Characterization of multipliers in pairs of Besov spaces

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Dedicated to the memory of Erhard Meister

Abstract

We give necessary and sufficient conditions for a function to be a multiplier from one Besov space $B^m_p(\mathbf{R}^n)$ into another $B^l_p(\mathbf{R}^n)$ where $0 < l \leq m$ and $p \in (1, \infty)$. We also show that the space of multipliers acting from the Sobolev space $W^m_p(\mathbf{R}^n)$ into a distribution Sobolev space $W^{-k}_p(\mathbf{R}^n)$ is isomorphic to $W^{-k}_{p,\mathrm{unif}}(\mathbf{R}^n) \cap W^{-m}_{p',\mathrm{unif}}(\mathbf{R}^n)$ for either $k \geq m > 0$, $k > n/p'$, or $m \geq k > 0$, $m > n/p$, where $p \in (1, \infty)$ and $p + p' = pp'$.

1 Introduction

By a multiplier acting from one Banach function space S_1 into another S_2 we call a function γ such that $\gamma u \in S_2$ for any $u \in S_1$. By $M(S_1 \to S_2)$ we denote the space of multipliers $\gamma : S_1 \to S_2$ with the norm

$$\|\gamma\|_{M(S_1 \to S_2)} = \sup\{\|\gamma u\|_{S_2} : \quad \|u\|_{S_1} \leq 1\}.$$

We write MS instead of $M(S \to S)$.

A theory of pointwise multipliers was developed in our book [MS], where a complete bibliography and description of related results obtained before 1985 can be found. In particular, [MS] contains characterisation of the spaces $M(H^m_p(\mathbf{R}^n) \to H^l_p(\mathbf{R}^n))$ with $1 < p < \infty$, where $H^k_p(\mathbf{R}^n)$ is the Bessel potential space. We also

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described multipliers $M(W_p^m(\mathbf{R}^n) \to W_p^l(\mathbf{R}^n))$ in Sobolev (k integer)-Slobodetskii (k noninteger) spaces with $1 \le p < \infty$ and both m and l being either integer or noninteger.

We mention known results on multipliers preserving a certain Besov space. Necessary and sufficient conditions for a function to belong to $MB_p^l(\mathbf{R}^n)$, $1 < p < \infty$, $0 < l < \infty$, are given in [MS]. Recently a characterization of $MB_{p,q}^s(\mathbf{R}^n)$ for $1 \le p \le q \le \infty$, $s > n/p$, was obtained by Sickel and Smirnov [SS]. The spaces $MB_{\infty,1}^0(\mathbf{R}^n)$ and $MB_{\infty,\infty}^0(\mathbf{R}^n)$ were described by Koch and Sickel [KS].

The main goal of the present paper is to characterize the space $M(B_p^m(\mathbf{R}^n) \to B_p^l(\mathbf{R}^n))$ for $m \ge l > 0$, $p \in (1, \infty)$.

A sufficient condition for inclusion into the space $M(W_p^m(\mathbf{R}^n) \to W_p^{-k}(\mathbf{R}^n))$ of Sobolev multipliers can be found in Sect.1.5 [MS]. Recently Maz'ya and Verbitsky [MV2], [MV3] described the spaces $M(W_2^1(\mathbf{R}^n) \to W_2^{-1}(\mathbf{R}^n))$ and $M(W_2^{1/2}(\mathbf{R}^n) \to W_2^{-1/2}(\mathbf{R}^n))$, solving the problem of the form boundedness of the Schrödinger and the relativistic Schrödinger operators (see [MV2] and [MV4] for further results in the same vein). We conclude the present paper by showing that the space $M(W_p^m(\mathbf{R}^n) \to W_p^{-k}(\mathbf{R}^n))$ is isomorphic to $W_{p,\mathrm{unif}}^{-k}(\mathbf{R}^n) \cap W_{p',\mathrm{unif}}^{-m}(\mathbf{R}^n)$ provided $k \ge m > 0$, $k > n/p'$ or $m \ge k > 0$, $m > n/p$, where where $p \in (1, \infty)$ and $p + p' = pp'$. This is a straightforward corollary of the above mentioned sufficient condition from Sect. 1.5 [MS]. However, the result seems to be new even for $n = 1$, except for the case $k = m = 1$ treated in [MV4].

Let $s = k + \alpha$, where $\alpha \in (0, 1]$ and k is a nonnegative integer. Further, let

$$\Delta_h^{(2)} u(x) = u(x + 2h) - 2u(x + h) + u(x)$$

and

$$(C_{p,s}u)(x) = \Big(\int_{\mathbf{R}^n} |\Delta_h^{(2)} \nabla_k u(x)|^p |h|^{-n-p\alpha} dh \Big)^{1/p},$$

where ∇_k stands for the gradient of order k, i.e. $\nabla_k u = \{\partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}\}$, $\alpha_1 + \dots + \alpha_n = k$. The Besov space $B_p^s(\mathbf{R}^n)$ is introduced as the completion of $C_0^\infty(\mathbf{R}^n)$ in the norm

$$\|C_{p,s}u; \mathbf{R}^n\|_{L_p} + \|u; \mathbf{R}^n\|_{L_p}.$$

Let $\{s\}$ and $[s]$ denote the fractional and integer parts of a positive number s and let

$$(D_{p,s}u)(x) = \Big(\int |\Delta_h \nabla_{[s]} u(x)|^p |h|^{-n-p\{s\}} dh \Big)^{1/p},$$

where $\Delta_h v(x) = v(x + h) - v(x)$. The fractional Sobolev space W_p^s is defined as the closure of C_0^∞ in the norm

$$\|D_{p,s}u\|_{L_p} + \|u\|_{L_p}.$$

(Here and in the sequel, we omit $\mathbf{R}^n$ in the notation of norms, spaces, and in the range of integration.) For $\{s\} > 0$ the spaces B_p^s and W_p^s have the same elements and their norms are equivalent since

$$(2-2^{\{s\}})D_{p,s}u \le C_{p,s}u \le (2+2^{\{s\}})D_{p,s}u \tag{1}$$

which follows directly from the identity

$$2[u(x+h)-u(x)] = -[u(x+2h)-2u(x+h)+u(x)] + [u(x+2h)-u(x)].$$

In what follows the equivalence $a \sim b$ means that there exist positive constants c_1, c_2 such that $c_1 b \le a \le c_2 b$.

With any Banach space S of functions on $\mathbf{R}^n$ one can associate the spaces

$$S_{\text{loc}} = \{u : \eta u \in S \quad \text{for all} \;\; \eta \in C_0^\infty\}$$

and

$$S_{\text{unif}} = \{u : \sup_{z\in\mathbf{R}^n} \|\eta_z u\|_S < \infty\},$$

where $\eta_z(x) = \eta(x-z)$, $\eta \in C_0^\infty$, $\eta = 1$ on $\mathcal{B}_1$. Here and in what follows $\mathcal{B}_r(x)$ is the ball $\{y \in \mathbf{R}^n : |y-x| < r\}$ and $\mathcal{B}_r = \mathcal{B}_r(0)$. The space S_{unif} is endowed with the norm

$$\|u\|_{S_{\text{unif}}} = \sup_{z\in\mathbf{R}^n} \|\eta_z u\|_S.$$

The obvious consequence of the definition of the multiplier space $M(S_1 \to S_2)$ is the imbedding

$$M(S_1 \to S_2) \subset S_{2,\text{unif}}.$$

Let Λ^μ be the operator defined for any $\mu \in \mathbf{R}$ by

$$\Lambda^\mu = (-\Delta+1)^{\mu/2} = F^{-1}\big(1+|\xi|^2\big)^{\mu/2}F,$$

where F is the Fourier transform in $\mathbf{R}^n$ and F^{-1} is the inverse of F. By J_l we denote the Bessel potential of order l, that is the operator Λ^{-l}. Throughout the paper we assume that $m > 0$ and use the notion of the (p,m)-capacity $\text{cap}_{p,m}(e)$ of a compact set $e \subset \mathbf{R}^n$ which is defined by

$$\text{cap}_{p,m}(e) = \inf\{\|f\|^p_{L_p} : \; f \in L_p, \; f \ge 0 \text{ and } J_m f(x) \ge 1 \text{ for all } x \in e\}.$$

For properties of this capacity see [M], Ch. 7 and [AH], Ch. 2 and Sect. 4.4. In particular, it is well known that if $0 < r \le 1$, then

$$\text{cap}_{p,m}(\mathcal{B}_r) \sim \begin{cases} r^{n-mp} & \text{for } mp < n, \\ \big(\log\frac{2}{r}\big)^{1-p} & \text{for } mp = n, \\ 1 & \text{for } mp > n, \end{cases} \tag{2}$$

and if e is a compact set in $\mathbf{R}^n$ with $\text{diam}(e) \le 1$, then

$$\text{cap}_{p,m}(e) \ge \begin{cases} c(\text{mes}_n e)^{(n-mp)/n} & \text{for } mp < n, \\ \big(\log\frac{2^n}{\text{mes}_n e}\big)^{1-p} & \text{for } mp = n. \end{cases} \tag{3}$$

The following assertion is the main result of this article.

Theorem 1. *Let* $0 < l \leq m$, $p \in (1,\infty)$, *and let* $\gamma \in B^l_{p,\mathrm{loc}}$. *There holds the equivalence relation*

$$\|\gamma\|_{M(B^m_p \to B^l_p)} \sim \sup_e \frac{\|C_{p,l}\gamma; e\|_{L_p}}{[\mathrm{cap}_{p,m}(e)]^{1/p}} + \begin{cases} \|\gamma\|_{L_{1,\mathrm{unif}}}, & m > l, \\ \|\gamma\|_{L_\infty}, & m = l, \end{cases} \tag{4}$$

where e *is an arbitrary compact set in* $\mathbf{R}^n$. *The finiteness of the right-hand side in* (4) *is necessary and sufficient for* $\gamma \in M(B^m_p \to B^l_p)$.

The relation (4) *remains valid if one adds the condition* $\mathrm{diam}(e) \leq 1$.

