Sharp Bohr's type real part estimates

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Abstract. We consider analytic functions f in the unit disk D with Taylor coefficients $c_0, c_1, \dots$ and derive estimates with sharp constants for the l_q-norm (quasi-norm for $0 < q < 1$) of the remainder of their Taylor series, where $q \in (0, \infty]$. As the main result, we show that given a function f with $\Re f$ in the Hardy space $h_1(D)$ of harmonic functions on D, the inequality

$$\left\{ \sum_{n=m}^{\infty} |c_n z^n|^q \right\}^{1/q} \leq \frac{2r^m}{(1-r^q)^{1/q}} \, ||\Re f||_{h_1}$$

holds with the sharp constant, where $r = |z| < 1$, $m \geq 1$. This estimate implies sharp inequalities for l_q-norms of the Taylor series remainder for bounded analytic functions, analytic functions with bounded $\Re f$, analytic functions with $\Re f$ bounded from above, as well as for analytic functions with $\Re f > 0$. In particular, we prove that

$$\left\{ \sum_{n=m}^{\infty} |c_n z^n|^q \right\}^{1/q} \leq \frac{2r^m}{(1-r^q)^{1/q}} \sup_{|\zeta|<1} \Re\{f(\zeta) - f(0)\}.$$

As corollary of the above estimate with $||\Re f||_{h_1}$ in the right-hand side, we obtain some sharp Bohr's type modulus and real part inequalities.

Keywords. Taylor series, Bohr's inequality, Hadamard's real part theorem.

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0. Introduction

The present article is connected with two classical assertions of the analytic functions theory, namely with Hadamard's real part theorem and Bohr's theorem on the majorant of a Taylor's series.

Hadamard's real part theorem states that the inequality

$$(0.1) \qquad |f(z) - f(0)| \leq \frac{C|z|}{1-|z|} \sup_{|\zeta|<1} \Re\{f(\zeta) - f(0)\}$$

holds for analytic functions in the disk $D = \{z : |z| < 1\}$. This inequality for analytic functions vanishing at $z = 0$ was first obtained by Hadamard with $C = 4$ in 1892 [28]. The sharp constant $C = 2$ in (0.1) was found by Borel [17, 18] and Carathéodory (see

Landau [31, 32]). A detailed historical survey on these and other related inequalities for analytic functions can be found in the paper by Jensen [29].

A refined Bohr's theorem [14], as stated by M. Riesz, I. Schur, F. Wiener (see Landau [33]), claims that any function

$$(0.2) \qquad f(z) = \sum_{n=0}^{\infty} c_n z^n,$$

analytic and bounded in D, obeys the inequality

$$(0.3) \qquad \sum_{n=0}^{\infty} |c_n z^n| \leq \sup_{|\zeta|<1} |f(\zeta)|,$$

where $|z| \leq 1/3$. Moreover, the value $1/3$ of the radius cannot be improved.

In the present paper we deal, similarly to Aizenberg, Grossman and Korobeinik [7], Bénéteau, Dahlner and Khavinson [11], Djakov and Ramanujan [25], with the value of l_q-norm (quasi-norm, for $0 < q < 1$) of the remainder of the Taylor series (0.2).

In Section 1, we prove the inequality

$$(0.4) \qquad \left\{ \sum_{n=m}^{\infty} |c_n z^n|^q \right\}^{1/q} \leq \frac{2r^m}{(1-r^q)^{1/q}} \, ||\Re f||_{h_1},$$

with the sharp constant, where $r = |z| < 1$, $m \geq 1$, $0 < q \leq \infty$, and

$$||g||_{h_1} = \sup_{\varrho<1} \frac{1}{2\pi\varrho} \int_{|\zeta|=\varrho} |g(\zeta)||d\zeta|$$

is the norm in the Hardy space $h_1(D)$ of harmonic functions on D.

Section 2 contains corollaries of (0.4) for analytic functions f in D with bounded $\Re f$, with $\Re f$ bounded from above, with $\Re f > 0$, as well as for bounded analytic functions. In particular, we obtain the estimate

$$\left\{ \sum_{n=m}^{\infty} |c_n z^n|^q \right\}^{1/q} \leq \frac{2r^m}{(1-r^q)^{1/q}} \, \sup_{|\zeta|<1} \Re\{f(\zeta) - f(0)\},$$

with the best possible constant. This estimate, taken with $q = 1$, $m = 1$, is an improvement of (0.1) with $C = 2$. Other inequalities, which follow from (0.4), contain the supremum of $|\Re f(\zeta)| - |\Re f(0)|$ or $|f(\zeta)| - |f(0)|$ in D, as well as $\Re f(0)$ in the case $\Re f > 0$ on D. Each of these estimates specified for $q = 1$ and $m = 1$ improves a certain sharp Hadamard's type inequality.

