_p(s) \ge \Big(\int_0^s \frac{dr}{\lambda(r)^{p'}}\Big)^{1-p} \tag{2.9}$$

for $s \in (0, m_n(\Omega)/2)$ with $p' = p/(p-1)$, which follows along the same lines as in [Ma2], Proposition 4.3.4/1. It is shown in [Ma2], Lemma 3.2.4, that $\lambda(s) > 0$ for $s > 0$. Owing to inequalities (2.8) and (2.9), we have $\nu_p(s) > 0$ and $\pi_p(s) > 0$ for $s \in (0, m_n(\Omega)/2)$ as well.

Clearly,

$$\lambda(s) \le c_n\, s^{(n-1)/n}. \tag{2.10}$$

Moreover, if Ω is bounded and Lipschitz, then

$$\lambda(s) \approx s^{(n-1)/n} \quad \text{near } s = 0. \tag{2.11}$$

Here and in what follows, the notation

$$f \approx g \qquad \text{near } 0 \tag{2.12}$$

for functions $f, g : (0, \infty) \to [0, \infty)$ means that there exist positive constants c_1, c_2, and s_0 such that

$$c_1 g(c_1 s) \le f(s) \le c_2 g(c_2 s) \quad \text{if } s \in (0, s_0). \tag{2.13}$$

Furthermore, for any Ω,

$$\nu_p(s) \le c_{n,p}\, s^{(n-p)/n} \quad \text{for } n > p \tag{2.14}$$

and

$$\nu_p(s) \le c_n \left(\log \frac{1}{s}\right)^{1-n} \quad \text{for } n = p. \tag{2.15}$$

Both inequalities can be verified by setting appropriate test functions in the above definition of the p-capacity.

If Ω is bounded and Lipschitz, then, by (2.11) and (2.8),

$$\nu_p(s) \approx \begin{cases} s^{(n-p)/n} & \text{if } n > p, \\ \left(\log \frac{1}{s}\right)^{1-n} & \text{if } n = p \end{cases} \tag{2.16}$$

near $s = 0$.

Similarly to (2.14) and (2.15), we obtain the equality

$$\pi_p(s) = 0 \qquad \text{for } n \ge p. \tag{2.17}$$

Finally, if Ω is bounded and Lipschitz, then

$$\pi_p(s) \approx s^{(n-p)/n} \qquad \text{for } p > n. \tag{2.18}$$

In what follows we use the notations

$$\mathcal{G}_\sigma = \{x \in \Omega : u(x) > \sigma\},$$

$$\mathcal{L}_\sigma = \{x \in \Omega : u(x) < \sigma\},$$

$$\mathcal{E}_\sigma = \{x \in \Omega : u(x) = \sigma\}.$$

3 Main results for arbitrary Ω

Theorem 1 *For every $\varepsilon \in (0, m_n(\Omega)/2)$ and for all $u \in L^{1,p}(\Omega) \cap L^{1,r}(\Omega)$, $p \ge 1$, $r \ge 1$,*

$$\mathrm{osc}_\Omega u \le \pi_p(\varepsilon)^{-1/p} \|\nabla u\|_{L^p(\Omega)} + \nu_p(\varepsilon)^{-1/r} \|\nabla u\|_{L^r(\Omega)}, \tag{3.1}$$

where

$$\mathrm{osc}_\Omega u = \operatorname{ess\,sup}_\Omega u - \operatorname{ess\,inf}_\Omega u.$$

Proof. Let the numbers T and t be chosen so that

$$m_n\{x : u(x) > T\} \le m_n(\Omega)/2 \le m_n\{x : u(x) \ge T\},$$

$$m_n\{x : u(x) > t\} \le \varepsilon \le m_n\{x : u(x) \ge t\}.$$

Furthermore, let

$$S := \operatorname{ess\,sup}_\Omega u.$$

Then

$$S - t \le \left(c_p(\mathcal{E}_S, \mathcal{G}_t)\right)^{-1/p} \left(\int_{\mathcal{G}_t} |\nabla u|^p dx\right)^{1/p}$$

and

$$t - T \le \left(c_r(\mathcal{G}_t, \mathcal{G}_T)\right)^{-1/r} \left(\int_{\mathcal{G}_T \setminus \mathcal{G}_t} |\nabla u|^r dx\right)^{1/r}$$

which implies

$$\operatorname{ess\,sup}_\Omega u - T \le \pi_p(\varepsilon)^{-1/p} \|\nabla u\|_{L^p(\mathcal{G}_T)} + \nu_p(\varepsilon)^{-1/r} \|\nabla u\|_{L^r(\mathcal{G}_T)}.$$

An analogous estimate for $T - \text{ess inf}_\Omega u$ is proved in the same way. Adding both estimates, we arrive at (3.1). $\square$

We obtain a direct consequence of Theorem 1 and the above lower estimates for the isocapacitary functions which is formulated in terms of the isoperimetric function λ.