For $mp > n$ the statement of the above theorem simplifies. Namely, the relation (4) is equivalent to

$$\|\gamma\|_{M(B^m_p \to B^l_p)} \sim \|\gamma\|_{B^l_{p,\mathrm{unif}}} \quad \text{for } m \geq l, \tag{5}$$

and for $lp > n$

$$\|\gamma\|_{MB^l_p} \sim \|C_{p,l}\gamma\|_{L_{p,\mathrm{unif}}} + \|\gamma\|_{L_\infty}. \tag{6}$$

From results of Kerman and Saywer [KeS] and Maz'ya and Verbitsky [MV1] it follows that the supremum in the right-hand side of (4) is equivalent to each of the suprema

$$\sup_{\{Q\}} \frac{\|J_m \chi_Q (C_{p,l}\gamma)^p; Q\|_{L_{p/(p-1)}}}{\|C_{p,l}\gamma; Q\|^{p-1}_{L_p}}, \tag{7}$$

where $\{Q\}$ is the collection of all cubes, χ_Q is the characteristic function of Q, and

$$\sup_{x \in \mathbf{R}^n} \frac{J_m (J_m (C_{p,l}\gamma)^p)^{p/(p-1)}(x)}{J_m (C_{p,l}\gamma)^p(x)}. \tag{8}$$

From (4), (7), and (8) one can deduce various precise upper and lower estimates for the norm in $M(B^m_p \to B^l_p)$ formulated in more conventional terms (compare with [MS], Ch. 3).

2 Preliminaries

In this section, we collect some auxiliary assertions used in the sequel.

Lemma 1. (see [St], Sect. 5.1) *There holds the equivalence relation*

$$\|u\|_{B^k_p} \sim \|\Lambda^\alpha u\|_{B^{k-\alpha}_p}, \tag{9}$$

where $p \in (1,\infty)$ *and* $\alpha \in (0,k)$.

By H^k_p, $k \geq 0$, $p \in (1,\infty)$, we denote the space of Bessel potentials defined as the completion of C^∞_0 in the norm

$$\|u\|_{H^k_p} = \|\Lambda^k u\|_{L_p}. \tag{10}$$

The following relations are well known

$$\|\gamma\|_{M(B^k_p\to L_p)} \sim \|\gamma\|_{M(H^k_p\to L_p)} \sim$$

$$\sup_e \frac{\|\gamma;e\|_{L_p}}{[\mathrm{cap}_{p,k}(e)]^{1/p}} \sim \sup_{e,\mathrm{diam}(e)\le 1} \frac{\|\gamma;e\|_{L_p}}{[\mathrm{cap}_{p,k}(e)]^{1/p}} \tag{11}$$

(see [MS], Lemma 2.2.2/1, Corollary 3.2.1/1, Remark 3.2.1/1 and [AH], Sect. 4.4).

Using estimates (2) for the capacity of a ball, one obtains the following relations from (11)

$$\|\gamma\|_{M(B^k_p\to L_p)} \sim \|\gamma\|_{L_{p,\mathrm{unif}}} \quad \text{for } pk>n, \tag{12}$$

$$\|\gamma\|_{M(B^k_p\to L_p)} \ge c \sup_{x\in\mathbf{R}^n, r\in(0,1)} r^{k-n/p}\|\gamma;\mathcal{B}_r(x)\|_{L_p} \quad \text{for } pk<n, \tag{13}$$

$$\|\gamma\|_{M(B^k_p\to L_p)} \ge c \sup_{x\in\mathbf{R}^n, r\in(0,1)} \big(\log\frac{2}{r}\big)^{(p-1)/p}\|\gamma;\mathcal{B}_r(x)\|_{L_p} \quad \text{for } pk=n. \tag{14}$$

Lemma 2. *Let γ_ρ denote a mollifier of a function γ which is defined as*

$$\gamma_\rho(x) = \rho^{-n}\int K\big(\rho^{-1}(x-\xi)\big)\gamma(\xi)d\xi,$$

where $K\in C_0^\infty(\mathcal{B}_1)$, $K\ge 0$, and $\|K\|_{L_1}=1$. The inequalities

$$\|\gamma_\rho\|_{M(B^m_p\to B^l_p)} \le \|\gamma\|_{M(B^m_p\to B^l_p)} \le \liminf_{\rho\to 0}\|\gamma_\rho\|_{M(B^m_p\to B^l_p)},$$

$$\|\gamma_\rho\|_{M(B^m_p\to L_p)} \le \|\gamma\|_{M(B^m_p\to L_p)} \le \liminf_{\rho\to 0}\|\gamma_\rho\|_{M(B^m_p\to L_p)},$$

and

$$\sup_e \frac{\|C_{p,l}\gamma_\rho;e\|_{L_p}}{[\mathrm{cap}_{p,m}(e)]^{1/p}} \le \sup_e \frac{\|C_{p,l}\gamma;e\|_{L_p}}{[\mathrm{cap}_{p,m}(e)]^{1/p}}$$

are valid.

Proof. The proof of two-sided estimates is the same as in Lemma 3.2.1/1 [MS]. By Minkowski's inequality

$$\frac{\|C_{p,l}\gamma_\rho;e\|_{L_p}}{[\mathrm{cap}_{p,m}(e)]^{1/p}} \le \frac{\displaystyle\int K(z)\Big(\int_e \big(C_{p,l}\gamma(x-\rho z)\big)^p dx\Big)^{1/p}dz}{[\mathrm{cap}_{p,m}(e)]^{1/p}}$$

$$\le \frac{\displaystyle\int_{\mathcal{B}_1} K(z)\Big(\int_E \big(C_{p,l}\gamma(\xi)\big)^p d\xi\Big)^{1/p}dz}{[\mathrm{cap}_{p,m}(E)]^{1/p}} \le \|K\|_{L_1}\sup_e \frac{\|C_{p,l}\gamma;e\|_{L_p}}{[\mathrm{cap}_{p,m}(e)]^{1/p}}$$

where $E=\{x-\rho z : x\in e, z\in\mathcal{B}_1\}$.

Below we use the interpolation properties

$$B_p^{m-k} = \Big(B_p^m,\ H_p^{m-l}\Big)_{k/l,p} \tag{15}$$

and

$$B_p^{m-k} = \Big(B_p^m,\ B_p^{m-l}\Big)_{k/l,p}, \tag{16}$$

where $l < k < m$ (see, [Tr], Th. 2.4.2). In particular, (16) implies

$$\|\gamma\|_{MB_p^r} \le c\ \|\gamma\|^{\theta}_{MB_p^\sigma}\ \|\gamma\|^{1-\theta}_{MB_p^\rho}\ , \tag{17}$$

where $p \in (1, \infty)$, $\sigma > \rho > 0$, $0 < \theta < 1$, and $r = \theta\sigma + (1-\theta)\rho$. It follows from (11) and (16) that $\gamma \in M(B_p^m \to B_p^l) \cap M(B_p^{m-l} \to L_p)$ implies $\gamma \in M(B_p^{m-k} \to B_p^{l-k})$ for $0 < k < l$. Moreover,

$$\|\gamma\|_{M(B_p^{m-k} \to B_p^{l-k})} \le c\|\gamma\|^{1-k/l}_{M(B_p^m \to B_p^l)}\|\gamma\|^{k/l}_{M(B_p^{m-l} \to L_p)} \tag{18}$$

for $0 < k < l < m$ and

$$\|\gamma\|_{MB_p^{l-k}} \le c\|\gamma\|^{1-k/l}_{MB_p^l}\|\gamma\|^{k/l}_{L_\infty} \tag{19}$$

for $0 < k < l$.

In what follows we shall use five following assertions proved in the book [MS].

Lemma 3. (see [MS], Lemma 3.1.2/1) *Let $\mathcal{M}$ be the Hardy-Littlewood maximal operator defined by*

$$\mathcal{M}v(x) = \sup_{t>0} \frac{1}{\mathrm{mes}_n \mathcal{B}_t} \int_{\mathcal{B}_t(x)} |v(y)| dy.$$

Also let $J_r^{(n+s)}$ denote the Bessel potential in $\mathbf{R}^{n+s}$, $s \ge 1$. Then, for any nonnegative function $f \in L_p(\mathbf{R}^{n+s})$

$$(J^{(n+s)}_{r\theta+s/p} f)(x,0) \le c\left((J^{(n+s)}_{r+s/p} f)(x,0)\right)^{\theta} \left(\mathcal{M}F(x)\right)^{1-\theta},$$

where $F(x) = \|f(x,\cdot); \mathbf{R}^s\|_{L_p}$ and $0 < \theta < 1$.

Lemma 4. (see [MS], Lemma 3.2.1/3) *For any nonnegative function $\varphi \in L_{p\mu,loc}$, $p \in (1, \infty)$, and $0 < \lambda \le \mu$, there holds*

$$\sup_e \left(\frac{\int_e \varphi^{\lambda p}(x) dx}{\mathrm{cap}_{p,\lambda}(e)}\right)^{1/\lambda} \le c\ \sup_e \left(\frac{\int_e \varphi^{\mu p}(x) dx}{\mathrm{cap}_{p,\mu}(e)}\right)^{1/\mu}. \tag{20}$$

Lemma 5. (see [MS], Lemma 3.1.1./1) *For any positive $\alpha > 0$ and $\beta > 0$ there holds inequalities*

$$(C_{p,\alpha} u)(x) \le \big(J_\beta C_{p,\alpha} \Lambda^\beta u\big)(x), \tag{21}$$

$$(D_{p,\alpha}u)(x) \le \big(J_\beta D_{p,\alpha}\Lambda^\beta u\big)(x). \qquad (22)$$

Lemma 6. (see Lemma 3.9.1 [MS]). *For* $\delta \in (0,1)$ *and any* $k \ge 1$ *there holds*

$$\Big(\int\int |\Delta_h\gamma(x)\Delta_h u(x)|^p |h|^{-n-p} dh dx\Big)^{1/p} \le c \sup_e \frac{\|C_{p,\delta}\gamma; e\|_{L_p}}{[\mathrm{cap}_{p,k-1+\delta}(e)]^{1/p}} \|u\|_{B_p^k}. \quad (23)$$

Lemma 7. (see Lemma 3.1.1/2 [MS]) *For any* α, $\beta > 0$ *with* $\alpha + \beta < 1$ *there holds*

$$\|D_{p,\alpha}D_{p,\beta}u\|_{L_p} \le c\,\|D_{p,\alpha+\beta}u\|_{L_p}.$$

3 Lower estimates of the norm in $M(B_p^m \to B_p^l)$

The following is the main result of this section.

