

CONDUCTOR INEQUALITIES AND CRITERIA FOR SOBOLEV-LORENTZ TWO-WEIGHT INEQUALITIES

ȘERBAN COSTEA AND VLADIMIR MAZ'YA

In memory of S. L. Sobolev

ABSTRACT. In this paper we present integral conductor inequalities connecting the Lorentz p, q -(quasi)norm of a gradient of a function to a one-dimensional integral of the p, q -capacitance of the conductor between two level surfaces of the same function. These inequalities generalize an inequality obtained by the second author in the case of the Sobolev norm. Such conductor inequalities lead to necessary and sufficient conditions for Sobolev-Lorentz type inequalities involving two arbitrary measures.

1. Introduction

During the last decades Sobolev-Lorentz function spaces, which include classical Sobolev spaces, attracted attention not only as an interesting mathematical object, but also as a tool for a finer tuning of properties of solutions to partial differential equations. (See [Alb], [AFT1], [AFT2], [BBGGPV], [Cia], [CP], [Cos], [DHM], [HL], [KKM], [ST], *et al.*)

In the present paper we generalize the inequality

$$(1) \quad \int_0^\infty \text{cap}_p(\overline{M_{at}}, M_t) d(t^p) \leq c(a, p) \int_\Omega |\nabla f|^p dx$$

to Sobolev-Lorentz spaces. Here $f \in Lip_0(\Omega)$, i.e. f is an arbitrary Lipschitz function compactly supported in the open set $\Omega \subset \mathbf{R}^n$, while M_t is the set $\{x \in \Omega : |f(x)| > t\}$ with $t > 0$. Inequality (1) was obtained in [M1]. (See also [M3, Chapter 2].) It has various extensions and applications to the theory of Sobolev-type spaces on domains in \mathbf{R}^n , Riemannian manifolds, metric and topological spaces, to linear and nonlinear partial differential equations, Dirichlet forms, and Markov processes etc. (See [Ad], [AH], [AP], [AX1], [AX2], [Ai], [CS], [Dah], [DKX], [Fi], [FU1], [FU2], [Gr], [Ha.j], [Han], [HMV], [Ka], [Ko1], [Ko2], [Mal], [M1], [M2], [M4], [M5], [MN], [MP], [Ne], [Ra], [Ta], [V1], [V2], [Vo], *et al.*) In the sequel, we prove the inequalities

$$(2) \quad \int_0^\infty \text{cap}(\overline{M_{at}}, M_t) d(t^p) \leq c(a, p, q) \|\nabla f\|_{L^{p,q}(\Omega, m_n; \mathbf{R}^n)}^p \text{ when } 1 \leq q \leq p$$

and

$$(3) \quad \int_0^\infty \text{cap}_{p,q}(\overline{M_{at}}, M_t)^{q/p} d(t^q) \leq c(a, p, q) \|\nabla f\|_{L^{p,q}(\Omega, m_n; \mathbf{R}^n)}^q \text{ when } p < q < \infty$$

for all $f \in Lip_0(\Omega)$.

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The proof of (2) and (3) is based on the superadditivity of the p, q -capacitance, also justified in this paper.

From (2) and (3) we derive necessary and sufficient conditions for certain two-weight inequalities involving Sobolev-Lorentz norms, generalizing results obtained in [M4] and [M5]. Specifically, let μ and ν be two locally finite nonnegative measures on Ω and let p, q, r, s be real numbers such that $1 < s \leq \max(p, q) \leq r < \infty$ and $q \geq 1$. We characterize the inequality

$$(4) \quad \|f\|_{L^{r, \max(p, q)}(\Omega, \mu)} \leq A \left(\|\nabla f\|_{L^{p, q}(\Omega, m_n; \mathbf{R}^n)} + \|f\|_{L^{s, \max(p, q)}(\Omega, \nu)} \right)$$

restricted to functions $f \in Lip_0(\Omega)$ by requiring the condition

$$(5) \quad \mu(g)^{1/r} \leq K(\text{cap}_{p, q}(\bar{g}, G)^{1/p} + \nu(G)^{1/s})$$

to be valid for all open bounded sets g and G subject to $\bar{g} \subset G$, $\bar{G} \subset \Omega$. When $n = 1$ inequality (4) becomes

$$(6) \quad \|f\|_{L^{r, \max(p, q)}(\Omega, \mu)} \leq A \left(\|f'\|_{L^{p, q}(\Omega, m_1)} + \|f\|_{L^{s, \max(p, q)}(\Omega, \nu)} \right).$$

The requirement that (6) be valid for all functions $f \in Lip_0(\Omega)$ when $n = 1$ is shown to be equivalent to the condition

$$(7) \quad \mu(\sigma_d(x))^{1/r} \leq K(\tau^{(1-p)/p} + \nu(\sigma_{d+\tau}(x))^{1/s})$$

whenever x, d and τ are such that $\overline{\sigma_{d+\tau}(x)} \subset \Omega$. Here and throughout the paper $\sigma_d(x)$ denotes the open interval $(x - d, x + d)$ for every $d > 0$.