Note that a sharp estimate of the full majorant series by the supremum modulus of f was obtained by Bombieri [15] for $r \in [1/3, 1/\sqrt{2}]$.

In Section 3 we give modifications of Bohr's theorem as consequences of our inequalities with sharp constants derived in Section 2. For example, if a function (0.2) is analytic on D, then for any $q \in (0, \infty]$, integer $m \geq 1$ and $|z| \leq r_{m,q}$ the inequality

$$(0.5) \qquad \left\{ \sum_{n=m}^{\infty} |c_n z^n|^q \right\}^{1/q} \leq \sup_{|\zeta|<1} \Re\{e^{-i\arg f(0)} f(\zeta)\} - |f(0)|$$

holds, where $r_{m,q}$ is the root of the equation

$$2^q r^{mq} + r^q - 1 = 0$$

in the interval $(0,1)$, and $r_{m,q}$ is the radius of the largest disk centered at $z = 0$ in which (0.5) takes place.

In particular,

(0.6) $\qquad r_{1,q} = (1 + 2^q)^{-1/q} \quad \text{and} \quad r_{2,q} = 2^{1/q}\big(1 + \sqrt{1 + 2^{q+2}}\,\big)^{-1/q}.$

Some of the inequalities presented in Section 3 contain known analogues of Bohr's theorem with $\Re f$ in the right-hand side (see Aizenberg, Aytuna and Djakov [4], Paulsen, Popescu and Singh [36], Sidon [39], Tomić [41]).

Multi-dimensional analogues and other generalizations of Bohr's theorem are treated in the papers by Aizenberg [1, 2, 3], Aizenberg, Aytuna and Djakov [4, 5], Aizenberg and Tarkhanov [6], Aizenberg, Liflyand and Vidras [8], Aizenberg and Vidras [9], Boas and Khavinson [12], Boas [13], Dineen and Timoney [22, 23], Djakov and Ramanujan [25].

Various interesting recent results related to Bohr's inequalities were obtained by Aizenberg, Grossman and Korobeinik [7], Bénéteau, Dahlner and Khavinson [11], Bombieri and Bourgain [16], Guadarrama [26], Defant and Frerick [21].

Certain problems of the operator theory connected with Bohr's theorem are examined by Defant, Garcia and Maestre [20], Dixon [24], Glazman and Ljubič [27], Nikolski [35], Paulsen, Popescu and Singh [36].

1. Sharp estimate of l_q -norm for the remainder of Taylor series by the norm of $\Re f$ in the Hardy space $h_1(D)$

In the sequel, we use the notation $r = |z|$ and $D_\varrho = \{z \in \mathbb{C} : |z| < \varrho\}$.

We start with a sharp inequality for an analytic function f in the disk $D = \{z \in \mathbb{C} : |z| < 1\}$. The right-hand side of the inequality contains the norm in the Hardy space $h_1(D)$ of harmonic functions on the disk

$$\|g\|_{h_1} = \sup_{\varrho < 1} M_1(g, \varrho),$$

where $M_1(g, \varrho)$ is the integral mean value of the function g on the circle $|\zeta| = \varrho < 1$ defined by

$$M_1(g, \varrho) = \frac{1}{2\pi\varrho} \int_{|\zeta|=\varrho} |g(\zeta)||d\zeta|.$$

Proposition 1. *Let the function (0.2) be analytic on D with $\Re f \in h_1(D)$, and let $q > 0$, $m \geq 1$, $|z| = r < 1$. Then the inequality*

$$(1.1) \qquad \left\{ \sum_{n=m}^{\infty} |c_n z^n|^q \right\}^{1/q} \leq \frac{2r^m}{(1-r^q)^{1/q}} \, ||\Re f||_{h_1}$$

holds with the sharp constant.