Lemma 8. *Let* $0 < l \le m$ *and* $p \in (1,\infty)$. *Then*

$$\|\gamma\|_{L_\infty} \le \|\gamma\|_{MB_p^l} \qquad \textit{for } m = l \qquad (24)$$

and

$$\|\gamma\|_{M(B_p^{m-l}\to L_p)} \le c\,\|\gamma\|_{M(B_p^m \to B_p^l)} \qquad \textit{for } m > l. \qquad (25)$$

Proof. Let $u \in B_p^l$ and let N be a positive integer. Clearly,

$$\|\gamma^N u\|_{L_p}^{1/N} \le \|\gamma^N u\|_{B_p^l}^{1/N} \le \|\gamma\|_{MB_p^l}\,\|u\|_{B_p^l}^{1/N}.$$

Passing to the limit as $N \to \infty$ we arrive at (24).

Now suppose $0 < l < m$. Let γ_ρ be the mollification of $\gamma \in M(B_p^m \to B_p^l)$. By Lemma 2, it suffices to prove (25) for γ_ρ. To simplify the notation we write γ in place of γ_ρ.

We consider two cases: $m \ge 2l$ and $2l > m > l$. Assume first that $m \ge 2l$. Let $U \in H_p^{m-l+1/p}(\mathbf{R}^{n+1})$ denote an extension of the function $u \in B_p^{m-l}(\mathbf{R}^n)$ to $\mathbf{R}^{n+1}$ such that

$$\|U;\mathbf{R}^{n+1}\|_{H_p^{m-l+1/p}} \le c\|u;\mathbf{R}^n\|_{B_p^{m-l}}. \qquad (26)$$

It is standard that the converse estimate

$$\|u;\mathbf{R}^n\|_{B_p^{m-l}} \le c\|U;\mathbf{R}^{n+1}\|_{H_p^{m-l-1/p}} \qquad (27)$$

holds for all extensions U. Let us represent U as the Bessel potential $J_{m-l+1/p}^{(n+1)}f$ with density $f \in L_p(\mathbf{R}^{n+1})$. By Lemma 3,

$$|u(x)| \le c\,\big((J_{m+1/p}^{(n+1)}|f|)(x,0)\big)^{(m-l)/l}\big(\mathcal{M}F(x)\big)^{l/m},$$

where $F(x) = \|f(x,\cdot);\mathbf{R}\|_{L_p}$. Therefore,

$$\|\gamma u\|_{L_p} \leq c\,\|f;\mathbf{R}^{n+1}\|^{l/m}_{L_p}\,\|\,|\gamma|^{m/(m-l)}\big(J^{(n+1)}_{m+1/p}|f|\big)(\cdot,0)\|^{(m-l)/m}_{L_p}.$$

The right-hand side does not exceed

$$c\,\|f;\mathbf{R}^{n+1}\|^{l/m}_{L_p}\,\|\gamma\,\big(J^{(n+1)}_{m+1/p}|f|\big)(\cdot,0)\|^{(m-l)/m}_{B^l_p}\,\sup_e\left(\frac{\int_e|\gamma|^{\frac{pl}{m-l}}dx}{\mathrm{cap}_{p,l}(e)}\right)^{(m-l)/mp}. \quad (28)$$

Setting $\varphi = |\gamma|^{\frac{1}{m-l}}$, $\lambda = l$, $\mu = m-l$ in Lemma 4, we find that in the case $m \geq 2l$ the supremum in (28) is dominated by

$$c\left(\sup_e\frac{\int_e|\gamma|^p dx}{\mathrm{cap}_{p,m-l}(e)}\right)^{l/mp} \leq c\,\|\gamma\|^{l/m}_{M(B^{m-l}_p\to L_p)}.$$

Hence and by (28)

$$\|\gamma u\|_{L_p} \leq c\|f;\mathbf{R}^{n+1}\|^{l/m}_{L_p}\|\gamma\|^{(m-l)/m}_{M(B^m_p\to B^l_p)}\|J^{(n+1)}_{m+1/p}|f|(\cdot,0)\|^{(m-l)/m}_{B^m_p}\|\gamma\|^{l/m}_{M(B^{m-l}_p\to L_p)}.$$

Using first (27) and then (10) and (26), we obtain

$$\|J^{(n+1)}_{m+1/p}|f|(\cdot,0)\|_{B^m_p} \leq c\,\|J^{(n+1)}_{m+1/p}|f|;\mathbf{R}^{n+1}\|_{H^{m+1/p}_p} = c\|f;\mathbf{R}^{n+1}\|_{L_p}$$

$$= c\|U;\mathbf{R}^{n+1}\|_{H^{m-l+1/p}_p} \leq c\,\|u;\mathbf{R}^n\|_{B^{m-l}_p}.$$

Thus,

$$\|\gamma u\|_{L_p} \leq c\,\|\gamma\|^{l/m}_{M(B^{m-l}_p\to L_p)}\|\gamma\|^{(m-l)/m}_{M(B^m_p\to B^l_p)}\|u\|_{B^{m-l}_p},$$

which implies (25) for $m \geq 2l$.

Suppose $2l > m > l$. Let μ be an arbitrary positive number less than $m-l$. By (18) with $k = l-\mu$,

$$\|\gamma\|_{M(B^{m-l+\mu}_p\to B^\mu_p)} \leq c\,\|\gamma\|^{(l-\mu)/l}_{M(B^{m-l}_p\to L_p)}\|\gamma\|^{\mu/l}_{M(B^m_p\to B^l_p)}.$$

Since $m-l+\mu > 2\mu$, it follows from the first part of the proof that there holds inequality (25) with m and l replaced by $m-l+\mu$ and μ, respectively, i.e.

$$\|\gamma\|_{M(B^{m-l}_p\to L_p)} \leq c\,\|\gamma\|_{M(B^{m-l+\mu}_p\to B^\mu_p)}.$$

Consequently,

$$\|\gamma\|_{M(B^{m-l}_p\to L_p)} \leq c\,\|\gamma\|^{(l-\mu)/l}_{M(B^{m-l}_p\to L_p)}\|\gamma\|^{\mu/l}_{M(B^m_p\to B^l_p)}$$

and (25) is proved for $2l > m > l$ as well.

By Lemma 8 and (11), the following assertion holds.

Corollary 1. *Let* $\gamma \in M(B_p^m \to B_p^l)$, $0 < l < m$. *Then*

$$\sup_e \frac{\|\gamma; e\|_{L_p}}{[\mathrm{cap}_{p,m-l}(e)]^{1/p}} \leq c\, \|\gamma\|_{M(B_p^m \to B_p^l)}.$$

Lemma 8 in combination with (18) and (19) implies

Corollary 2. *Let* $\gamma \in M(B_p^m \to B_p^l)$, $0 < l \leq m$. *Then* $\gamma \in M(B_p^{m-k} \to B_p^{l-k})$, $0 < k < l$, *and*

$$\|\gamma\|_{M(B_p^{m-k} \to B_p^{l-k})} \leq c \|\gamma\|_{M(B_p^m \to B_p^l)}.$$

The following assertion contains an estimate for derivatives of a multiplier.

Lemma 9. *Let* $\gamma \in M(B_p^m \to B_p^l)$, $0 < l \leq m$. *Then* $D^\alpha \gamma \in M(B_p^m \to B_p^{l-|\alpha|})$ *for any multi-index* α *of order* $|\alpha| \leq l$. *The inequality holds*

$$\|D^\alpha \gamma\|_{M(B_p^m \to B_p^{l-|\alpha|})} \leq c\, \|\gamma\|_{M(B_p^m \to B_p^l)}.$$

Proof. It suffices to consider the case $|\alpha| = 1$, $l \geq 1$. Clearly,

$$\|u \nabla \gamma\|_{B_p^{l-1}} \leq \|u\gamma\|_{B_p^l} + \|\gamma \nabla u\|_{B_p^{l-1}}$$

$$\leq \big(\|\gamma\|_{M(B_p^m \to B_p^l)} + \|\gamma\|_{M(B_p^{m-1} \to B_p^{l-1})} \big) \|u\|_{B_p^m}.$$

This and Corollary 2 imply

$$\|u \nabla \gamma\|_{B_p^{l-1}} \leq c\, \|\gamma\|_{M(B_p^m \to B_p^l)} \|u\|_{B_p^m}$$

which completes the proof.

Lemmas 8 and 9 imply the following

Corollary 3. *Let* $\gamma \in M(B_p^m \to B_p^l)$, $0 < l \leq m$. *Then, for any multi index* α *of order* $|\alpha| \leq l$, $D^\alpha \gamma \in M(B_p^{m-|\alpha|} \to L_p)$. *The inequality holds*

$$\|D^\alpha \gamma\|_{M(B_p^{m-l+|\alpha|} \to L_p)} \leq c\, \|\gamma\|_{M(B_p^m \to B_p^l)}.$$

4 Proof of necessity in Theorem 1

In this section we derive the inequalities

$$\sup_e \frac{\|C_{p,l}\gamma; e\|_{L_p}}{[\mathrm{cap}_{p,m}(e)]^{1/p}} + \sup_{x \in \mathbf{R}^n} \|\gamma; \mathcal{B}_1(x)\|_{L_p} \leq c \|\gamma\|_{M(B_p^m \to B_p^l)}, \quad m > l \tag{29}$$

and

$$\sup_e \frac{\|C_{p,l}\gamma; e\|_{L_p}}{[\mathrm{cap}_{p,l}(e)]^{1/p}} + \|\gamma\|_{L_\infty} \leq c \|\gamma\|_{MB_p^l}. \tag{30}$$

The core of the proof is the following assertion.

Lemma 10. *Let $\gamma \in M(B_p^m \to B_p^l)$, where $0 < l \le m$ and $p \in (1, \infty)$. Then*

$$\sup_e \frac{\|C_{p,l}\gamma; e\|_{L_p}}{[\mathrm{cap}_{p,m}(e)]^{1/p}} \le c\,\|\gamma\|_{M(B_p^m \to B_p^l)}. \tag{31}$$

Proof. We use induction in l and start by showing that (31) is valid for $l \in (0,1]$.