2. Preliminaries

Let us introduce some notation, to be used in the sequel. By Ω we denote a nonempty open subset of \mathbf{R}^n , whereas m_n stands for the Lebesgue n -measure in \mathbf{R}^n , where $n \geq 1$ is integer. For a Lebesgue measurable $u : \Omega \rightarrow \mathbf{R}$, $\text{supp } u$ is the smallest closed set such that u vanishes outside $\text{supp } u$. We also define

$$Lip(\Omega) = \{\varphi : \Omega \rightarrow \mathbf{R} : \varphi \text{ is Lipschitz}\}$$

$$Lip_0(\Omega) = \{\varphi : \Omega \rightarrow \mathbf{R} : \varphi \text{ is Lipschitz and with compact support in } \Omega\}.$$

If $\varphi \in Lip(\Omega)$, we write $\nabla \varphi$ for the gradient of φ . This notation makes sense, since by Rademacher's theorem ([Fed, Theorem 3.1.6]) every Lipschitz function on Ω is m_n -a.e. differentiable.

Throughout this section we will assume that $m \geq 1$ is a positive integer and that (Ω, μ) is a measure space. Let $f : \Omega \rightarrow \mathbf{R}^n$ be a μ -measurable function. We define $\mu_{[f]}$, the *distribution function* of f as follows (see [BS, Definition II.1.1]):

$$\mu_{[f]}(t) = \mu(\{x \in \Omega : |f(x)| > t\}), \quad t \geq 0.$$

We define f^* , the *nonincreasing rearrangement* of f by

$$f^*(t) = \inf\{v : \mu_{[f]}(v) \leq t\}, \quad t \geq 0.$$

(See [BS, Definition II.1.5].) We notice that f and f^* have the same distribution function. Moreover, for every positive α we have

$$(|f|^\alpha)^* = (|f^*|^\alpha)$$

and if $|g| \leq |f|$ a.e. on Ω , then $g^* \leq f^*$. (See [BS, Proposition II.1.7].) We also define f^{**} , the *maximal function* of f^* by

$$f^{**}(t) = m_{f^*}(t) = \frac{1}{t} \int_0^t f^*(s) ds, \quad t > 0.$$

(See [BS, Definition II.3.1].)

Throughout this paper, we will denote the Hölder conjugate of $p \in [1, \infty]$ by p' .

The *Lorentz space* $L^{p,q}(\Omega, \mu; \mathbf{R}^n)$, $1 < p < \infty$, $1 \leq q \leq \infty$, is defined as follows:

$$L^{p,q}(\Omega, \mu; \mathbf{R}^n) = \{f : \Omega \rightarrow \mathbf{R}^n : f \text{ is } \mu\text{-measurable and } \|f\|_{L^{p,q}(\Omega, \mu; \mathbf{R}^n)} < \infty\},$$

where

$$\|f\|_{L^{p,q}(\Omega, \mu; \mathbf{R}^n)} = \| |f| \|_{p,q} = \begin{cases} \left(\int_0^\infty (t^{1/p} f^*(t))^q \frac{dt}{t} \right)^{1/q} & 1 \leq q < \infty \\ \sup_{t>0} t \mu_{[f]}(t)^{1/p} = \sup_{s>0} s^{1/p} f^*(s) & q = \infty. \end{cases}$$

(See [BS, Definition IV.4.1].) We omit \mathbf{R}^n in the notation of function spaces for the scalar case, i.e. for $n = 1$.

If $1 \leq q \leq p$, then $\|\cdot\|_{L^{p,q}(\Omega, \mu; \mathbf{R}^n)}$ represents a norm, but for $p < q \leq \infty$ it represents a quasinorm, equivalent to the norm $\|\cdot\|_{L^{(p,q)}(\Omega, \mu; \mathbf{R}^n)}$, where

$$\|f\|_{L^{(p,q)}(\Omega, \mu; \mathbf{R}^n)} = \| |f| \|_{(p,q)} = \begin{cases} \left(\int_0^\infty (t^{1/p} f^{**}(t))^q \frac{dt}{t} \right)^{1/q} & 1 \leq q < \infty \\ \sup_{t>0} t^{1/p} f^{**}(t) & q = \infty. \end{cases}$$

(See [BS, Definition IV.4.4].) Namely, from [BS, Lemma IV.4.5] we have that

$$\| |f| \|_{L^{p,q}(\Omega, \mu)} \leq \| |f| \|_{L^{(p,q)}(\Omega, \mu)} \leq p' \| |f| \|_{L^{p,q}(\Omega, \mu)}$$

for every $q \in [1, \infty]$ and every μ -measurable function $f : \Omega \rightarrow \mathbf{R}^n$.