Corollary 1 *Let $p \geq 1$, $r \geq 1$. Then, for every $\varepsilon \in (0, m_n(\Omega)/2)$ and for all $u \in L^{1,p}(\Omega) \cap L^{1,r}(\Omega)$*

$$\text{osc}_\Omega u \leq \Big(\int_0^\varepsilon \frac{d\mu}{\lambda(\mu)^{p'}}\Big)^{1/p'} \|\nabla u\|_{L^p(\Omega)} + \Big(\int_\varepsilon^{m_n(\Omega)/2} \frac{d\mu}{\lambda(\mu)^{r'}}\Big)^{1/r'} \|\nabla u\|_{L^r(\Omega)}. \tag{3.2}$$

Proof. The result follows from the lower estimates (5.6) and (5.7) inserted into (3.1).

Remark 1. One can add the inequality $p > n$ in Theorem 1 since (3.1) has no sense for $n \geq p$ because of (2.17). $\square$

In the case $r = 1$ the estimate (3.2) is simplified as

$$\text{osc}_\Omega u \leq \Big(\int_0^\varepsilon \frac{d\mu}{\lambda(\mu)^{p'}}\Big)^{1/p'} \|\nabla u\|_{L^p(\Omega)} + \lambda(\varepsilon)^{-1} \|\nabla u\|_{L^1(\Omega)}, \tag{3.3}$$

where $p > n$.

4 Domains of the class $\mathcal{J}_\alpha$

We say that a domain belongs to the class $\mathcal{J}_\alpha$, $\alpha > 0$, if there is a constant $\mathcal{K}_\alpha$ such that

$$\lambda(\mu) \geq \mathcal{K}_\alpha \, \mu^\alpha$$

for $\mu \in (0, m_n(\Omega)/2)$. This class was introduced in [Ma1] and studied in detail in [Ma2], [Ma3].

Corollary 1 can be made more visible for $\Omega \in \mathcal{J}_\alpha$.

Corollary 2 *Let $p > n$, $r \geq 1$ and let*

$$\frac{1}{p'} > \alpha > \frac{1}{r'}.$$

Then, for every $\varepsilon \in (0, m_n(\Omega)/2)$ and for all $u \in L^{1,p}(\Omega)$

$$\begin{aligned} \text{osc}_\Omega u \;&\leq \mathcal{K}_\alpha^{-1} \Big((1 - \alpha p')^{-1/p'} \varepsilon^{-\alpha + 1/p'} \|\nabla u\|_{L^p(\Omega)} \\ &+ (\alpha r' - 1)^{-1/r'} \varepsilon^{-\alpha + 1/r'} \|\nabla u\|_{L^r(\Omega)} \Big). \end{aligned} \tag{4.1}$$

Taking the minimum value of the right-hand side in ε, we arrive at the following alternatives.

Corollary 3 *Let the conditions of Corollary 2 hold.*

(i) If

$$\|\nabla u\|_{L^r(\Omega)} \leq \frac{r'(1 - \alpha p')^{1/p}}{p'(\alpha r' - 1)^{1/r}} \Big(\frac{m_n(\Omega)}{2}\Big)^{1/r - 1/p} \|\nabla u\|_{L^p(\Omega)},$$

then

$$\mathrm{osc}_\Omega u \le C_{\alpha,p,r}\,\mathcal{K}_\alpha^{-1}\|\nabla u\|_{L^p(\Omega)}^{\frac{\alpha-1/r'}{1/r-1/p}}\,\|\nabla u\|_{L^r(\Omega)}^{\frac{-\alpha+1/p'}{1/r-1/p}}$$

with

$$C_{\alpha,p,r} = (1-\alpha p')^{\frac{-\alpha+1/r'}{p'(1/r-1/p)}}(\alpha r'-1)^{\frac{\alpha-1/p'}{r'(1/r-1/p)}}$$
$$\times\Big(\Big(\frac{\alpha-1/r'}{-\alpha+1/p'}\Big)^{\frac{-\alpha+1/p'}{1/r-1/p}}+\Big(\frac{-\alpha+1/p'}{\alpha-1/r'}\Big)^{\frac{\alpha-1/r'}{1/r-1/p}}\Big).$$