(i) Let $l \in (0,1)$. We have

$$\|uC_{p,l}\gamma\|_{L_p} \le c\big(\|\gamma u\|_{B_p^l} + \|\gamma C_{p,l} u\|_{L_p}\big)$$

$$\le c\big(\|\gamma\|_{M(B_p^m \to B_p^l)}\|u\|_{B_p^l} + \|\gamma C_{p,l} u\|_{L_p}\big). \tag{32}$$

Consider first the case $m = l$. Clearly, $\|\gamma C_{p,l} u\|_{L_p} \le \|\gamma\|_{L_\infty}\|u\|_{B_p^l}$ which together with (32) and (24) gives

$$\|uC_{p,l}\gamma\|_{L_p} \le c\,\|\gamma\|_{MB_p^l}\|u\|_{B_p^l}.$$

Therefore, $\|C_{p,l}\gamma\|_{M(B_p^l \to L_p)} \le c\|\gamma\|_{MB_p^l}$ and, in view of (11), we obtain (31).

Suppose now that $l < m$. By (21)

$$\|\gamma C_{p,l} u\|_{L_p} \le \|\gamma\|_{M(B_p^{m-l} \to L_p)}\|J_{m-l}C_{p,l}\Lambda^{m-l}u\|_{B_p^{m-l}}. \tag{33}$$

Owing to Lemma 1, the last norm does not exceed

$$c\,\|C_{p,l}\Lambda^{m-l}u\|_{L_p} \le c\,\|\Lambda^{m-l}u\|_{B_p^l} \le c\,\|u\|_{B_p^m}$$

which in combination with with (33) implies

$$\|\gamma C_{p,l} u\|_{L_p} \le c\,\|\gamma\|_{M(B_p^{m-l} \to L_p)}\|u\|_{B_p^m}. \tag{34}$$

Using (32), (34) and Lemma 8, we arrive at

$$\|uC_{p,l}\gamma\|_{L_p} \le c\,\|\gamma\|_{M(B_p^m \to B_p^l)}\|u\|_{B_p^m}.$$

Thus,

$$\|C_{p,l}\gamma\|_{M(B_p^m \to L_p)} \le c\,\|\gamma\|_{M(B_p^m \to B_p^l)}$$

which together with (11) gives (31).

(ii) Let $l = 1$. In view of the identity

$$\Delta_h^{(2)}(\gamma u) = \gamma\Delta_h^{(2)}u + u\Delta_h^{(2)}\gamma + \Delta_{2h}\gamma\Delta_{2h}u - 2\Delta_h\gamma\Delta_h u \tag{35}$$

one has

$$\|uC_{p,1}\gamma\|_{L_p} \le \|\gamma u\|_{B_p^1} + \|\gamma C_{p,1} u\|_{L_p}$$

$$+4\Big(\int\int |\Delta_h\gamma(x)\Delta_h u(x)|^p|h|^{-n-p}dhdx\Big)^{1/p} \tag{36}$$

for any $u\in C_0^\infty$.

We proceed separately for $m=1$ and $m>1$. Let first $m=1$. Using (23) with $k=1$ and $\delta\in(0,1)$ together with (36) and (24), we find

$$\|uC_{p,1}\gamma\|_{L_p}\le c\,\Big(\|\gamma\|_{MB_p^1}+\sup_e\frac{\|C_{p,\delta}\gamma;e\|_{L_p}}{[\mathrm{cap}_{p,\delta}(e)]^{1/p}}\Big)\|u\|_{B_p^1}. \tag{37}$$

In view of part (i) of this proof, the last supremum is majorized by $c\|\gamma\|_{MB_p^\delta}$. Hence (37) leads to the inequality

$$\sup_e\frac{\|C_{p,1}\gamma;e\|_{L_p}}{[\mathrm{cap}_{p,1}(e)]^{1/p}}\le c\,(\|\gamma\|_{MB_p^1}+\|\gamma\|_{MB_p^\delta}). \tag{38}$$

Since by Corollary 2 there holds $\|\gamma\|_{MB_p^\delta}\le c\|\gamma\|_{MB_p^1}$, we arrive at (31) for $m=l=1$.

Next we estimate the right-hand side of (36) for $m>1$. By (21), its second term is majorized by

$$\begin{aligned}\|\gamma J_{m-1}C_{p,1}\Lambda^{m-1}u\|_{L_p}&\le c\,\|\gamma\|_{M(B_p^{m-1}\to L_p)}\|J_{m-1}C_{p,1}\Lambda^{m-1}u\|_{B_p^{m-1}}\\&\le c\,\|\gamma\|_{M(B_p^{m-1}\to L_p)}\|C_{p,1}\Lambda^{m-1}u\|_{L_p}\\&\le c\,\|\gamma\|_{M(B_p^{m-1}\to L_p)}\|\Lambda^{m-1}u\|_{B_p^1}\le c\,\|\gamma\|_{M(B_p^m\to B_p^1)}\|u\|_{B_p^m}.\end{aligned} \tag{39}$$

The last inequality in this chain follows from (9) and (25). We estimate the third term in the right-hand side of (36) using (23) with $k=m>1$ and (31) with $l=\delta<1$. Then this term does not exceed

$$c\,\sup_e\frac{\|C_{p,\delta}\gamma;e\|_{L_p}}{[\mathrm{cap}_{p,m-1+\delta}(e)]^{1/p}}\|u\|_{B_p^m}\le c\,\|\gamma\|_{M(B_p^{m-1+\delta}\to B_p^\delta)}\|u\|_{B_p^m}. \tag{40}$$

Furthermore, by Corollary 2

$$\|\gamma\|_{M(B_p^{m-1+\delta}\to B_p^\delta)}\le c\,\|\gamma\|_{M(B_p^m\to B_p^1)}.$$

Therefore, the third term on the right in (36) is dominated by $c\,\|\gamma\|_{M(B_p^m\to B_p^1)}\|u\|_{B_p^m}$. This along with (36) and (39) implies

$$\|uC_{p,1}\gamma\|_{L_p}\le c\,\|\gamma\|_{M(B_p^m\to B_p^1)}\|u\|_{B_p^m}$$

and thus (31) holds for $l=1$.

(iii) Suppose that l is a positive integer and that the lemma is proved for $\gamma\in M(B_p^m\to B_p^k)$, where k is any positive integer not exceeding $l-1$. Applying (35), we find

$$\|uC_{p,l}\gamma\|_{L_p}\le\|\gamma u\|_{B_p^l}+c\sum_{j=0}^{l-1}\|\,|\nabla_j\gamma|C_{p,l-j}u\|_{L_p}+c\sum_{j=1}^{l-1}\|\,|\nabla_j u|C_{p,l-j}\gamma\|_{L_p}$$

$$+c\sum_{j=0}^{l-1}\Big(\int\int|\Delta_h\nabla_j\gamma(x)|^p|\Delta_h\nabla_{l-1-j}u|^p|h|^{-n-p}dhdx\Big)^{1/p}. \tag{41}$$

By (21) with $\alpha = l-j$, $\beta = m-l+j$ we have

$$(C_{p,l-j}u)(x) \le (J_{m-l+j}C_{p,l-j}\Lambda^{m-l+j}u)(x).$$

Therefore, for $j = 1,\dots,l-1$ and $m \ge l$,

$$\|\,|\nabla_j\gamma|C_{p,l-j}u\|_{L_p} \le c\,\|\nabla_j\gamma\|_{M(B_p^{m-l+j}\to L_p)}\|J_{m-l+j}C_{p,l-j}\Lambda^{m-l+j}u\|_{B_p^{m-l+j}}$$

$$\le c\,\|\nabla_j\gamma\|_{M(H_p^{m-l+j}\to L_p)}\|C_{p,l-j}\Lambda^{m-l+j}u\|_{L_p}. \tag{42}$$

According to (9),

$$\|C_{p,l-j}\Lambda^{m-l+j}u\|_{L_p} \le \|\Lambda^{m-l+j}u\|_{B_p^{l-j}} \le c\,\|u\|_{B_p^m}. \tag{43}$$

By Corollary 3,

$$\|\nabla_j\gamma\|_{M(H_p^{m-l+j}\to L_p)} \le c\,\|\gamma\|_{M(B_p^m\to B_p^l)}, \quad j = 1,\dots,l-1,\ \ m \ge l. \tag{44}$$

For $j = 0$ by Lemma 8 we obtain

$$\|\gamma C_{p,l}u\|_{L_p} \le \|\gamma\|_{M(B_p^m\to B_p^l)}\|u\|_{B_p^m}. \tag{45}$$

Unifying (42)-(45), we find that for all $j = 0,\dots,l-1$ and $1 \le l \le m$,

$$\|\,|\nabla_j\gamma|C_{p,l-j}u\|_{L_p} \le c\,\|\gamma\|_{M(B_p^m\to B_p^l)}\|u\|_{B_p^m}. \tag{46}$$

For any $j = 1,\dots,l-1$ we have

$$\|\,|\nabla_j u|C_{p,l-j}\gamma\|_{L_p} \le c\sup_e \frac{\|C_{p,l-j}\gamma; e\|_{L_p}}{[\text{cap} - p, m-j(e)]^{1/p}}\|u\|_{B_p^m}. \tag{47}$$

From the induction assumption and Corollary 2 it follows that for $m \ge l$ one has

$$\sup_e \frac{\|C_{p,l-j}\gamma; e\|_{L_p}}{[\text{cap}_{p,m-j}(e)]^{1/p}} \le c\,\|\gamma\|_{M(B_p^{m-j}\to B_p^{l-j})} \le c\,\|\gamma\|_{M(B_p^m\to B_p^l)} \tag{48}$$

which together with (47) implies

$$\|\,|\nabla_j u|C_{p,l-j}\gamma\|_{L_p} \le c\,\|\gamma\|_{M(B_p^m\to B_p^l)}\|u\|_{B_p^m}, \quad j = 1,\dots,l-1. \tag{49}$$

Next we estimate the last sum in (41). Let $\delta \in (0,1)$ be such that $m+\delta$ is a noninteger. By (23) with γ replaced by $\nabla_j\gamma$, u replaced by $\nabla_{l-1-j}u$, and $k = m-l+j+1$ each term of the last sum in (41) does not exceed

$$c\sup_e \frac{\|C_{p,j+\delta}\gamma; e\|_{L_p}}{[\text{cap}_{p,m-l+j+\delta}(e)]^{1/p}}\|\nabla_{l-1-j}u\|_{B_p^{m-l+j+1}} \tag{50}$$

By the induction assumption and Corollary 2 this implies

$$\Big(\int\int|\Delta_h\nabla_j\gamma(x)|^p|\Delta_h\nabla_{l-1-j}u|^p|h|^{-n-p}dhdx\Big)^{1/p}$$

$$\leq c\|\gamma\|_{M(B_p^{m-l+j+\delta}\to B_p^{j+\delta})}\|u\|_{B_p^m}\leq c\|\gamma\|_{M(B_p^m\to B_p^l)}\|u\|_{B_p^m}. \tag{51}$$

Combining this with (49) and (47), we obtain from (41)

$$\|uC_{p,l}\gamma\|_{L_p}\leq c\,\|\gamma\|_{M(B_p^m\to B_p^l)}\|u\|_{B_p^m} \tag{52}$$

and thus (31) follows for all integer l.