It is known that $(L^{p,q}(\Omega, \mu; \mathbf{R}^n), \|\cdot\|_{L^{p,q}(\Omega, \mu; \mathbf{R}^n)})$ is a Banach space for $1 \leq q \leq p$, while $(L^{p,q}(\Omega, \mu; \mathbf{R}^n), \|\cdot\|_{L^{(p,q)}(\Omega, \mu; \mathbf{R}^n)})$ is a Banach space for $1 < p < \infty$, $1 \leq q \leq \infty$.

Remark 2.1. It is also known (see [BS, Proposition IV.4.2]) that for every $p \in (1, \infty)$ and $1 \leq r < s \leq \infty$ there exists a constant $C(p, r, s)$ such that

$$(8) \quad \| |f| \|_{L^{p,s}(\Omega, \mu)} \leq C(p, r, s) \| |f| \|_{L^{p,r}(\Omega, \mu)}$$

for all measurable functions $f \in L^{p,r}(\Omega, \mu; \mathbf{R}^n)$ and all integers $n \geq 1$. In particular, the embedding $L^{p,r}(\Omega, \mu; \mathbf{R}^n) \hookrightarrow L^{p,s}(\Omega, \mu; \mathbf{R}^n)$ holds.

2.1. The subadditivity and superadditivity of the Lorentz quasinorms. In the second part of this paper, we will prove a few results by relying on the superadditivity of the Lorentz p, q -quasinorm. Therefore we recall the known results and present new results concerning the superadditivity and the subadditivity of the Lorentz p, q -quasinorm.

The superadditivity of the Lorentz p, q -norm in the case $1 \leq q \leq p$ was stated in [CHK, Lemma 2.5].

Proposition 2.2. (See [CHK, Lemma 2.5].) Let (Ω, μ) be a measure space. Suppose $1 \leq q \leq p$. Let $\{E_i\}_{i \geq 1}$ be a collection of pairwise disjoint measurable subsets of Ω with $E_0 = \cup_{i \geq 1} E_i$ and let $f \in L^{p,q}(\Omega, \mu)$. Then

$$\sum_{i \geq 1} \|\chi_{E_i} f\|_{L^{p,q}(\Omega, \mu)}^p \leq \|\chi_{E_0} f\|_{L^{p,q}(\Omega, \mu)}^p.$$

We obtain a similar result concerning the superadditivity in the case $1 < p < q < \infty$.

Proposition 2.3. Let (Ω, μ) be a measure space. Suppose $1 < p < q < \infty$. Let $\{E_i\}_{i \geq 1}$ be a collection of pairwise disjoint measurable subsets of Ω with $E_0 = \cup_{i \geq 1} E_i$ and let $f \in L^{p,q}(\Omega, \mu)$. Then

$$\sum_{i \geq 1} \|\chi_{E_i} f\|_{L^{p,q}(\Omega, \mu)}^q \leq \|\chi_{E_0} f\|_{L^{p,q}(\Omega, \mu)}^q.$$

Proof. For every $i = 0, 1, 2, \dots$ we let $f_i = \chi_{E_i} f$, where χ_{E_i} is the characteristic function of E_i . We can assume without loss of generality that all the functions f_i are nonnegative. We have (see [KKM, Proposition 2.1])

$$\|f_i\|_{L^{p,q}(\Omega, \mu)}^q = p \int_0^\infty s^{q-1} \mu_{[f_i]}(s)^{q/p} ds,$$

where $\mu_{[f_i]}$ is the distribution function of f_i , $i = 0, 1, 2, \dots$. From the definition of f_0 we obviously have

$$(9) \quad \mu_{[f_0]}(s) = \sum_{i \geq 1} \mu_{[f_i]}(s) \text{ for every } s > 0$$

which implies, since $1 < p < q < \infty$, that

$$\mu_{[f_0]}(s)^{q/p} \geq \sum_{i \geq 1} \mu_{[f_i]}(s)^{q/p} \text{ for every } s > 0.$$

This yields

$$\begin{aligned} \|f_0\|_{L^{p,q}(\Omega, \mu)}^q &= p \int_0^\infty s^{q-1} \mu_{[f_0]}(s)^{q/p} ds \geq p \int_0^\infty s^{q-1} \left(\sum_{i \geq 1} \mu_{[f_i]}(s)^{q/p} \right) ds \\ &= \sum_{i \geq 1} p \int_0^\infty s^{q-1} \mu_{[f_i]}(s)^{q/p} ds = \sum_{i \geq 1} \|f_i\|_{L^{p,q}(\Omega, \mu)}^q. \end{aligned}$$

This finishes the proof of the superadditivity in the case $1 < p < q < \infty$. □

We have a similar result for the subadditivity of the Lorentz p, q -quasinorm. When $1 < p < q \leq \infty$ we obtain a result that generalizes [Cos, Theorem 2.5].