(ii) If

$$\|\nabla u\|_{L^r(\Omega)} > \frac{r'(1-\alpha p')^{1/p}}{p'(\alpha r'-1)^{1/r}}\Big(\frac{m_n(\Omega)}{2}\Big)^{1/r-1/p}\|\nabla u\|_{L^p(\Omega)},$$

then

$$\begin{aligned}\mathrm{osc}_\Omega u \ &\le \mathcal{K}_\alpha^{-1}\Big((1-\alpha p')^{-1/p'}\Big(\frac{m_n(\Omega)}{2}\Big)^{-\alpha+1/p'}\|\nabla u\|_{L^p(\Omega)}\\ &+(\alpha r'-1)^{-1/r'}\Big(\frac{m_n(\Omega)}{2}\Big)^{-\alpha+1/r'}\|\nabla u\|_{L^r(\Omega)}\Big).\end{aligned}$$

Remark 2. Without taking care of constant factors, we can deduce the following inequality of Gagliardo-Nirenberg type (cf. [Ga], [Ni]) from Corollary 3:

$$\mathrm{osc}_\Omega u \le c_{\alpha,p,r}\,\mathcal{K}_\alpha^{-1}\|\nabla u\|^{\gamma}_{L^p(\Omega)}\|\nabla u\|^{1-\gamma}_{L^r(\Omega)} \tag{4.2}$$

for an arbitrary function u in $L^{1,p}(\Omega)$, where Ω is a domain of the class $\mathcal{J}_\alpha$ with $1/p' > \alpha > 1/r'$, $p > n$, $r \ge 1$, and

$$\gamma = \frac{\alpha-1/r'}{1/r-1/p}.$$

5 The case $\Omega \in \mathcal{J}_{1/r'}$

We turn to the critical case when Ω belongs to the class $\mathcal{J}_{1/r'}$.

Corollary 4 *Let $p > n$, $p > r \ge 1$ and let $\Omega \in \mathcal{J}_{1/r'}$. Then, for every $\varepsilon \in (0, m_n(\Omega)/2)$ and for all $u \in L^{1,p}(\Omega)$,*

$$\begin{aligned}\mathrm{osc}_\Omega u \ &\le \mathcal{K}_{1/r'}^{-1}\Big(\Big(\frac{(p-1)r}{p-r}\Big)^{1/p'}\varepsilon^{(p-r)/pr}\|\nabla u\|_{L^p(\Omega)}\\ &+\Big(\log\frac{m_n(\Omega)}{2\varepsilon}\Big)^{1/r'}\|\nabla u\|_{L^r(\Omega)}\Big).\end{aligned} \tag{5.1}$$

Corollary 5 *Let $p > n$, $p > r \ge 1$ and let $\Omega \in \mathcal{J}_{1/r'}$. Then, for all $u \in L^{1,p}(\Omega)$,*

$$\begin{aligned}\mathrm{osc}_\Omega u \ &\le 2^{1/r}\mathcal{K}_{1/r'}^{-1}\min\Big\{\Big(\frac{pr}{p-1}\Big)^{1/r'}\|\nabla u\|_{L^r(\Omega)}\Big(\log Q\frac{\|\nabla u\|_{L^p(\Omega)}}{\|\nabla u\|_{L^r(\Omega)}}\Big)^{1/r'},\\ &\Big(\frac{(p-1)r}{p-r}\Big)^{1/p'}\Big(\frac{m_n(\Omega)}{2}\Big)^{\frac{p-r}{p(r-1)}}\|\nabla u\|_{L^p(\Omega)}\Big\},\end{aligned} \tag{5.2}$$

where

$$Q = \Big(\frac{m_n(\Omega)}{2}\Big)^{\frac{p-r}{pr}}\Big(\frac{e(p-r)}{p(r-1)}\Big)^{1/r'}\Big(\frac{(p-1)r}{p-r}\Big)^{1/p'}. \tag{5.3}$$