Proof. 1. Proof of inequality (1.1). Let a function f, analytic in D with $\Re f \in h_1(D)$ be given by (0.2). Since by the Schwarz formula

$$f^{(n)}(0) = \frac{n!}{\pi i} \int_{|\zeta|=\varrho} \frac{\Re f(\zeta)}{\zeta^{n+1}} d\zeta,$$

$0 < \varrho < 1$, $n = 1, 2, \ldots$, which leads to

$$|f^{(n)}(0)| \leq \frac{n!}{\pi \varrho^{n+1}} \int_{|\zeta|=\varrho} |\Re f(\zeta)||d\zeta| = \frac{2n!}{\varrho^n} \, M_1(\Re f, \varrho).$$

Hence, by the arbitrariness of $\varrho \in (0,1)$ we obtain

$$(1.2) \qquad |c_n| \leq 2 \, ||\Re f||_{h_1}$$

for any $n \geq 1$.

Using (1.2), we find

$$\left\{ \sum_{n=m}^{\infty} |c_n z^n|^q \right\}^{1/q} \leq 2 \left\{ \sum_{n=m}^{\infty} r^{nq} \right\}^{1/q} ||\Re f||_{h_1} = \frac{2r^m}{(1-r^q)^{1/q}} \, ||\Re f||_{h_1}$$

for any z with $|z| = r < 1$.

2. Sharpness of the constant in (1.1). By (1.1), proved above, the sharp constant $C(r)$ in

$$(1.3) \qquad \left\{ \sum_{n=m}^{\infty} |c_n z^n|^q \right\}^{1/q} \leq C(r) \, ||\Re f||_{h_1}$$

satisfies

$$(1.4) \qquad C(r) \leq \frac{2r^m}{(1-r^q)^{1/q}}.$$

We show that the converse inequality for $C(r)$ holds as well.

Let $\rho > 1$. Consider the families of analytic functions in $\overline{D}$

$$(1.5) \qquad g_\rho(z) = \frac{\rho}{z - \rho}, \qquad f_\rho(z) = g_\rho(z) - \beta_\rho,$$

depending on the parameter ρ, with the real constant β_ρ defined by

$$M_1(\Re g_\rho - \beta_\rho, 1) = \min_{c \in \mathbb{R}} M_1(\Re g_\rho - c, 1).$$

Then, for any real constant c

$$M_1(\Re f_\rho - c, 1) \geq M_1(\Re f_\rho, 1) = ||\Re f_\rho||_{h_1}.$$

Setting here

$$c = \mathcal{A}(f_\rho) = \max_{|\zeta|=1} \Re f_\rho(\zeta)$$

and taking into account

$$
\begin{aligned}
M_1\big(\Re f_\rho - \mathcal{A}(f_\rho), 1\big) &= \frac{1}{2\pi} \int_{|\zeta|=1} \big[\mathcal{A}(f_\rho) - \Re f_\rho(\zeta)\big]|d\zeta| \\
&= \mathcal{A}(f_\rho) - \Re f_\rho(0) = \max_{|\zeta|=1} \Re\{f_\rho(\zeta) - f_\rho(0)\},
\end{aligned}
$$

we arrive at

$$\text{(1.6)} \qquad \max_{|\zeta|=1} \Re\{f_\rho(\zeta) - f_\rho(0)\} \geq ||\Re f_\rho||_{h_1}.$$

In view of

$$c_n(\rho) = \frac{f_\rho^{(n)}(0)}{n!} = -\frac{1}{\rho^n} \quad \text{for} \ \ n \geq 1,$$

we find

$$\text{(1.7)} \qquad \sum_{n=m}^{\infty} |c_n(\rho) z^n|^q = \sum_{n=m}^{\infty} \left(\frac{r}{\rho}\right)^{nq} = \frac{r^{mq}}{\rho^{(m-1)q}(\rho^q - r^q)}.$$

Putting $z = \zeta = e^{i\vartheta}$ in (1.5), we obtain

$$\text{(1.8)} \qquad \max_{|\zeta|=1} \Re\{f_\rho(\zeta) - f_\rho(0)\} = \frac{1}{2} \max_{\vartheta} \left(1 - \frac{\rho^2 - 1}{\rho^2 - 2\rho\cos\vartheta + 1}\right) = \frac{1}{\rho + 1}.$$