(iv) Now let l be noninteger. Suppose that

$$\sup_e\frac{\|C_{p,l}\gamma;e\|_{L_p}}{[\mathrm{cap}_{p,m}(e)]^{1/p}}\leq c\,\|\gamma\|_{M(B_p^m\to B_p^l)}$$

for all noninteger $l\in(0,N)$, where N is integer. Let $N<l<N+1$. In view of the equivalence $C_{p,l}\gamma\sim D_{p,l}\gamma$ we have

$$\|uD_{p,l}\gamma\|_{L_p}\leq\|\gamma u\|_{B_p^l}+c\sum_{j=0}^N\|\,|\nabla_j\gamma|D_{p,l-j}u\|_{L_p}+c\sum_{j=1}^N\|\,|\nabla_ju|D_{p,l-j}\gamma\|_{L_p}. \tag{53}$$

Let $t\in(0,m-l+j)$ if $m>l$ or $m=l$, $j>0$ and let $t=0$ if $m=l$ and $j=0$. By (22) with $\alpha=l-j$ and $\beta=t$ one has

$$\big(D_{p,l-j}u\big)(x)\leq\big(J_tD_{p,l-j}\Lambda^tu\big)(x).$$

Hence

$$\|\,|\nabla_j\gamma|D_{p,l-j}u\|_{L_p}\leq\|\nabla_j\gamma\|_{M(W_p^{m-l+j}\to L_p)}\|J_tD_{p,l-j}\Lambda^tu\|_{W_p^{m-l+j}}$$

$$\leq c\,\|\nabla_j\gamma\|_{M(B_p^{m-l+j}\to L_p)}\|D_{p,l-j}\Lambda^tu\|_{W_p^{m-l+j-t}}. \tag{54}$$

By definition of the operator $D_{p,l}$ and the space W_p^l,

$$\|D_{p,l-j}v\|_{W_p^{m-l+j-t}}=\|D_{p,m-l+j-t}D_{p,\{l\}}\nabla_{[l-j]}v\|_{L_p}+\|D_{p,l-j}v\|_{L_p}.$$

We use Lemma 7 with $\alpha=m-l+j-t$, $\beta=\{l\}$ assuming t to be so close to $m-l+j$ that $0<m-t-[l]+j<1$. Then

$$\|D_{p,m-l+j-t}D_{p,\{l\}}\nabla_{[l]-j}v\|_{L_p}\leq c\|D_{p,m-t-[l]-j}\nabla_{[l]-j}v\|_{L_p}\leq c\|v\|_{W_p^{m-t}}. \tag{55}$$

We may also choose t in such a way that $m-t$ is noninteger so that $W_p^{m-t}=B_p^{m-t}$. Then (54) and (55) with $v=\Lambda^tu$, together with Corollary 3 imply

$$\|\,|\nabla_j\gamma|D_{p,l-j}u\|_{L_p}\leq c\,\|\nabla_j\gamma\|_{M(B_p^{m-l+j}\to L_p)}\|\Lambda^tu\|_{B_p^{m-t}}$$

$$\leq c\,\|\gamma\|_{M(B_p^m\to B_p^l)}\|u\|_{B_p^m}. \tag{56}$$

By the induction hypothesis, we have for $j=1,\dots,N$

$$\|\,|\nabla_j u|D_{p,l-j}\gamma\|_{L_p}\leq c\sup_e \frac{\|D_{p,l-j}\gamma;e\|_{L_p}}{[\mathrm{cap}_{p,m-j}(e)]^{1/p}}\|\nabla_j u\|_{B_p^{m-j}}$$

$$\leq c\,\|\gamma\|_{M(B_p^{m-j}\to B_p^{l-j})}\|u\|_{B_p^m} \tag{57}$$

which together with Corollary 2 implies

$$\|\,|\nabla_j u|D_{p,l-j}\gamma\|_{L_p}\leq c\,\|\gamma\|_{M(B_p^m\to B_p^l)}\|u\|_{B_p^m}.$$

Hence and by (56) it follows from (53) that

$$\|uD_{p,l}\gamma\|_{L_p}\leq c\,\|\gamma\|_{M(B_p^m\to B_p^l)}\|u\|_{B_p^m}.$$

The proof is complete.

The following simple corollary contains the required lower estimate of the norm in $M(B_p^m\to B_p^l)$ in Theorem 1. It also finishes the proof of necessity in Theorem 1.

Corollary 4. *Let $\gamma\in M(B_p^m\to B_p^l)$, where $0<l\leq m$ and $p\in(1,\infty)$. Then*

$$c\Big(\sup_e \frac{\|C_{p,l}\gamma;e\|_{L_p}}{[\mathrm{cap}_{p,m}(e)]^{1/p}}+\sup_{x\in\mathbf{R}^n}\|\gamma;\mathcal{B}_1(x)\|_{L_p}\Big)\leq\|\gamma\|_{M(B_p^m\to B_p^l)}. \tag{58}$$

For $m=l$ the second term on the left should be replaced by $\|\gamma\|_{L_\infty}$.

Proof. Since $\gamma\in M(B_p^m\to B_p^l)$ it follows that

$$\|\gamma\eta\|_{L_p}\leq\|\gamma\|_{M(B_p^m\to B_p^l)}\|\eta\|_{B_p^m}$$

for any $\eta\in C_0^\infty(\mathcal{B}_2(x))$, $\eta=1$ on $\mathcal{B}_1(x)$, where x is an arbitrary point of $\mathbf{R}^n$. Therefore,

$$\sup_{x\in\mathbf{R}^n}\|\gamma;\mathcal{B}_1(x)\|_{L_p}\leq c\,\|\gamma\|_{M(B_p^m\to B_p^l)}.$$

The result follows by combining this with Lemma 10.

The next corollary contains one more lower estimate for the norm in the space $M(B_p^m\to B_p^l)$.

Corollary 5. *Let $\gamma\in M(B_p^m\to B_p^l)$, where $0<l\leq m$, $p\in(1,\infty)$. Then, for any $k=0,\dots,[l]$ there holds the inclusion $C_{p,l-k}\gamma\in M(B_p^{m-k}\to L_p)$ and*

$$\|C_{p,l-k}\gamma\|_{M(B_p^{m-k}\to L_p)}\leq c\,\|\gamma\|_{M(B_p^m\to B_p^l)}.$$

Proof. By Corollaries 4 and 2,

$$\sup_e \frac{\|C_{p,l-k}\gamma;e\|_{L_p}}{[\mathrm{cap}_{p,m-k}(e)]^{1/p}}\leq c\,\|\gamma\|_{M(B_p^{m-k}\to B_p^{l-k})}\leq c\,\|\gamma\|_{M(B_p^m\to B_p^l)}. \tag{59}$$

It remains to make use of (11).

5 Proof of sufficiency in Theorem 1

The aim of this section is to prove the upper estimate of $\|\gamma\|_{M(B_p^m\to B_p^l)}$ in (4).

Lemma 11. *Let $\gamma\in B^l_{p,loc}$, $p\in(1,\infty)$. Then for $m>l$*

$$c\,\|\gamma\|_{M(B_p^m\to B_p^l)}\le \sup_{e,\mathrm{diam}(e)\le 1}\Big(\frac{\|C_{p,l}\gamma;e\|_{L_p}}{[\mathrm{cap}_{p,m}(e)]^{1/p}}+\frac{\|\gamma,e\|_{L_p}}{[\mathrm{cap}_{p,m-l}(e)]^{1/p}}\Big). \tag{60}$$

For $m=l$ the second term should be replaced by $\|\gamma\|_{L_\infty}$.

Proof. It follows from the finiteness of the right-hand side of (60) that $\gamma\in L_{1,\mathrm{unif}}$. Let γ_ρ denote the mollifyer of γ with radius ρ. From $\gamma\in L_{1,\mathrm{unif}}$ it follows that all derivatives of γ_ρ are bounded. Hence $\gamma_\rho\in M(B_p^m\to B_p^l)$.

For integer l we find by (35) that there holds the estimate

$$\|\gamma_\rho u\|_{B_p^l}\le c\Big(\sum_{j=0}^{l-1}\|\,|\nabla_j\gamma_\rho|C_{p,l-j}u\|_{L_p}+\sum_{j=0}^{l-1}\|\,|\nabla_j u|C_{p,l-j}\gamma_\rho\|_{L_p}$$

$$+\sum_{j=0}^{l-1}\Big(\int\int|\Delta_h\nabla_j\gamma_\rho(x)|^p|\Delta_h\nabla_{l-1-j}u|^p|h|^{-n-p}dhdx\Big)^{1/p}\Big). \tag{61}$$

By Corollary 3, for any $\alpha\in(0,1)$

$$\|\nabla_j\gamma_\rho\|_{M(B_p^{m-l+j}\to L_p)}\le c\,\|\gamma_\rho\|_{M(B_p^{m-l+j+\alpha}\to B_p^{j+\alpha})}. \tag{62}$$

In view of (18), for $m>l$ the right-hand side in (62) does not exceed

$$c\|\gamma_\rho\|^{(l-j-\alpha)/l}_{M(B_p^{m-l}\to L_p)}\|\gamma_\rho\|^{(j+\alpha)/l}_{M(B_p^m\to B_p^l)}$$

Combining this with (42) and (43) we obtain

$$\|\,|\nabla_j\gamma_\rho|C_{p,l-j}u\|_{L_p}\le\big(\varepsilon\|\gamma_\rho\|_{M(B_p^m\to B_p^l)}+c(\varepsilon)\|\gamma_\rho\|_{M(B_p^{m-l}\to L_p)}\big)\|u\|_{B_p^m}, \tag{63}$$

where $j=0,\dots,l-1$, and ε is an arbitrary positive number.