Proof. Since

$$t^{\varepsilon}+1\leq 2^{1-\varepsilon}(t+1)^{\varepsilon}$$

for $t\geq 1$ and $0\leq\varepsilon\leq 1$, we have

$$\mathrm{osc}_{\Omega}u\leq 2^{1/r}\mathcal{K}_{1/r'}^{-1}\Big(A^{r'}\varepsilon^{\frac{p-r}{p(r-1)}}+B^{r'}\log\frac{m_n(\Omega)}{2\varepsilon}\Big)^{1/r'}, \tag{5.4}$$

where

$$A=\Big(\frac{(p-1)r}{p-r}\Big)^{1/p'}\|\nabla u\|_{L^p(\Omega)} \tag{5.5}$$

and

$$B=\|\nabla u\|_{L^r(\Omega)}. \tag{5.6}$$

Hence

$$\mathrm{osc}_{\Omega}u\leq 2^{1/r}\mathcal{K}_{1/r'}^{-1}\Big(\min_{[0,m_n(\Omega)/2]}f(\varepsilon)\Big)^{1/r'}, \tag{5.7}$$

where

$$f(\varepsilon)=A^{r'}\varepsilon^{\frac{p-r}{p(r-1)}}+B^{r'}\log\frac{m_n(\Omega)}{2\varepsilon}.$$

The only root of $f'(\varepsilon)$ is given by

$$\varepsilon_0=\Big(\frac{p(r-1)}{p-r}\Big)^{\frac{p(r-1)}{p-r}}\Big(\frac{B}{A}\Big)^{\frac{pr}{p-r}}.$$

If $\varepsilon_0<m_n(\Omega)/2$, then the minimum value of f on $[0,m_n(\Omega)/2]$ is

$$f(\varepsilon_0)=B^{r'}\frac{p(r-1)}{p-r}\log\Big(\Big(\frac{m_n(\Omega)}{2}\Big)^{\frac{p-r}{p(r-1)}}\frac{e(p-r)}{p(r-1)}\Big(\frac{A}{B}\Big)^{r'}\Big). \tag{5.8}$$

If $\varepsilon_0\geq m_n(\Omega)/2$, then the minimum value of f on $[0,m_n(\Omega)/2]$ is

$$f(m_n(\Omega/2)=A^{r'}\Big(\frac{m_n(\Omega)}{2}\Big)^{\frac{p-r}{p(r-1)}}. \tag{5.9}$$

Combining this with (5.7), we arrive at

$$\begin{aligned}\mathrm{osc}_{\Omega}u\ &\leq 2^{1/r}\mathcal{K}_{1/r'}^{-1}\min\Big\{\Big(\frac{p(r-1)}{p-r}\Big)^{1/r'}B\Big(r'\log\Big(Q_0\frac{A}{B}\Big)\Big)^{1/r'},\\ &A\Big(\frac{m_n(\Omega)}{2}\Big)^{\frac{p-r}{pr}}\Big\}\end{aligned} \tag{5.10}$$

with the constant Q_0 given by

$$Q_0=\Big(\frac{m_n(\Omega)}{2}\Big)^{\frac{p-r}{pr}}\Big(\frac{e(p-r)}{p(r-1)}\Big)^{1/r'}.$$

(Note that $Q_0A\geq B$). In order to obtain (5.2), we put the values of A and B defined in (5.5) and (5.6) into (5.10). The result follows. □

Remark 3. Without taking into account the values of the constants in (5.1) and (5.2), we can write both inequalities in the form

$$\mathrm{osc}_{\Omega}u\leq c_0\Big(\varepsilon^{(p-r)/pr}\|\nabla u\|_{L^p(\Omega)}+\Big(\log\frac{m_n(\Omega)}{2\varepsilon}\Big)^{1/r'}\|\nabla u\|_{L^r(\Omega)}\Big), \tag{5.11}$$

where $\varepsilon \in (0, m_n(\Omega)/2)$ and c_0 depends only on $\mathcal{K}_{1/r'}$, p, r, and

$$\mathrm{osc}_\Omega u \leq c_0\big(1 + \big|\log(c_2\|\nabla u\|_{L^p(\Omega)})\big|\big)^{1/r'}, \tag{5.12}$$

provided $\|\nabla u\|_{L^r(\Omega)} = 1$. We recall that $p > n$, $p > r \geq 1$, and $\Omega \in \mathcal{J}_{1/r'}$. In particular, if Ω is a bounded Lipschitz domain in $\mathbb{R}^n$, then $r = n$ and, for example, (5.12) becomes

$$\mathrm{osc}_\Omega u \leq c_1\big(1 + \big|\log(c_2\|\nabla u\|_{L^p(\Omega)})\big|\big)^{(n-1)/n}, \tag{5.13}$$

with $\|\nabla u\|_{L^n(\Omega)} = 1$ (cf. [BG]).