It follows from (1.3), (1.6), (1.7) and (1.8) that

$$\text{(1.9)} \qquad C(r) \geq \frac{r^m(\rho + 1)}{\rho^{m-1}(\rho^q - r^q)^{1/q}}.$$

Passing to the limit as $\rho \downarrow 1$ in the last inequality, we obtain

$$\text{(1.10)} \qquad C(r) \geq \frac{2r^m}{(1 - r^q)^{1/q}},$$

which together with (1.4) proves the sharpness of the constant in (1.1). $\square$

Remark 1. In [30] we found an explicit formula for the constant $\mathcal{C}_p$ in the inequality

$$|c_n| \leq \mathcal{C}_p ||\Re f||_p$$

with $1 \leq p \leq \infty$. In particular, we showed that $\mathcal{C}_1 = 1/\pi$, $\mathcal{C}_2 = 1/\sqrt{\pi}$ and $\mathcal{C}_\infty = 4/\pi$.

2. Sharp estimates of l_q-norm for the remainder of Taylor series for certain classes of analytic functions

In this section we obtain estimates with sharp constants for the l_q-norm (quasi-norm for $0 < q < 1$) of the Taylor series remainder for bounded analytic functions and analytic functions whose real part is bounded or on-side bounded.

We start with a theorem concerning analytic functions with real part bounded from above which refines Hadamard's real part estimate (0.1).

Theorem 1. *Let the function* (0.2) *be analytic on D with*

$$\sup_{|\zeta|<1} \Re f(\zeta) < \infty,$$

and let $q > 0$, $m \geq 1$, $|z| = r < 1$. Then the inequality

$$(2.1) \qquad \left\{ \sum_{n=m}^{\infty} |c_n z^n|^q \right\}^{1/q} \leq \frac{2r^m}{(1-r^q)^{1/q}} \sup_{|\zeta|<1} \Re\{f(\zeta) - f(0)\}$$

holds with the sharp constant.

Proof. By dilation, we write (1.1) for the disk D_ϱ, $r < \varrho < 1$, with f replaced by $f - \omega$, where ω is an arbitrary real constant. Then

$$(2.2) \qquad \left\{ \sum_{n=m}^{\infty} |c_n z^n|^q \right\}^{1/q} \leq \frac{2r^m}{\varrho^{m-1}(\varrho^q - r^q)^{1/q}} \|\Re f - \omega\|_{h_1(D_\varrho)}.$$

Putting here

$$\omega = \boldsymbol{A}(f) = \sup_{|\zeta|<1} \Re f(\zeta)$$

and taking into account that

$$\|\Re f - \boldsymbol{A}(f)\|_{h_1(D_\varrho)} = M_1(\Re f - \boldsymbol{A}(f), \varrho) = \sup_{|\zeta|<1} \Re\{f(\zeta) - f(0)\},$$

we find

$$\left\{ \sum_{n=m}^{\infty} |c_n z^n|^q \right\}^{1/q} \leq \frac{2r^m}{\varrho^{m-1}(\varrho^q - r^q)^{1/q}} \sup_{|\zeta|<1} \Re\{f(\zeta) - f(0)\},$$

which implies (2.1) after the passage to the limit as $\varrho \uparrow 1$.

Hence, the sharp constant $C(r)$ in

$$(2.3) \qquad \left\{ \sum_{n=m}^{\infty} |c_n z^n|^q \right\}^{1/q} \leq C(r) \sup_{|\zeta|<1} \Re\{f(\zeta) - f(0)\}$$

obeys (1.4).

To get the lower estimate for $C(r)$, we shall use functions g_ρ given by (1.5). Taking into account equalities

$$g^{(n)}(0) = f^{(n)}(0), \quad \Re\{g_\rho(\zeta) - g_\rho(0)\} = \Re\{f_\rho(\zeta) - f_\rho(0)\}$$

as well as (2.3), (1.7) and (1.8), we arrive at (1.9). Passing to the limit in (1.9) as $\rho \downarrow 1$, we obtain (1.10), which together with (1.4) proves the sharpness of the constant

in (2.1). $\qquad\square$

Remark 2. Inequality (2.1) for $q = m = 1$ is well known (see, e.g. Polya and Szegö [37], III, Ch. 5, § 2). Adding $|c_0|$ and $|f(0)|$ to the left- and right-hand sides of (2.1) with $q = m = 1$, respectively, and replacing $-\Re f(0)$ by $|f(0)|$ in the resulting relation, we arrive at

$$\sum_{n=0}^{\infty} |c_n z|^n \leq \frac{1+r}{1-r}|f(0)| + \frac{2r}{1-r} \sup_{|\zeta|<1} \Re f(\zeta),$$

which is a refinement of the Hadamard-Borel-Carathéodory inequality

$$|f(z)| \leq \frac{1+r}{1-r}|f(0)| + \frac{2r}{1-r} \sup_{|\zeta|<1} \Re f(\zeta)$$

(see, e.g., Burckel [19], Ch. 6 and references there, Titchmarsh [40], Ch. 5).