In case $m=l$ inequalities (62) and (19) imply

$$\|\nabla_j\gamma_\rho\|_{M(B_p^j\to L_p)}\le c\|\gamma_\rho\|^{(l-j)/l}_{L_\infty}\|\gamma_\rho\|^{j/l}_{MB_p^l}.$$

unifying this with (42) and (43) for $m=l$ we obtain

$$\|\,|\nabla_j\gamma_\rho|C_{p,l-j}u\|_{L_p}\le\big(\varepsilon\|\gamma_\rho\|_{MB_p^l}+c(\varepsilon)\|\gamma_\rho\|_{L_\infty}\big)\|u\|_{B_p^l}. \tag{64}$$

It follows from (47), (48), and (18), (19) that for $j>0$

$$\|\,|\nabla_j u|C_{p,l-j}\gamma_\rho\|_{L_p}\le\big(\varepsilon\|\gamma_\rho\|_{M(B_p^m\to B_p^l)}+c(\varepsilon)\|\gamma_\rho\|_{M(B_p^{m-l}\to L_p)}\big)\|u\|_{B_p^m}, \tag{65}$$

if $m > l$ and

$$\| \, |\nabla_j u| C_{p,l-j}\gamma_\rho\|_{L_p} \leq \big(\varepsilon\|\gamma_\rho\|_{MB_p^l} + c(\varepsilon)\|\gamma_\rho\|_{L_\infty}\big)\|u\|_{B_p^l} \tag{66}$$

if $m = l$.

The third sum in the right-hand side of (61) is estimated by using (51) and (18), (19) and has the same majorant as the right-hand side of (65) for $m > l$ or (66) for $m = l$. Thus, for $m > l$ we find

$$\|\gamma_\rho u\|_{B_p^l} \leq \Big(\varepsilon\|\gamma_\rho\|_{M(B_p^m\to B_p^l)} + c(\varepsilon)\|\gamma_\rho\|_{M(B_p^{m-l}\to L_p)}$$
$$+c \sup_{e,\mathrm{diam}(e)\leq 1} \frac{\|C_{p,l}\gamma_\rho; e\|_{L_p}}{[\mathrm{cap}_{p,m}(e)]^{1/p}}\Big)\|u\|_{B_p^m}. \tag{67}$$

Similarly, for $m = l$,

$$\|\gamma_\rho u\|_{B_p^l} \leq \Big(\varepsilon\|\gamma_\rho\|_{MB_p^l} + c(\varepsilon)\|\gamma_\rho\|_{L_\infty} + c \sup_{e,\mathrm{diam}(e)\leq 1} \frac{\|C_{p,l}\gamma_\rho; e\|_{L_p}}{[\mathrm{cap}_{p,l}(e)]^{1/p}}\Big)\|u\|_{B_p^l}. \tag{68}$$

For noninteger l the following estimate, simpler than (61), holds

$$\|\gamma_\rho u\|_{B_p^l} \leq c\Big(\sum_{j=0}^{[l]-1} \| \, |\nabla_j \gamma_\rho| C_{p,l-j} u\|_{L_p} + \sum_{j=0}^{[l]-1} \| \, |\nabla_j u| C_{p,l-j}\gamma_\rho\|_{L_p}\Big)$$

Combining (56) with Corollary 3 and (18), (19), we arrive at (63) and (64) in the same way as for integer l. We also note that (57) and (18) for $m > l$ and (19) for $m = l$ imply (65) and (66) for noninteger l. Reference to (11) and Lemma 2 completes the proof.

The required upper estimate of $\|\gamma\|_{MB_p^l}$ in (4) is obtained in Lemma 11. In order to show that for $m > l$ the second term on the right in (60) can be replaced by $\|\gamma\|_{L_{1,\mathrm{unif}}}$, we need several auxiliary assertions. Let $\gamma(x, y)$ denote the Poisson integral of a function $\gamma \in L_{1,\mathrm{unif}}$.

Lemma 12. (see Lemma 5.1.2 [MS]) *Let l be noninteger and let $\gamma \in W_{1,\mathrm{loc}}^{[l]}$. Then*

$$\Big(\int_0^\infty \Big|\frac{\partial^{[l]+1}\gamma(x,y)}{\partial y^{[l]+1}}\Big|^p y^{p-1-p\{l\}} dy\Big)^{1/p} \leq c\,(D_{p,l}\gamma)(x).$$

Lemma 13. (Verbitsky, see Sect. 2.6 [MS]) *For any $k = 0, 1, \dots$ there holds the inequality*

$$|\gamma(x)| \leq c\Big(\|\gamma\|_{L_{1,\mathrm{unif}}} + \int_0^1 \Big|\frac{\partial^{k+1}\gamma(x,y)}{\partial y^{k+1}}\Big| y^k dy\Big). \tag{69}$$

The following two lemmas are similar to those due to Verbitsky as presented in Sect. 2.6 [MS].

Lemma 14. *Let* $\gamma \in W^{[l]}_{1,loc}$, $y \in (0,1]$. *Then*

$$\Big|\frac{\partial^{[l]+1}\gamma(x,y)}{\partial y^{[l]+1}}\Big| \le c y^{\{l\}-m-1} \sup_{x\in\mathbf{R}^n, r\in(0,1)} r^{m-n/p}\|D_{p,l}\gamma;\mathcal{B}_r(x)\|_{L_p}.$$

Proof. We introduce the notation

$$K = \sup_{x\in\mathbf{R}^n, r\in(0,1)} r^{m-n/p}\|D_{p,l}\gamma;\mathcal{B}_r(x)\|_{L_p}. \tag{70}$$

Let $r \in (0,1]$. By Lemma 12

$$\int_{\mathcal{B}_r(x)}\int_0^\infty \Big|\frac{\partial^{[l]+1}\gamma(x,y)}{\partial y^{[l]+1}}\Big|^p y^{p-1-p\{l\}} dy\, dt \le cK^p r^{n-mp}. \tag{71}$$

Applying the mean value theorem for harmonic functions we find for $\frac{r}{2} < y < \frac{2r}{3}$

$$\Big|\frac{\partial^{[l]+1}\gamma(x,y)}{\partial y^{[l]+1}}\Big| \le c\, r^{-n-1}\int_{\mathcal{B}_r(x)}\int_{r/4}^r \Big|\frac{\partial^{[l]+1}\gamma(t,\eta)}{\partial \eta^{[l]+1}}\Big|\, d\eta dt.$$

By Hölder's inequality the right-hand side is dominated by

$$c\, r^{\{l\}-1-n/p}\Big(\int_{\mathcal{B}_r(x)}\int_{r/4}^r \Big|\frac{\partial^{[l]+1}\gamma(t,\eta)}{\partial \eta^{[l]+1}}\Big|^p \eta^{p-1-p\{l\}} d\eta dt\Big)^{1/p}$$

which by (71) does not exceed $cr^{\{l\}-m-1}K$. The proof is complete.

Lemma 15. *Let* $\gamma \in W^{[l]}_{1,loc}$. *Then for all* $x \in \mathbf{R}^n$ *there holds inequality*

$$|\gamma(x)| \le c\Big(\big(\sup_{x\in\mathbf{R}^n, r\in(0,1)} r^{m-n/p}\|D_{p,l}\gamma;\mathcal{B}_r(x)\|_{L_p}\big)^{l/m}(D_{p,l}\gamma(x))^{(m-l)/m} + \|\gamma\|_{L_{1,\mathrm{unif}}}\Big).$$

Proof. We put

$$v(y) = \begin{cases} \Big|\dfrac{\partial^{[l]+1}\gamma(x,y)}{\partial y^{[l]+1}}\Big| & \text{for } 0 < y \le 1, \\ 0 & \text{for } y > 1. \end{cases}$$

Then, for any $R > 0$

$$\int_0^1 \Big|\frac{\partial^{[l]+1}\gamma(x,y)}{\partial y^{[l]+1}}\Big| y^{[l]} dy = \int_0^\infty v(y) y^{[l]} dy = \int_0^R v(y) y^{[l]} dy + \int_R^\infty v(y) y^{[l]} dy.$$

Applying Hölder's inequality, we find

$$\int_0^R v(y) y^{[l]} dy \le c\, R^l \Big(\int_0^R (v(y))^p y^{p-p\{l\}-1} dy\Big)^{1/p}.$$

By Lemma 14,
$$\Big|\frac{\partial^{[l]+1}\gamma(x,y)}{\partial y^{[l]+1}}\Big| \leq cKy^{\{l\}-m-1},$$
where K is defined by (70). Hence
$$\int_0^\infty v(y)y^{[l]}dy \leq c\Big(R^l\Big(\int_0^\infty (v(y))^p y^{p-p\{l\}-1}dy\Big)^{1/p} + R^{l-m}K\Big).$$
Putting here
$$R = K^{1/m}\Big(\int_0^\infty v(y)^p y^{p-p\{l\}-1}dy\Big)^{-1/pm},$$
we arrive at
$$\int_0^\infty v(y)y^{[l]}dy \leq cK^{l/m}\Big(\int_0^\infty v(y)^p y^{p-p\{l\}-1}dy\Big)^{(m-l)/pm}.$$
Combining this with (69) for $k = [l]$ we arrive at
$$|\gamma(x)| \leq \Big(K^{l/m}\Big(\int_0^\infty v(y)^p y^{p-p\{l\}-1}dy\Big)^{(m-l)/pm} + \|\gamma\|_{L_{1,\mathrm{unif}}}\Big).$$
Reference to Lemma 12 completes the proof.

Now, we are in a position to prove the principle result of this section.