6 Higher order Sobolev spaces

Theorem 2 *Let $\Omega \in \mathcal{J}_\alpha$, $\alpha < 1$, and let u denote an arbitrary function in $W^{l,q}(\Omega)$ with integer l and $q \geq 1$. Further let $r = 1/(1-\alpha)$ and*

$$l(1-\alpha) < 1/q. \tag{6.1}$$

If

$$\|u\|_{W^{1,r}(\Omega)} = 1,$$

then

$$\|u\|_{L^\infty(\Omega)} \leq C_{\alpha,q,l}\Big(1 + \big(\log(1 + \|u\|_{W^{l,q}(\Omega)})\big)^\alpha\Big). \tag{6.2}$$

Proof. Obviously, $W^{l-1,q}(\Omega) \subset L^p(\Omega)$ implies $W^{l,q}(\Omega) \subset W^{1,p}(\Omega)$. If

$$q(l-1)(1-\alpha) < 1 \tag{6.3}$$

then, by Corllary 4.9/1 [Ma2], the embedding $W^{l-1,q}(\Omega) \subset L^p(\Omega)$ holds with

$$p = \frac{q}{1 - q(l-1)(1-\alpha)}. \tag{6.4}$$

Since by (6.4)

$$\frac{1}{p} = \frac{1}{q} - (l-1)(1-\alpha),$$

(6.3) obviously holds. (That is the value of p given by (6.4) can be used in (5.11).) In the case

$$q(l-1)(1-\alpha) = 1$$

by Corollary 4.9/1, [Ma2], the role of p can be played by an arbitrary large number, and for

$$q(l-1)(1-\alpha) > 1$$

we can put $p = \infty$ in veiw of Theorem 5.6.5/2 in [Ma2]. Therefore, we always have

$$\|\nabla u\|_{L^p(\Omega)} \leq c\,\|u\|_{W^{l,q}(\Omega)}. \tag{6.5}$$

Putting the estimate (6.5) into the right-hand side of (5.11), we obtain

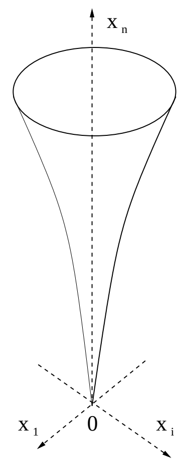
$$\|u\|_{L^\infty(\Omega)} \leq c_1\Big(\varepsilon^\sigma\|u\|_{W^{l,q}(\Omega)} + \Big(\log\frac{m_n(\Omega)}{2\varepsilon}\Big)^{1/r'}\|u\|_{W^{1,r}(\Omega)}\Big) \tag{6.6}$$

with $\sigma > 0$ and $\varepsilon \in (0, m_n(\Omega)/2)$. Minimizing the right-hand side in ε and arguing as in Corollary 5, we complete the proof.

7 Whirlpool domain

Example 1. Let Ω be the domain

$$\{x = (x', x_n) : |x'| < \varphi(x_n),\ 0 < x_n < 1\}, \tag{7.1}$$



where φ is a continuously differentiable convex function on $[0,1]$, $\varphi(0) = 0$. The area minimizing function satisfies

$$c\,[\varphi(t)]^{n-1} \le \lambda\big(v_{n-1}\int_0^t [\varphi(\tau)]^{n-1} d\tau\big) \le [\varphi(t)]^{n-1} \tag{7.2}$$

for sufficiently small t. (See [Ma2], p. 175-176). Here v_{n-1} is the volume of the unit ball in $\mathbb{R}^{n-1}$. Now the inequality (3.2) implies

$$\operatorname{osc}_\Omega u \le c\Big(\Big(\int_0^\delta \varphi(t)^{\frac{n-1}{p-1}} dt\Big)^{1/p'} \|\nabla u\|_{L^p(\Omega)} + \Big(\int_\delta^1 \varphi(t)^{\frac{n-1}{r-1}} dt\Big)^{1/r'} \|\nabla u\|_{L^r(\Omega)} \tag{7.3}$$

for sufficiently small $\delta > 0$.