The next assertion contains an sharp estimate for analytic functions with real part in the Hardy space $h_\infty(D)$ of harmonic functions bounded in D. It is a refinement of the inequality

$$|f(z) - f(0)| \leq \frac{2r}{1-r} \sup_{|\zeta|<1} \{|\Re f(\zeta)| - |\Re f(0)|\}$$

which follows from (0.1) with $C = 2$.

Theorem 2. *Let the function* (0.2) *be analytic on D with $\Re f \in h_\infty(D)$, and let $q > 0$, $m \geq 1$, $|z| = r < 1$. Then the inequality*

(2.4) $$\left\{\sum_{n=m}^{\infty} |c_n z^n|^q\right\}^{1/q} \leq \frac{2r^m}{(1-r^q)^{1/q}} \sup_{|\zeta|<1} \{|\Re f(\zeta)| - |\Re f(0)|\}$$

holds with the sharp constant.

Proof. Setting

$$\omega = \mathcal{R}(f) = \sup_{|\zeta|<1} |\Re f(\zeta)|$$

in (2.2) and making use of the equalities

$$\|\Re f - \mathcal{R}(f)\|_{h_1(D_\varrho)} = M_1(\Re f - \mathcal{R}(f), \varrho) = \sup_{|\zeta|<1} \{|\Re f(\zeta)| - \Re f(0)\},$$

we arrive at

$$\left\{\sum_{n=m}^{\infty} |c_n z^n|^q\right\}^{1/q} \leq \frac{2r^m}{\varrho^{m-1}(\varrho^q - r^q)^{1/q}} \sup_{|\zeta|<1} \{|\Re f(\zeta)| - \Re f(0)\}.$$

This estimate leads to

(2.5) $$\left\{\sum_{n=m}^{\infty} |c_n z^n|^q\right\}^{1/q} \leq \frac{2r^m}{(1-r^q)^{1/q}} \sup_{|\zeta|<1} \{|\Re f(\zeta)| - \Re f(0)\}$$

after the passage to the limit as $\varrho \uparrow 1$. Replacing f by $-f$ in the last inequality, we obtain

$$\left\{\sum_{n=m}^{\infty} |c_n z^n|^q\right\}^{1/q} \leq \frac{2r^m}{(1-r^q)^{1/q}} \sup_{|\zeta|<1} \{|\Re f(\zeta)| + \Re f(0)\},$$

which together with (2.5) results at (2.4).

Let us show that the constant in (2.4) is sharp. By $C(r)$ we denote the best constant in

$$(2.6) \qquad \left\{\sum_{n=m}^{\infty} |c_n z^n|^q\right\}^{1/q} \leq C(r) \sup_{|\zeta|<1} \{|\Re f(\zeta)| - |\Re f(0)|\}.$$

As shown above, $C(r)$ obeys (1.4).

We introduce the family of analytic functions in $\overline{D}$

$$(2.7) \qquad h_\rho(z) = \frac{\rho}{z - \rho} + \frac{\rho^2}{\rho^2 - 1},$$

depending on a parameter $\rho > 1$. Making elementary calculations, we arrive at

$$(2.8) \qquad \sup_{|\zeta|<1} \{|\Re h_\rho(\zeta)| - |\Re h_\rho(0)|\} = \frac{1}{\rho + 1}.$$

Taking into account that the functions (1.5) and (2.7) differ by a constant, and using (2.6), (1.7) and (2.8), we arrive at (1.9). Passing there to the limit as $\rho \downarrow 1$, we conclude that (1.10) holds, which together with (1.4) proves the sharpness of the constant in (2.4). $\qquad \square$

The following assertion contains an estimate with the sharp constant for analytic functions in the Hardy space $H_\infty(D)$. It gives a refinement of the estimate

$$|f(z) - f(0)| \leq \frac{2r}{1-r} \sup_{|\zeta|<1} \{|f(\zeta)| - |f(0)|\}$$

which follows from (0.1) with $C = 2$ and is valid for bounded analytic functions in D.