Proposition 2.4. Let (Ω, μ) be a measure space. Suppose $1 < p < q \leq \infty$. Let $\{E_i\}_{i \geq 1}$ be a collection of pairwise disjoint measurable subsets of Ω with $E_0 = \cup_{i \geq 1} E_i$ and let $f \in L^{p,q}(\Omega, \mu)$. Then

$$\sum_{i \geq 1} \|\chi_{E_i} f\|_{L^{p,q}(\Omega, \mu)}^p \geq \|\chi_{E_0} f\|_{L^{p,q}(\Omega, \mu)}^p.$$

Proof. Without loss of generality we can assume that all the functions f_i are nonnegative. We have to consider two cases, depending on whether $p < q < \infty$ or $q = \infty$.

Suppose $p < q < \infty$. We have (see [KKM, Proposition 2.1])

$$\|f_i\|_{L^{p,q}(\Omega,\mu)}^p = \left(p \int_0^\infty s^{q-1} \mu_{[f_i]}(s)^{q/p} ds \right)^{p/q},$$

where $\mu_{[f_i]}$ is the distribution function of f_i for $i = 0, 1, 2, \dots$. From (9) we obtain

$$\begin{aligned} \|f_0\|_{L^{p,q}(\Omega,\mu)}^p &= \left(p \int_0^\infty s^{q-1} \mu_{[f_0]}(s)^{q/p} ds \right)^{p/q} \leq \sum_{i \geq 1} \left(p \int_0^\infty s^{q-1} \mu_{[f_i]}(s)^{q/p} ds \right)^{p/q} \\ &= \sum_{i \geq 1} \|f_i\|_{L^{p,q}(\Omega,\mu)}^p. \end{aligned}$$

Suppose now $q = \infty$. From (9) we obtain

$$s^p \mu_{[f_0]}(s) = \sum_{i \geq 1} (s^p \mu_{[f_i]}(s))$$

for every $s \geq 0$, which implies

$$(10) \quad s^p \mu_{[f_0]}(s) \leq \sum_{i \geq 1} \|f_i\|_{L^{p,\infty}(\Omega,\mu)}^p$$

for every $s \geq 0$. By taking the supremum over all $s \geq 0$ in (10), we get the desired conclusion. This finishes the proof. \square

3. Sobolev-Lorentz p, q -capacitance

Suppose $1 < p < \infty$ and $1 \leq q \leq \infty$. Let $\Omega \subset \mathbf{R}^n$ be an open set, $n \geq 1$. Let $K \subset \Omega$ be compact. The Sobolev-Lorentz p, q -capacitance of the conductor (K, Ω) is denoted by

$$\text{cap}_{p,q}(K, \Omega) = \inf \{ \|\nabla u\|_{L^{p,q}(\Omega, m_n; \mathbf{R}^n)}^p : u \in W(K, \Omega) \},$$

where

$$W(K, \Omega) = \{ u \in \text{Lip}_0(\Omega) : u \geq 1 \text{ in a neighborhood of } K \}.$$

We call $W(K, \Omega)$ the *set of admissible functions for the conductor* (K, Ω) .

Since $W(K, \Omega)$ is closed under truncations from below by 0 and from above by 1 and since these truncations do not increase the p, q -quasinorm whenever $1 < p < \infty$ and $1 \leq q \leq \infty$, it follows that we can choose only functions $u \in W(K, \Omega)$ that satisfy $0 \leq u \leq 1$ when computing the p, q -capacitance of the conductor (K, Ω) .

Lemma 3.1. *If Ω is bounded, then we get the same p, q -capacitance for the conductor (K, Ω) if we restrict ourselves to a bigger set, namely*

$$W_1(K, \Omega) = \{ u \in \text{Lip}(\Omega) \cap C(\overline{\Omega}) : u \geq 1 \text{ on } K \text{ and } u = 0 \text{ on } \partial\Omega \}.$$

Proof. Let $u \in W_1(K, \Omega)$. We can assume without loss of generality that $0 \leq u \leq 1$. Moreover, we can also assume that $u = 1$ on an open neighborhood U of K . Let \tilde{U} be an open neighborhood of K such that $\tilde{U} \subset\subset U$. We choose a cutoff Lipschitz function η , $0 \leq \eta \leq 1$ such that $\eta = 1$ in $\Omega \setminus U$ and $\eta = 0$ in a neighborhood \tilde{U} of K . We notice

that $1 - \eta(1 - u) = u$. We notice that there exists a sequence of functions $\varphi_j \in Lip_0(\Omega)$ such that

$$\lim_{j \rightarrow \infty} (\|\varphi_j - u\|_{L^{p+1}(\Omega, m_n)} + \|\nabla \varphi_j - \nabla u\|_{L^{p+1}(\Omega, m_n; \mathbf{R}^n)}) = 0.$$