For the power β-cusp

$$\Omega = \Big\{x : \sum_{i=1}^{n-1} x_i^2 < x_n^{2\beta},\ 0 < x_n < 1\Big\}, \quad \beta > 1, \tag{7.4}$$

one has by (7.2)

$$c_1\, s^\alpha \le \lambda(s) \le c_2\, s^\alpha, \qquad \alpha = \frac{\beta(n-1)}{\beta(n-1)+1}.$$

For this particular case (7.3) takes the form

$$\operatorname{osc}_\Omega u \le c\Big(\delta^{(1+\frac{\beta(n-1)}{p-1})\frac{p-1}{p}} \|\nabla u\|_{L^p(\Omega)} + \delta^{(1+\frac{\beta(n-1)}{r-1})\frac{r-1}{r}} \|\nabla u\|_{L^r(\Omega)}\Big), \tag{7.5}$$

where $p-1+\beta(n-1) > 0$ and $r-1+\beta(n-1) < 0$. In the critical case $r-1+\beta(n-1) = 0$ one has for small $\delta > 0$

$$\operatorname{osc}_\Omega u \le c\Big(\delta^{(1+\frac{\beta(n-1)}{p-1})\frac{p-1}{p}} \|\nabla u\|_{L^p(\Omega)} + \big(\log \delta^{-1}\big)^{1/r'} \|\nabla u\|_{L^r(\Omega)}\Big). \tag{7.6}$$

Minimizing the right-hand side in (7.5) and (7.6), we arrive at inequalities independent of δ of Gagliardo-Nirenberg and Brezis-Wainger type for the β-cusp. For instance, the result of Theorem 2 for the β-cusp runs as follows.

Theorem 3 *Let Ω be the β-cusp (7.4) and let u denote an arbitrary function in $W^{l,q}(\Omega)$, where l is integer, $q \geq 1$, and*

$$ql > \beta(n-1).$$

If

$$\|u\|_{W^{1,1+\beta(n-1)}(\Omega)} = 1,$$

then

$$\|u\|_{L^\infty(\Omega)} \leq C_{\beta,q,l}\Big(1 + \Big(\log(1 + \|u\|_{W^{l,q}(\Omega)})\Big)^{\alpha}\Big) \tag{7.7}$$

with $\alpha = \beta(n-1)/[1+\beta(n-1)]$.

Remark 4. Let us show that the power α of the logarithm in (7.7) is the best possible. We choose

$$u(x) = \frac{\log \frac{1}{x_n+\delta}}{\Big(\log \frac{1}{\delta}\Big)^{1/[1+\beta(n-1)]}}$$

with a small $\delta > 0$. Then

$$\|u\|_{L^\infty(\Omega)} \approx \Big(\log \frac{1}{\delta}\Big)^{\beta(n-1)/[1+\beta(n-1)]}$$

and

$$\|\nabla u\|_{L^{1+\beta(n-1)}(\Omega)} \approx \Big(\log \frac{1}{\delta}\Big)^{-1/[1+\beta(n-1)]} \Big(\int_0^1 \frac{x_n^{\beta(n-1)}dx_n}{(x_n+\delta)^{1+\beta(n-1)}}\Big)^{1/[1+\beta(n-1)]} \approx 1.$$

Similarly,

$$\|u\|_{W^{l,q}(\Omega)} \approx \delta^{(1+\beta(n-1)-lq)/q}.$$

Using this information in the inequality (7.7), we see that α on its right-hand side cannot be diminished.

8 λ-John domains

We recall that a bounded domain $\Omega \subset \mathbb{R}^n$ is λ-John, $\lambda \geq 1$, if there is a constant $C > 0$ and a distinguished point $x_0 \in \Omega$ such that every $x \in \Omega$ can be joined to x_0 by a rectifiable arc $\gamma \subset \Omega$ along which

$$\operatorname{dist}(y, \partial\Omega) \geq C\,|\gamma(x,y)|^\lambda, \quad y \in \gamma,$$

where $|\gamma(x,y)|$ is the length of the portion of γ joining x to y. Clearly, the class of λ-John domains increases with λ. By Kilpeläinen and Malý [KM], every λ-John domain belongs to the class $\mathcal{J}_{\lambda(n-1)/n}$. This fact together with Theorem 2 implies inequality (6.2) with $\alpha = \lambda(n-1)/n$ for every λ-John domain.