Theorem 3. *Let the function* (0.2) *be analytic and bounded on* D, *and let* $q > 0$, $m \geq 1$, $|z| = r < 1$. *Then the inequality*

$$(2.9) \qquad \left\{\sum_{n=m}^{\infty} |c_n z^n|^q\right\}^{1/q} \leq \frac{2r^m}{(1-r^q)^{1/q}} \sup_{|\zeta|<1} \{|f(\zeta)| - |f(0)|\}$$

holds with the sharp constant.

Proof. Setting

$$\omega = \mathcal{M}(f) = \sup_{|\zeta|<1} |f(\zeta)|$$

in (2.2) and using the equalities

$$\|\Re f - \mathcal{M}(f)\|_{h_1(D_\varrho)} = M_1(\Re f - \mathcal{M}(f), \varrho) = \sup_{|\zeta|<1} \{|f(\zeta)| - \Re f(0)\},$$

we obtain

$$\left\{\sum_{n=m}^{\infty}|c_n z^n|^q\right\}^{1/q} \leq \frac{2r^m}{\varrho^{m-1}(\varrho^q - r^q)^{1/q}} \sup_{|\zeta|<1}\{|f(\zeta)| - \Re f(0)\}.$$

Passing here to the limit as $\varrho \uparrow 1$, we find

$$\left\{\sum_{n=m}^{\infty}|c_n z^n|^q\right\}^{1/q} \leq \frac{2r^m}{(1 - r^q)^{1/q}} \sup_{|\zeta|<1}\{|f(\zeta)| - \Re f(0)\}.$$

Replacing f by $fe^{i\alpha}$, we arrive at

$$\left\{\sum_{n=m}^{\infty}|c_n z^n|^q\right\}^{1/q} \leq \frac{2r^m}{(1 - r^q)^{1/q}} \sup_{|\zeta|<1}\{|f(\zeta)| - \Re\big(f(0)e^{i\alpha}\big)\},$$

which implies (2.9) by the arbitrariness of α.

Let us show that the constant in (2.9) is sharp. By $C(r)$ we denote the best constant in

$$(2.10) \qquad \left\{\sum_{n=m}^{\infty}|c_n z^n|^q\right\}^{1/q} \leq C(r) \sup_{|\zeta|<1}\{|f(\zeta)| - |f(0)|\}.$$

As shown above, $C(r)$ obeys (1.4).

We consider the family h_ρ of analytic functions in $\overline{D}$, defined by (2.7). After elementary calculations, we arrive at

$$(2.11) \qquad \sup_{|\zeta|<1}\{|h_\rho(\zeta)| - |h_\rho(0)|\} = \frac{1}{\rho + 1}.$$

Taking into account that the functions (1.5) and (2.7) differ by a constant, and using (2.10), (1.7) and (2.11), we arrive at (1.9). Passing there to the limit as $\rho \downarrow 1$, we obtain (1.10), which together with (1.4) proves the sharpness of the constant in (2.9). $\square$

Remark 3. The inequality

$$(2.12) \qquad |f^{(n)}(0)| \leq n!\, \frac{[\mathcal{M}(f)]^2 - |f(0)|^2}{\mathcal{M}(f)},$$

with $f \in H_\infty(D)$ and $\mathcal{M}(f)$ being the supremum of $|f(z)|$ on D, was obtained by Landau (see [31], pp. 305-306) for $n = 1$ and by F. Wiener (see Bohr [14], Jensen [29]) for all n.

A consequence of (2.12) is the inequality

$$(2.13) \qquad \left\{\sum_{n=m}^{\infty}|c_n z^n|^q\right\}^{1/q} \leq \frac{r^m}{(1 - r^q)^{1/q}} \frac{[\mathcal{M}(f)]^2 - |f(0)|^2}{\mathcal{M}(f)}$$

with the constant factor in the right-hand side twice as small as in (2.9) and sharp which can be checked using the sequence of functions given by (2.7) and the limit passage as $\rho \downarrow 1$.