Without loss of generality the sequence φ_j can be chosen such that $\varphi_j \rightarrow u$ and $\nabla \varphi_j \rightarrow \nabla u$ pointwise a.e. in Ω . Then $\psi_j = 1 - \eta(1 - \varphi_j)$ is a sequence belonging to $W(K, \Omega)$ and

$$\lim_{j \rightarrow \infty} (\|\psi_j - u\|_{L^{p+1}(\Omega, m_n)} + \|\nabla \psi_j - \nabla u\|_{L^{p+1}(\Omega, m_n; \mathbf{R}^n)}) = 0.$$

This, Hölder's inequality for Lorentz spaces, and the behaviour of the Lorentz p, q -quasinorm in q yields

$$\lim_{j \rightarrow \infty} (\|\psi_j - u\|_{L^{p,q}(\Omega, m_n)} + \|\nabla \psi_j - \nabla u\|_{L^{p,q}(\Omega, m_n; \mathbf{R}^n)}) = 0.$$

The desired conclusion follows. □

3.1. Basic properties of the p, q -capacitance. Usually, a capacitance is a monotone and subadditive set function. The following theorem will show, among other things, that this is true in the case of the p, q -capacitance. We follow [Cos] for (i)-(vi). In addition we will prove some superadditivity properties of the p, q -capacitance.

Theorem 3.2. *Suppose $1 < p < \infty$ and $1 \leq q \leq \infty$. Let $\Omega \subset \mathbf{R}^n$ be open. The set function $K \mapsto \text{cap}_{p,q}(K, \Omega)$, $K \subset \Omega$, K compact, enjoys the following properties:*

- (i) *If $K_1 \subset K_2$, then $\text{cap}_{p,q}(K_1, \Omega) \leq \text{cap}_{p,q}(K_2, \Omega)$.*
- (ii) *If $\Omega_1 \subset \Omega_2$ are open and K is a compact subset of Ω_1 , then*

$$\text{cap}_{p,q}(K, \Omega_2) \leq \text{cap}_{p,q}(K, \Omega_1).$$

- (iii) *If K_i is a decreasing sequence of compact subsets of Ω with $K = \bigcap_{i=1}^{\infty} K_i$, then*

$$\text{cap}_{p,q}(K, \Omega) = \lim_{i \rightarrow \infty} \text{cap}_{p,q}(K_i, \Omega).$$

- (iv) *If Ω_i is an increasing sequence of open sets with $\bigcup_{i=1}^{\infty} \Omega_i = \Omega$ and K is a compact subset of Ω_1 , then*

$$\text{cap}_{p,q}(K, \Omega) = \lim_{i \rightarrow \infty} \text{cap}_{p,q}(K, \Omega_i).$$

- (v) *Suppose $p \leq q \leq \infty$. If $K = \bigcup_{i=1}^k K_i \subset \Omega$ then*

$$\text{cap}_{p,q}(K, \Omega) \leq \sum_{i=1}^k \text{cap}_{p,q}(K_i, \Omega),$$

where $k \geq 1$ is a positive integer.

- (vi) *Suppose $1 \leq q < p$. If $K = \bigcup_{i=1}^k K_i \subset \Omega$ then*

$$\text{cap}_{p,q}(K, \Omega)^{q/p} \leq \sum_{i=1}^k \text{cap}_{p,q}(K_i, \Omega)^{q/p},$$

where $k \geq 1$ is a positive integer.

- (vii) *Suppose $1 \leq q \leq p$. Suppose $\Omega_1, \dots, \Omega_k$ are k pairwise disjoint open sets and K_i are compact subsets of Ω_i for $i = 1, \dots, k$. Then*

$$\text{cap}_{p,q}(\bigcup_{i=1}^k K_i, \bigcup_{i=1}^k \Omega_i) \geq \sum_{i=1}^k \text{cap}_{p,q}(K_i, \Omega_i).$$

(viii) Suppose $p < q < \infty$. Suppose $\Omega_1, \dots, \Omega_k$ are k pairwise disjoint open sets and K_i are compact subsets of Ω_i for $i = 1, \dots, k$. Then

$$\text{cap}_{p,q}(\cup_{i=1}^k K_i, \cup_{i=1}^k \Omega_i)^{q/p} \geq \sum_{i=1}^k \text{cap}_{p,q}(K_i, \Omega_i)^{q/p}.$$

(ix) Suppose $1 \leq q < \infty$. If Ω_1 and Ω_2 are two disjoint open sets and $K \subset \Omega_1$, then

$$\text{cap}_{p,q}(K, \Omega_1 \cup \Omega_2) = \text{cap}_{p,q}(K, \Omega_1).$$

Proof. Properties (i)-(vi) are proved by duplicating the proof of [Cos, Theorem 3.2], so we will prove only (vii)-(ix).