References

[BC] I.Benjamini and J.Cao, *A new isoperimetric theorem for surfaces of variable curvature*, Duke Math. J. **85** (1996), 359–396.

[BG] H.Brezis and T.Gallouet, *Nonlinear Schrödinger evolution equations*, Nonlinear Analysis, Theory, Methods and Applications, **4** (1980), no.4, 677–681.

[BW] H.Brezis and S. Wainger, *A note on limiting cases of Sobolev embeddings*, Comm. Partial Diff. Equations, **5** (1980), 773–789.

[CF] I.Chavel and E.A.Feldman, *Modified isoperimetric constants, and large time heat diffusion in Riemannian manifolds*, Duke Math. J. **64** (1991), 473–499.

[CGL] T.Coulhon, A.Grigoryan and D.Levin, *On isoperimetric profiles of product spaces*, Comm. Anal. Geom. **11** (2003), 85–120.

[En] H.Engler, *An alternative proof of the Brezis-Wainger inequality*, Comm. Partial Diff. Equations, **14**, no. 4 (1989), 541–544.

[Ga] E.Gagliardo, *Ulteriori proprieta di alcune classi di funzioni in piu variabili*, Ric. Mat. **8** (1959) 24–51.

[Go] P. Górka, *Brezis-Wainger inequality on Riemannian manifolds*, J. Inequal. Appl. (2008), Art. ID 715961, 6 pp.

[Gr] A.Grigoryan, *Isoperimetric inequalities and capacities on Riemannian manifolds*, in The Mazya anniversary collection, Vol. 1 (Rostock, 1998), 139–153, Oper. Theory Adv. Appl., **109**, Birkhäuser, Basel, 1999.

[GP] R.Grimaldi and P.Pansu, *Calibrations and isoperimetric profiles*, Amer. J. Math. **129** (2007), 315–350.

[KM] T.Kilpeläinen and J. Malý, *Sobolev inequalities on sets with irregular boundaries*, Z. Anal. Anwend. **19** (2000), no. 2, 369–380.

[Kle] B.Kleiner, *An isoperimetric comparison theorem*, Invent. Math. **108** (1992), 37–47.

[Ma1] V.Maz'ya, *Classes of regions and imbedding theorems for function spaces*, Dokl. Akad. Nauk. SSSR **133** (1960), 527–530 (Russian); English translation: Soviet Math. Dokl. **1** (1960), 882–885.

[Ma2] V.Maz'ya, *Sobolev Spaces*, Springer, Berlin, 1985.

[Ma3] V.Maz'ya, *Sobolev Spaces with Applications to Elliptic Partial Differential Equations*, Springer, Berlin, 2010, Grundlehren der Mathematischen Wissenschaften, to appear.

[MHH] F.Morgan, H.Howards and M.Hutchings, *The isoperimetric problem on surfaces of revolution of decreasing Gauss curvature*, Trans. Amer. Math. Soc. **352** (2000), 4889–4909.

[MJ] F.Morgan and D.L.Johnson, *Some sharp isoperimetric theorems for Riemannian manifolds*, Indiana Univ. Math. J. **49** (2000), 1017–1041.

[Ni] L.Nirenberg, *On elliptic partial differential equations (Lecture II)*, Ann. Sc. Norm. Super. Pisa, **3** (1959), no.13, 115–162.

[Pi] Ch.Pittet, *The isoperimetric profile of homogeneous Riemannian manifolds*, J. Diff. Geom. **54** (2000), 255–302.

[Ri] M.Ritoré, *Constant geodesic curvature curves and isoperimetric domains in rotationally symmetric surfaces*, Comm. Anal. Geom. **9** (2001), 1093–1138.

[Sa] Y.Sawano, *Brezis-Gallouet-Wainger type inequality for Besov-Morrey spaces*, Studia Math. **196**, no.1 (2010), 91–101.

[Ta] M.Taylor, *Partial Differential Equations, III. Nonlinear Equations*, Applied Mathematical Sciences, **117**, Springer, 1997.