A proof of (2.12), different from that given by Wiener was found by Paulsen, Popescu and Singh [36] who derived (2.13) for $q = 1, m = 1$ and $\mathcal{M}(f) \leq 1$. A generalization of (2.12) is due to Jensen [29]. Sharp estimates for derivatives of f at an arbitrary point of the disk, involving (2.12), are due to Ruscheweyh [38] who applied classical methods. A different approach to these estimates and their generalizations was worked out by Anderson and Rovnyak [10], Bénéteau, Dahlner and Khavinson [11] as well as MacCluer, Stroethoff and Zhao [34]. Extensions to several variables can be found in the articles by Bénéteau, Dahlner and Khavinson [11] and MacCluer, Stroethoff and Zhao [34].

The next assertion refines the inequality

$$|f(z) - f(0)| \leq \frac{2r}{1 - r} \, \Re f(0)$$

resulting from (0.1) with $C = 2$ and valid for analytic functions in D with $\Re f(0) > 0$.

Theorem 4. *Let the function* (0.2) *be analytic with positive* $\Re f$ *on* D, *and let* $q > 0$, $m \geq 1$, $|z| = r < 1$. *Then the inequality*

$$(2.14) \qquad \left\{ \sum_{n=m}^{\infty} |c_n z^n|^q \right\}^{1/q} \leq \frac{2r^m}{(1 - r^q)^{1/q}} \, \Re f(0)$$

holds with the sharp constant.

Proof. Setting $\omega = 0$ in (2.2), with f such that $\Re f > 0$ in D, we obtain

$$\left\{ \sum_{n=m}^{\infty} |c_n z^n|^q \right\}^{1/q} \leq \frac{2r^m}{\varrho^{m-1}(\varrho^q - r^q)^{1/q}} \, \Re f(0),$$

which leads to (2.14) as $\varrho \uparrow 1$.

Thus, the sharp constant $C(r)$ in

$$(2.15) \qquad \left\{ \sum_{n=m}^{\infty} |c_n z^n|^q \right\}^{1/q} \leq C(r) \, \Re f(0)$$

obeys the estimate (1.4).

To show the sharpness of the constant in (2.14), consider the family of analytic functions in $\overline{D}$

$$(2.16) \qquad w_\rho(z) = \frac{\rho}{\rho - z} - \frac{\rho}{\rho + 1},$$

depending on the parameter $\rho > 1$. The real part of w_ρ is positive in D. Taking into account that the functions (1.5) and (2.16) differ by a constant and using (2.15), (1.7) and $\Re w_\rho(0) = (\rho + 1)^{-1}$, we arrive at (1.9). Passing there to the limit as $\rho \downarrow 1$, we obtain (1.10), which together with (1.4) proves the sharpness of the constant in (2.14). $\square$

3. Bohr's type modulus and real part theorems

In this section we collect some corollaries of the theorems in Sect. 2.

Corollary 1. *Let the function* (0.2) *be analytic on D, and let*

$$\sup_{|\zeta|<1} \Re\{e^{-i\arg f(0)} f(\zeta)\} < \infty,$$

where $\arg f(0)$ is replaced by zero if $f(0) = c_0 = 0$.

Then for any $q \in (0, \infty]$, integer $m \geq 1$ and $|z| \leq r_{m,q}$ the inequality

$$(3.1) \qquad \left\{\sum_{n=m}^{\infty} |c_n z^n|^q\right\}^{1/q} \leq \sup_{|\zeta|<1} \Re\{e^{-i\arg f(0)} f(\zeta)\} - |f(0)|$$

holds, where $r_{m,q}$ is the root of the equation $2^q r^{mq} + r^q - 1 = 0$ in the interval $(0,1)$. Here $r_{m,q}$ is the radius of the largest disk centered at $z = 0$ in which (3.1) *takes place for all f. In particular,* (0.6) *holds.*

Proof. The condition

$$\frac{2r^m}{(1-r^q)^{1/q}} \leq 1$$

ensuring the sharpness of the constant in (2.1) holds if $|z| \leq r_{m,q}$, where $r_{m,q}$ is the root of the equation $2^q r^{mq} + r^q - 1 = 0$ in the interval $(0,1)$.

The disk of radius $r_{m,q}$ centered at $z = 0$ is the largest disk, where the inequality

$$(3.2) \qquad \left\{\sum_{n=m}^{\infty} |c_n z^n|^q\right\}^{1/q} \leq \sup_{|\zeta|<1} \Re f(\zeta) - \Re f(0)$$

holds for all f. The last inequality coincides with (3.1) for $f(0) = c_0 = 0$.