In order to prove (vii) and (viii), it is enough to assume that $k = 2$. A finite induction on k would prove each of these claims. So we assume that $k = 2$. Let $u \in Lip_0(\Omega_1 \cup \Omega_2)$ and let $u_i = \chi_{\Omega_i} u, i = 1, 2$. We let v_i be the restriction of u to Ω_i for $i = 1, 2$. Then $v_i \in Lip_0(\Omega_i)$ for $i = 1, 2$. We notice that u_i can be regarded as the extension of v_i by 0 to $\Omega_1 \cup \Omega_2$ for $i = 1, 2$. We notice that $u \in W(K_1 \cup K_2, \Omega_1 \cup \Omega_2)$ if and only if $v_i \in W(K_i, \Omega_i)$ for $i = 1, 2$.

Suppose first that $1 \leq q \leq p$. Since Ω_1 and Ω_2 are disjoint and $u = u_1 + u_2$ with the functions u_i supported in Ω_i for $i = 1, 2$, we obtain via Proposition 2.2

$$\begin{aligned} \|\nabla u\|_{L^{p,q}(\Omega_1 \cup \Omega_2, m_n; \mathbf{R}^n)}^p &\geq \|\nabla u_1\|_{L^{p,q}(\Omega_1 \cup \Omega_2, m_n; \mathbf{R}^n)}^p + \|\nabla u_2\|_{L^{p,q}(\Omega_1 \cup \Omega_2, m_n; \mathbf{R}^n)}^p \\ &= \|\nabla v_1\|_{L^{p,q}(\Omega_1, m_n; \mathbf{R}^n)}^p + \|\nabla v_2\|_{L^{p,q}(\Omega_2, m_n; \mathbf{R}^n)}^p. \end{aligned}$$

This proves (vii).

Suppose now that $p < q < \infty$. Since Ω_1 and Ω_2 are disjoint and $u = u_1 + u_2$ with the functions u_i supported in Ω_i for $i = 1, 2$, we obtain via Proposition 2.3

$$\begin{aligned} \|\nabla u\|_{L^{p,q}(\Omega_1 \cup \Omega_2, m_n; \mathbf{R}^n)}^q &\geq \|\nabla u_1\|_{L^{p,q}(\Omega_1 \cup \Omega_2, m_n; \mathbf{R}^n)}^q + \|\nabla u_2\|_{L^{p,q}(\Omega_1 \cup \Omega_2, m_n; \mathbf{R}^n)}^q \\ &= \|\nabla v_1\|_{L^{p,q}(\Omega_1, m_n; \mathbf{R}^n)}^q + \|\nabla v_2\|_{L^{p,q}(\Omega_2, m_n; \mathbf{R}^n)}^q. \end{aligned}$$

This proves (viii).

We see that (ix) follows from (vii) and (ii) when $1 \leq q \leq p$. (We use (vii) with $k = 2$ by taking $K_1 = K$ and $K_2 = \emptyset$.) When $p < q < \infty$, (ix) follows from (viii) and (ii). (We use (viii) with $k = 2$ by taking $K_1 = K$ and $K_2 = \emptyset$.) This finishes the proof of the theorem. □

Remark 3.3. The definition of the p, q -capacitance implies

$$\text{cap}_{p,q}(K, \Omega) = \text{cap}_{p,q}(\partial K, \Omega)$$

whenever K is a compact set in Ω . Moreover, if $n = 1$ and Ω is an open interval of \mathbf{R} , then

$$\text{cap}_{p,q}(K, \Omega) = \text{cap}_{p,q}(H, \Omega),$$

where H is the smallest compact interval containing K .

4. Conductor inequalities

Lemma 4.1. *Suppose $\Omega \subset \mathbf{R}^n$ is open. Let $f \in Lip_0(\Omega)$ and let $a > 1$ be a constant. For $t > 0$ we denote $M_t = \{x \in \Omega : |f(x)| > t\}$. Then the function $t \mapsto \text{cap}_{p,q}(\overline{M_{at}}, M_t)$ is upper semicontinuous.*

Proof. Let $t_0 > 0$ and $\varepsilon > 0$. Let $u \in W(\overline{M_{at_0}}, M_{t_0})$ be chosen such that

$$\|\nabla u\|_{L^{p,q}(\Omega, m_n; \mathbf{R}^n)}^p < \text{cap}_{p,q}(\overline{M_{at_0}}, M_{t_0}) + \varepsilon.$$