Lemma 16. *Let $0 < l < m$, $p \in (1,\infty)$. Then*
$$\|\gamma\|_{M(B_p^m \to B_p^l)} \leq c\Big(\sup_{e,\mathrm{diam}(e)\leq 1} \frac{\|C_{p,l}\gamma; e\|_{L_p}}{[\mathrm{cap}_{p,m}(e)]^{1/p}} + \|\gamma\|_{L_{1,\mathrm{unif}}}\Big). \tag{72}$$

Proof. By (20) with $\varphi = |\gamma_\rho|^{\frac{1}{m-l}}$, $\lambda = m-l$, $\mu = m-\varepsilon$, where ε is a positive number less than l such that both $l-\varepsilon$ and $m-\varepsilon$ are nonintegers, we find
$$\sup_e \frac{\displaystyle\int_e |\gamma_\rho|^p(x)dx}{\mathrm{cap}_{p,m-l}(e)} \leq c\,\sup_e\left(\frac{\displaystyle\int_e |\gamma_\rho|^{\frac{m-\varepsilon}{m-l}p}(x)dx}{\mathrm{cap}_{p,m-\varepsilon}(e)}\right)^{\frac{m-l}{m-\varepsilon}} \tag{73}$$
Owing to Lemma 15 with l replaced by $l-\varepsilon$ and m replaced by $m-\varepsilon$
$$\int_e |\gamma_\rho|^{\frac{(m-\varepsilon)p}{m-l}}dx \leq c\Big(\Big(\sup_{x\in\mathbf{R}^n, r\in(0,1)} r^{m-\varepsilon-\frac{n}{p}}\|D_{p,l-\varepsilon}\gamma_\rho; \mathcal{B}_r(x)\|_{L_p}\Big)^{\frac{(l-\varepsilon)p}{m-l}} \times$$
$$\int_e |D_{p,l-\varepsilon}\gamma_\rho|^p dx + \|\gamma_\rho\|_{L_{1,\mathrm{unif}}}^{\frac{(m-\varepsilon)p}{m-l}}\ \mathrm{mes}_n e\Big).$$
Hence

$$\left(\frac{\int_e |\gamma_\rho|^{\frac{(m-\varepsilon)p}{m-l}}(x)dx}{\mathrm{cap}_{p,m-\varepsilon}(e)}\right)^{\frac{m-l}{(m-\varepsilon)p}} \le c\Big\{\Big(\sup_{x\in\mathbf{R}^n, r\in(0,1)} r^{m-\varepsilon-\frac{n}{p}}\|D_{p,l-\varepsilon}\gamma_\rho;\mathcal{B}_r(x)\|_{L_p}\Big)^{\frac{l-\varepsilon}{m-\varepsilon}}\times$$

$$\Big(\sup_e \frac{\|D_{p,l-\varepsilon}\gamma_\rho; e\|_{L_p}}{[\mathrm{cap}_{p,m-\varepsilon}(e)]^{1/p}}\Big)^{\frac{m-l}{m-\varepsilon}} + \|\gamma_\rho\|_{L_{1,\mathrm{unif}}}\Big\}. \tag{74}$$

By Corollary 2

$$\sup_e \frac{\|D_{p,l-\varepsilon}\gamma_\rho; e\|_{L_p}}{[\mathrm{cap}_{p,m-\varepsilon}(e)]^{1/p}} \le c\,\|\gamma_\rho\|_{M(W_p^{m-\varepsilon}\to W_p^{l-\varepsilon})}$$

$$= c\,\|\gamma_\rho\|_{M(B_p^{m-\varepsilon}\to B_p^{l-\varepsilon})} \le c\,\|\gamma_\rho\|_{M(B_p^m\to B_p^l)}.$$

Thus, the left-hand side of (74) has the majorant

$$c\Big(\big(\sup_{x\in\mathbf{R}^n, r\in(0,1)} r^{m-\varepsilon-\frac{n}{p}}\|D_{p,l-\varepsilon}\gamma_\rho;\mathcal{B}_r(x)\|_{L_p}\big)^{\frac{l-\varepsilon}{m-\varepsilon}}\|\gamma_\rho\|^{\frac{m-l}{m-\varepsilon}}_{M(B_p^m\to B_p^l)} + \|\gamma_\rho\|_{L_{1,\mathrm{unif}}}\Big)$$

which together with (73) implies the inequality

$$\sup_e\left(\frac{\int_e |\gamma_\rho|^p(x)dx}{\mathrm{cap}_{p,m-l}(e)}\right)^{1/p} \le c(\delta)\sup_{x\in\mathbf{R}^n, r\in(0,1)} r^{m-\varepsilon-\frac{n}{p}}\|D_{p,l-\varepsilon}\gamma_\rho;\mathcal{B}_r(x)\|_{L_p}$$

$$+\delta\|\gamma_\rho\|_{M(B_p^m\to B_p^l)} + c\,\|\gamma_\rho\|_{L_{1,\mathrm{unif}}}, \tag{75}$$

where δ is an arbitrary positive number.

Next we show that

$$\sup_{x\in\mathbf{R}^n, r\in(0,1)} r^{m-\varepsilon-\frac{n}{p}}\|D_{p,l-\varepsilon}\gamma_\rho;\mathcal{B}_r(x)\|_{L_p}$$

$$\le c(\sigma)\sup_{x\in\mathbf{R}^n, r\in(0,1)} r^{m-\frac{n}{p}}\|C_{p,l}\gamma_\rho;\mathcal{B}_r(x)\|_{L_p} + \sigma\,\|\gamma_\rho\|_{M(B_p^m\to B_p^l)} \tag{76}$$

where σ is an arbitrary positive number. We note that by (1) $D_{p,l-\varepsilon}\gamma_\rho$ can be replaced by $C_{p,l-\varepsilon}\gamma_\rho$. Let ω denote a positive number to be chosen later. Further, let $k = l-1$ and $\lambda = 1$ for integer l and $k = [l]$ and $\lambda = \{l\}$ for noninteger l. We have

$$\int_{\mathcal{B}_r(x)} dy \int_{\mathcal{B}_{\omega r}} \frac{|\nabla_k\gamma_\rho(y+2h) - 2\nabla_k\gamma_\rho(y+h) + \nabla_k\gamma_\rho(y)|^p}{|h|^{n+p(\lambda-\varepsilon)}}dh$$

$$\le (\omega r)^{p\varepsilon}\int_{\mathcal{B}_r(x)}dy\int_{\mathcal{B}_{\omega r}}\frac{|\nabla_k\gamma_\rho(y+2h)-2\nabla_k\gamma_\rho(y+h)+\nabla_k\gamma_\rho(y)|^p}{|h|^{n+p\lambda}}dh$$

$$\le (\omega r)^{p\varepsilon}\|C_{p,l}\gamma_\rho;\mathcal{B}_r(x)\|^p_{L_p}. \tag{77}$$

Besides,

$$\int_{\mathcal{B}_r(x)}dy\int_{\mathbf{R}^n\backslash\mathcal{B}_{\omega r}}\frac{|\nabla_k\gamma_\rho(y+2h)-2\nabla_k\gamma_\rho(y+h)+\nabla_k\gamma_\rho(y)|^p}{|h|^{n+p(\lambda-\varepsilon)}}dh$$

$$\le c\Big(\int_{\mathcal{B}_r(x)}dy\int_{\mathbf{R}^n\backslash\mathcal{B}_{\omega r}}\frac{|\nabla_k\gamma_\rho(y+2h)|^p}{|h|^{n+p(\lambda-\varepsilon)}}dh+\int_{\mathcal{B}_r(x)}dy\int_{\mathbf{R}^n\backslash\mathcal{B}_{\omega r}}\frac{|\nabla_k\gamma_\rho(y+h)|^p}{|h|^{n+p(\lambda-\varepsilon)}}dh$$

$$+(\omega r)^{p(\varepsilon-\lambda)}\|\nabla_k\gamma_\rho;\mathcal{B}_r(x)\|^p_{L_p}\Big). \tag{78}$$

Further, we have

$$\int_{\mathcal{B}_r(x)}dy\int_{\mathbf{R}^n\backslash\mathcal{B}_{\omega r}}\frac{|\nabla_k\gamma_\rho(y+2h)|^p}{|h|^{n+p(\lambda-\varepsilon)}}dh$$

$$\le\int_{\mathbf{R}^n\backslash\mathcal{B}_{\omega r}}\frac{dh}{|h|^{n+p(\lambda-\varepsilon)}}\int_{\mathcal{B}_r(x+2h)}|\nabla_k\gamma_\rho(z)|^pdz$$

$$\le c\omega^{p(\varepsilon-\lambda)}r^{n-pm+p\varepsilon}\sup_{x\in\mathbf{R}^n,r\in(0,1)}r^{p(m-\lambda)-n}\|\nabla_k\gamma_\rho;\mathcal{B}_r(x)\|^p_{L_p}.$$

By (12)–(14) the last supremum is dominated by

$$c\,\|\nabla_k\gamma_\rho\|^p_{M(W_p^{m-\lambda}\to L_p)}$$

which by Corollary 3 does not exceed $c\,\|\gamma_\rho\|^p_{M(B_p^m\to B_p^l)}$.

Clearly, the second term in the right in the right-hand side of (78) is estimated in the same way. Similarly, the third term does not exceed

$$c\,\omega^{p(\varepsilon-\lambda)}r^{n-pm+p\varepsilon}\|\gamma_\rho\|^p_{M(B_p^m\to B_p^l)}.$$

Hence

$$\int_{\mathcal{B}_r(x)}dy\int_{\mathbf{R}^n\backslash\mathcal{B}_{\omega r}}\frac{|\nabla_k\gamma_\rho(y+2h)-2\nabla_k\gamma_\rho(y+h)+\nabla_k\gamma_\rho(y)|^p}{|h|^{n+p(\lambda-\varepsilon)}}dh$$

$$\le c\,\omega^{p(\varepsilon-\lambda)}r^{n-pm+p\varepsilon}\|\gamma_\rho\|^p_{M(B_p^m\to B_p^l)}. \tag{79}$$

From (77) and (79) we obtain

$$r^{m-\varepsilon-n/p}\|D_{p,l-\varepsilon}\gamma_\rho\|_{L_p}\le c\big(\omega^\varepsilon r^{m-n/p}\|C_{p,l}\gamma_\rho;\mathcal{B}_r(x)\|_{L_p}+\omega^{\varepsilon-\lambda}\|\gamma_\rho\|_{M(B_p^m\to B_p^l)}\big).$$

Setting $\sigma = c\omega^{\varepsilon-\lambda}$ we arrive at (76).