Suppose now that $f(0) \neq 0$. Setting $e^{-i\arg f(0)} f$ in place of f in (3.2) and noting that the coefficients $|c_n|$ in the left-hand side of (3.2) do not change, when $\Re f(0)$ is replaced by $|f(0)| = |c_0|$, we arrive at (3.1). □

Inequality (3.1) with $q = 1, m = 1$ becomes

$$(3.3) \qquad \sum_{n=1}^{\infty} |c_n z^n| \leq \sup_{|\zeta|<1} \Re\{e^{-i\arg f(0)} f(\zeta)\} - |f(0)|$$

with $|z| \leq 1/3$, where $1/3$ is the radius of the largest disk centered at $z = 0$ in which (3.3) takes place. Note that (3.3) is equivalent to a sharp inequality obtained by Sidon [39] in his proof of Bohr's theorem and to the inequality derived by Paulsen, Popescu and Singh [36].

For $q = 1, m = 2$ inequality (3.1) is

$$(3.4) \qquad \sum_{n=2}^{\infty} |c_n z^n| \leq \sup_{|\zeta|<1} \Re\{e^{-i\arg f(0)} f(\zeta)\} - |f(0)|,$$

where $|z| \leq 1/2$ and $1/2$ is the radius of the largest disk about $z = 0$ in which (3.4) takes place.

The next assertion follows from Theorem 3. For $q = 1, m = 1$ it contains Bohr's inequality (0.3).

Corollary 2. *Let the function* (0.2) *be analytic and bounded on* D. *Then for any* $q \in (0, \infty]$, *integer* $m \geq 1$ *and* $|z| \leq r_{m,q}$ *the inequality*

$$(3.5) \qquad \left\{ \sum_{n=m}^{\infty} |c_n z^n|^q \right\}^{1/q} \leq \sup_{|\zeta|<1} |f(\zeta)| - |f(0)|$$

holds, where $r_{m,q}$ *is the root of the equation* $2^q r^{mq} + r^q - 1 = 0$ *in the interval* $(0, 1)$. *Here* $r_{m,q}$ *is the radius of the largest disk centered at* $z = 0$ *in which* (3.5) *takes place for all* f. *In particular,* (0.6) *holds.*

For $q = 1, m = 2$ inequality (3.5) takes the form

$$(3.6) \qquad |c_0| + \sum_{n=2}^{\infty} |c_n z^n| \leq \sup_{|\zeta|<1} |f(\zeta)|,$$

where $|z| \leq 1/2$. The value $1/2$ of the radius of the disk where (3.6) is valid cannot be improved. Note that the inequality

$$(3.7) \qquad |c_0|^2 + \sum_{n=1}^{\infty} |c_n z^n| \leq 1,$$

was obtained by Paulsen, Popescu and Singh [36] for functions (0.2) satisfying the condition $|f(\zeta)| \leq 1$ in D and is valid for $|z| \leq 1/2$. The value $1/2$ of the radius of the disk where (3.7) holds is largest. Comparison of (3.6) and (3.7) shows that none of these inequalities is a consequence of the other one.

We conclude this section by an assertion which follows from Theorem 4.

Corollary 3. *Let the function* (0.2) *be analytic, and* $\Re\{e^{-i \arg f(0)} f\} > 0$ *on* D. *Then for any* $q \in (0, \infty]$, *integer* $m \geq 1$ *and* $|z| \leq r_{m,q}$ *the inequality*

$$(3.8) \qquad \left\{ \sum_{n=m}^{\infty} |c_n z^n|^q \right\}^{1/q} \leq |f(0)|$$

holds, where $r_{m,q}$ *is the root of the equation* $2^q r^{mq} + r^q - 1 = 0$ *in the interval* $(0, 1)$. *Here* $r_{m,q}$ *is the radius of the largest disk centered at* $z = 0$ *in which* (3.5) *takes place for all* f. *In particular,* (0.6) *holds.*

Note that the inequality (3.8) for $q = 1, m = 1$ with $|z| \leq 1/3$ was obtained by Aizenberg, Aytuna and Djakov [4] (see also Aizenberg, Grossman and Korobeinik [7]).

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