Let g be an open neighborhood of $\overline{M_{at_0}}$ such that $u \geq 1$ on g . Since g contains the compact set $\overline{M_{at_0}}$, there exists $\delta_1 > 0$ small such that $g \supset \overline{M_{a(t_0-\delta_1)}}$. Let G be an open set such that $\text{supp } u \subset G \subset \subset M_{t_0}$. There exists a small $\delta_2 > 0$ such that $\overline{G} \subset M_{t_0+\delta_2}$. Thus we have $\overline{M_{a(t_0-\delta)}} \subset g$ and $\overline{G} \subset M_{t_0+\delta}$ for every $\delta \in (0, \min\{\delta_1, \delta_2\})$. From the choice of g and G we have that $u \in W(K, \Omega)$ whenever $K \subset g$ and $\overline{G} \subset \Omega$. This and the choice of u imply that

$$\text{cap}_{p,q}(\overline{M_{a(t_0-\delta)}}, M_{t_0+\delta}) \leq \text{cap}_{p,q}(\overline{M_{at_0}}, M_{t_0}) + \varepsilon$$

for every $\delta \in (0, \min\{\delta_1, \delta_2\})$. Using the monotonicity of $\text{cap}_{p,q}$ again, we deduce that

$$\text{cap}_{p,q}(\overline{M_{at}}, M_t) \leq \text{cap}_{p,q}(\overline{M_{at_0}}, M_{t_0}) + \varepsilon$$

for every t sufficiently close to t_0 . The result follows. \square

Theorem 4.2. *Let Φ denote an increasing convex (not necessarily strictly convex) function given on $[0, \infty)$, $\Phi(0) = 0$. Suppose $a > 1$ is a constant.*

(i) *If $1 \leq q \leq p$, then*

$$\Phi^{-1} \left(\int_0^\infty \Phi(t^p \text{cap}_{p,q}(\overline{M_{at}}, M_t)) \frac{dt}{t} \right) \leq c(a, p, q) \|\nabla \varphi\|_{L^{p,q}(\Omega, m_n; \mathbf{R}^n)}^p$$

for every $\varphi \in Lip_0(\Omega)$.

(ii) *If $p < q < \infty$, then*

$$\Phi^{-1} \left(\int_0^\infty \Phi(t^q \text{cap}_{p,q}(\overline{M_{at}}, M_t)^{q/p}) \frac{dt}{t} \right) \leq c(a, p, q) \|\nabla \varphi\|_{L^{p,q}(\Omega, m_n; \mathbf{R}^n)}^q$$

for every $\varphi \in Lip_0(\Omega)$.

Proof. The proof follows [M4]. When $p = q$ we are in the case of the p -capacitance and for that case the result was proved in [M4, Theorem 1]. So we can assume without loss of generality that $p \neq q$. Let $\varphi \in Lip_0(\Omega)$. We set

$$\Lambda_t(\varphi) = \frac{1}{(a-1)t} \min\{(|\varphi| - t)_+, (a-1)t\}.$$

From Lemma 3.1 we notice that

$$(11) \quad \Lambda_t(\varphi) \in W_1(\overline{M_{at}}, M_t) \text{ and } |\nabla \Lambda_t(\varphi)| = \frac{1}{(a-1)t} \chi_{M_t \setminus M_{at}} |\nabla \varphi| \text{ } m_n\text{-a.e.}$$

The proof splits now, depending on whether $1 \leq q < p$ or $p < q < \infty$.

We assume first that $1 \leq q < p$. From (11) we have

$$t^p \text{cap}_{p,q}(\overline{M_{at}}, M_t) \leq \frac{1}{(a-1)^p} \|\chi_{M_t \setminus M_{at}} \nabla \varphi\|_{L^{p,q}(\Omega, m_n; \mathbf{R}^n)}^p.$$

Hence

$$\int_0^\infty \Phi(t^p \text{cap}_{p,q}(\overline{M_{at}}, M_t)) \frac{dt}{t} \leq \int_0^\infty \Phi\left(\frac{1}{(a-1)^p} \|\chi_{M_t \setminus M_{at}} \nabla \varphi\|_{L^{p,q}(\Omega, m_n; \mathbf{R}^n)}^p\right) \frac{dt}{t}.$$

Let γ denote a locally integrable function on $(0, \infty)$ such that there exist the limits $\gamma(0)$ and $\gamma(\infty)$. Then the identity

$$(12) \quad \int_0^\infty (\gamma(t) - \gamma(at)) \frac{dt}{t} = (\gamma(0) - \gamma(\infty)) \log a$$

holds.

We set

$$\gamma(t) = \Phi\left(\frac{1}{(a-1)^p} \|\chi_{M_t} \nabla \varphi\|_{L^{p,q}(\Omega, m_n; \mathbf{R}^n)}^p\right).$$

Using the convexity of Φ and Proposition 2.2, we see that

$$\Phi\left(\frac{1}{(a-1)^p} \|\chi_{M_t \setminus M_{at}} \nabla \varphi\|_{L^{p,q}(\Omega, m_n; \mathbf{R}^n)}^p\right) \leq \Phi(\gamma(t)) - \Phi(\gamma(at)).$$

Since

$$\gamma(0) = \Phi\left(\frac{1}{(a-1)^p} \|\nabla \varphi\|_{L^{p,q}(\Omega, m_n; \mathbf{R}^n)}^p\right) \text{ and } \gamma(\infty) = 0,$$

we get

$$\int_0^\infty \Phi(t^p \text{cap}_{p,q}(\overline{M_{at}}, M_t)) \frac{dt}{t} \leq \log a \cdot \Phi\left(\frac{1}{(a-1)^p} \|\nabla \varphi\|_{L^{p,q}(\Omega, m_n; \mathbf{R}^n)}^p\right).$$

This finishes the proof of the case $1 \leq q < p$.