By (12)–(14) and (76),

$$\sup_{x\in\mathbf{R}^n, r\in(0,1)} r^{m-\varepsilon-\frac{n}{p}} \|D_{p,l-\varepsilon}\gamma_\rho; \mathcal{B}_r(x)\|_{L_p}$$
$$\leq c(\sigma) \sup_e \frac{\|C_{p,l}\gamma_\rho; e\|_{L_p}}{[\mathrm{cap}_{p,m}(e)]^{1/p}} + \sigma\|\gamma_\rho\|_{M(B_p^m\to B_p^l)},$$

which together with (75) and Lemma 11 results at

$$\|\gamma_\rho\|_{M(B_p^m\to B_p^l)} \leq c\Big(\sup_e \frac{\|C_{p,l}\gamma_\rho; e\|_{L_p}}{[\mathrm{cap}_{p,m}(e)]^{1/p}} + \|\gamma_\rho\|_{L_{1,\mathrm{unif}}}\Big). \tag{80}$$

Estimating the right-hand side of (80) by Lemma 2 and using the equivalence (see, Proposition 2.1. 5 [MS])

$$\mathrm{cap}_{p,m}(e) \sim \sum_{j\geq 1} \mathrm{cap}_{p,m}(e\cap\mathcal{B}^{(j)}),$$

where $\{\mathcal{B}^{(j)}\}_{j\geq 0}$ is a covering of $\mathbf{R}^n$ by balls of diameter one with multiplicity depending only on n, we complete the proof.

6 The case $mp > n$

For $mp > n$ Theorem 1 admits a simpler formulation.

Corollary 6. *Let $0 < l < m$, $mp > n$, and $p \in (1,\infty)$. Then*

$$\|\gamma\|_{M(B_p^m\to B_p^l)} \sim \sup_{x\in\mathbf{R}^n} \big(\|C_{p,l}\gamma; \mathcal{B}_1(x)\|_{L_p} + \|\gamma; \mathcal{B}_1(x)\|_{L_p}\big). \tag{81}$$

For $m = l$ the second term on the right should be replaced by $\|\gamma\|_{L_\infty}$.

Proof. The lower estimate of $\|\gamma\|_{M(B_p^m\to B_p^l)}$ follows from the relation

$$\mathrm{cap}_{p,m}(e) \sim 1 \tag{82}$$

valid for $mp > n$ and e with $\mathrm{diam}(e) \leq 1$, combined with Corollary 4. The upper estimate results from

$$\|\gamma\|_{M(B_p^m\to B_p^l)} \leq \|\gamma\|_{MB_p^l} \leq c\,\big(\sup_{e,\mathrm{diam}(e)\leq 1} \|C_{p,l}\gamma; e\|_{L_p} + \|\gamma\|_{L_\infty}\big)$$
$$\leq c \sup_{x\in\mathbf{R}^n} \big(\|C_{p,l}\gamma; \mathcal{B}_1(x)\| + \|\gamma; \mathcal{B}_1(x)\|_{L_p}\big).$$

The proof is complete.

Remark 1. One can easily verify that the right-hand side in (81) is equivalent to the norm of γ in $B^l_{p,\mathrm{unif}}$. Hence $M(B_p^m \to B_p^l)$ is isomorphic to $B^l_{p,\mathrm{unif}}$ for $0 < l < m$, $mp > n$, $p \in (1,\infty)$.

7 The space $M(W_p^m \to W_p^{-k})$

Let W_p^m denote the usual Sobolev space with $p \in (1,\infty)$ and integer m, and let W_p^{-k} stand for the dual space $(W_{p'}^k)^\star$, $p+p' = pp'$. In [MS], the following sufficient condition for inclusion into the distribution space $M(W_p^m \to W_p^{-k})$ can be found. We supply it with the proof for completeness and reader's convenience.

Theorem 2. (see Sect. 1.5 [MS]) (i) *Let* $p \in (1,\infty)$*,* $m \le k$*. If*

$$\gamma = \sum_{|\alpha|\le k} D^\alpha \gamma_\alpha \tag{83}$$

with

$$\gamma_\alpha \in M(W_{p'}^k \to W_{p'}^{k-m}) \cap M(W_p^m \to L_p), \tag{84}$$

then $\gamma \in M(W_p^m \to W_p^{-k})$.

(ii) *Let* $p \in (1,\infty)$*,* $m \ge k$*. If*

$$\gamma = \sum_{|\alpha|\le m} D^\alpha \gamma_\alpha$$

with

$$\gamma_\alpha \in M(W_p^m \to W_p^{m-k}) \cap M(W_{p'}^k \to L_{p'}),$$

then $\gamma \in M(W_p^m \to W_p^{-k})$.

Proof. It suffices to prove only (i) since (ii) follows from (i) by duality.

Let $u \in W_p^m$, $m \le k$. Since

$$uD^\alpha\gamma_\alpha = \sum_{\lambda\le\alpha} c_{\lambda\alpha} D^\lambda(\gamma_\alpha D^{\alpha-\lambda} u), \quad c_{\lambda\alpha} = \text{const},$$

we have

$$\|\gamma u\|_{W_p^{-k}} \le c \sum_{|\lambda|\le|\alpha|\le k} \|\gamma_\alpha D^{\alpha-\lambda} u\|_{W_p^{|\lambda|-k}}$$

$$\le c \sum_{|\lambda|\le|\alpha|\le k} \|\gamma_\alpha\|_{M(W_p^{m-k+|\lambda|} \to W_p^{|\lambda|-k})} \|u\|_{W_p^{m+|\alpha|+k}}. \tag{85}$$

Applying the interpolation inequality

$$\|\gamma_\alpha\|_{M(W_p^{m-k+|\lambda|} \to W_p^{|\lambda|-k})} \le c\|\gamma_\alpha\|^{(k-|\lambda|)/k}_{M(W_p^{m-k} \to W_p^{-k})} \|\gamma_\alpha\|^{|\lambda|/k}_{M(W_p^m \to L_p)}$$

which follows from the interpolation property of Sobolev spaces (see [Tr], Sect. 2.4) we obtain from from (85)

$$\|\gamma u\|_{W_p^{-k}} \le c\,(\|\gamma_\alpha\|_{M(W_p^{m-k} \to W_p^{-k})} + \|\gamma_\alpha\|_{M(W_p^m \to L_p)})\|u\|_{W_p^m}.$$

It remains to note that

$$\|\gamma_\alpha\|_{M(W_p^{m-k}\to W_p^{-k})} = \|\gamma_\alpha\|_{M(W_{p'}^k\to W_{p'}^{k-m})}.$$

The following assertion shows that this theorem provides a complete characterisation of $M(W_p^m \to W_p^{-k})$ which holds under some conditions involving k, m, p, and n.

Corollary 7. *Let k and m be positive integers and let either $k \geq m > 0$ and $k > n/p'$ or $m \geq k > 0$ and $m > n/p$. Then $\gamma \in M(W_p^m \to W_p^{-k})$ if and only if*

$$\gamma \in W_{p,\mathrm{unif}}^{-k} \cap W_{p',\mathrm{unif}}^{-m}. \tag{86}$$

In particular, if $\max\{k,m\} > n/2$ then $M(W_2^m \to W_2^{-k})$ is isomorphic to $W_2^{-\min\{m,k\}}$.

Proof. It suffices to consider the case $k \geq m > 0$, $k > n/p'$, because the case $m \geq k > 0$, $m > n/p$ results by duality.

Necessity. It follows from the inclusion $\gamma \in M(W_p^m \to W_p^{-k})$ that $\gamma \in W_{p,\mathrm{unif}}^{-k}$. Since $M(W_p^m \to W_p^{-k})$ is isomorphic to $M(W_{p'}^k \to W_{p'}^{-m})$, we have $\gamma \in W_{p',\mathrm{unif}}^{-m}$ as well.

Sufficiency. It is standard and easily proved (compare with Sect. 1.1.14 [M]) that $\gamma \in W_{p,\mathrm{unif}}^{-k} \cap W_{p',\mathrm{unif}}^{-m}$ if and only if (83) holds with $\gamma_\alpha \in L_{p,\mathrm{unif}} \cap W_{p',\mathrm{unif}}^{k-m}$. Since $M(W_{p'}^k \to W_{p'}^{k-m})$ is isomorphic to $W_{p',\mathrm{unif}}^{k-m}$ for $p'k > n$, it follows that $\gamma_\alpha \in M(W_{p'}^k \to W_{p'}^{k-m})$.

It remains to show that $\gamma_\alpha \in M(W_p^m \to L_p)$. We choose q and r to satisfy

$$1/q > \max\{0, 1/p - m/n\} > -\varepsilon + 1/q,$$

$$1/r > \max\{0, 1/p' - (k-m)/n\} > -\varepsilon + 1/r$$

with a sufficiently small ε. Since $1/p > 1 - k/n$, we have $1/p > 1/q + 1/r$. By Hölder's inequality

$$\|\gamma_\alpha u\|_{L_{p,\mathrm{unif}}} \leq c\, \|\gamma_\alpha\|_{L_{r,\mathrm{unif}}} \|u\|_{L_{q,\mathrm{unif}}}$$

and by Sobolev's imbedding theorem

$$\|\gamma_\alpha u\|_{L_{p,\mathrm{unif}}} \leq c\, \|\gamma_\alpha\|_{W_{p',\mathrm{unif}}^{k-m}} \|u\|_{W_{p,\mathrm{unif}}^m}.$$

This means that $\gamma_\alpha \in M(W_p^m \to L_p)$. The proof is completed by reference to assertion (i) of Theorem 2.

Remark 2. Note that by Sobolev's imbedding theorems $W_{p',\mathrm{unif}}^{-m} \subset W_{p,\mathrm{unif}}^{-k}$, $k \geq m$, if and only if either $n \leq (k-m)p$ or

$$n > (k-m)p, \quad \frac{k-m}{n} \geq \frac{2-p}{p}.$$

Under these conditions, $M(W_p^m \to W_p^{-k})$ is isomorphic to $W_{p',\mathrm{unif}}^{-m}$ if $kp' > n$. Analogously, if $m \geq k$, $mp > n$ and either $n \leq (m-k)p'$ or

$$n > (m-k)p', \quad \frac{m-k}{n} \geq \frac{p-2}{p},$$

then $M(W_p^m \to W_p^{-k})$ is isomorphic to $W_{p,\mathrm{unif}}^{-k}$.

We finish by stating a direct but important application of Corollary 8 to the theory of differential operators.

Corollary 9. *Let k and m be integers and let $\mathcal{L}_{m+k}(D)$ denote a differential operator of order $m+k$ with constant coefficients. If either $k \geq m$ and $kp' > n$, or $m \geq k > 0$ and $mp > n$ then the operator*

$$W_p^m \ni u \to \mathcal{L}(D)u + \gamma(x)u \in W_p^{-k}$$

is continuous if and only if

$$\gamma \in W_{p,\mathrm{unif}}^{-k} \cap W_{p',\mathrm{unif}}^{-m}.$$

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