We assume now that $p < q < \infty$. From (11) we have

$$t^q \text{cap}_{p,q}(\overline{M_{at}}, M_t)^{q/p} \leq \frac{1}{(a-1)^q} \|\chi_{M_t \setminus M_{at}} \nabla \varphi\|_{L^{p,q}(\Omega, m_n; \mathbf{R}^n)}^q.$$

Hence

$$\int_0^\infty \Phi(t^q \text{cap}_{p,q}(\overline{M_{at}}, M_t)^{q/p}) \frac{dt}{t} \leq \int_0^\infty \Phi\left(\frac{1}{(a-1)^q} \|\chi_{M_t \setminus M_{at}} \nabla \varphi\|_{L^{p,q}(\Omega, m_n; \mathbf{R}^n)}^q\right) \frac{dt}{t}.$$

As before, we let γ denote a locally integrable function on $(0, \infty)$ such that there exist the limits $\gamma(0)$ and $\gamma(\infty)$. We set

$$\gamma(t) = \Phi\left(\frac{1}{(a-1)^q} \|\chi_{M_t} \nabla \varphi\|_{L^{p,q}(\Omega, m_n; \mathbf{R}^n)}^q\right).$$

Using the convexity of Φ and Proposition 2.3, we see that

$$\Phi\left(\frac{1}{(a-1)^q} \|\chi_{M_t \setminus M_{at}} \nabla \varphi\|_{L^{p,q}(\Omega, m_n; \mathbf{R}^n)}^q\right) \leq \Phi(\gamma(t)) - \Phi(\gamma(at)).$$

Since

$$\gamma(0) = \Phi\left(\frac{1}{(a-1)^q} \|\nabla \varphi\|_{L^{p,q}(\Omega, m_n; \mathbf{R}^n)}^q\right) \text{ and } \gamma(\infty) = 0,$$

we get

$$\int_0^\infty \Phi(t^q \text{cap}_{p,q}(\overline{M_{at}}, M_t)^{q/p}) \frac{dt}{t} \leq \log a \cdot \Phi\left(\frac{1}{(a-1)^q} \|\nabla \varphi\|_{L^{p,q}(\Omega, m_n; \mathbf{R}^n)}^q\right).$$

This finishes the proof of the case $p < q < \infty$. The theorem is proved. \square

Choosing $\Phi(t) = t$, we arrive at the inequalities mentioned in the beginning of this paper.

Corollary 4.3. *Suppose $1 < p < \infty$ and $1 \leq q < \infty$. Let $a > 1$ be a constant. Then (2) and (3) hold for every $\varphi \in Lip_0(\Omega)$.*

5. Necessary and sufficient conditions for two-weight embeddings

We derive now necessary and sufficient conditions for Sobolev-Lorentz type inequalities involving two measures, generalizing results obtained in [M4] and [M5].

Theorem 5.1. *Let p, q, r, s be chosen such that $1 < p < \infty$, $1 \leq q < \infty$ and $1 < s \leq \max(p, q) \leq r < \infty$. Let Ω be an open set in \mathbf{R}^n and let μ and ν be two nonnegative locally finite measures on Ω .*

(i) *Suppose that $1 \leq q \leq p$. The inequality*

$$(13) \quad \|f\|_{L^{r,p}(\Omega,\mu)} \leq A \left(\|\nabla f\|_{L^{p,q}(\Omega,m_n;\mathbf{R}^n)} + \|f\|_{L^{s,p}(\Omega,\nu)} \right)$$

holds for every $f \in Lip_0(\Omega)$ if and only if there exists a constant $K > 0$ such that the inequality (5) is valid for all open bounded sets g and G that are subject to $\bar{g} \subset G \subset \bar{G} \subset \Omega$.

(ii) *Suppose that $p < q < \infty$. The inequality*

$$(14) \quad \|f\|_{L^{r,q}(\Omega,\mu)} \leq A \left(\|\nabla f\|_{L^{p,q}(\Omega,m_n;\mathbf{R}^n)} + \|f\|_{L^{s,q}(\Omega,\nu)} \right)$$

holds for every $f \in Lip_0(\Omega)$ if and only if there exists a constant $K > 0$ such that the inequality (5) is valid for all open bounded sets g and G that are subject to $\bar{g} \subset G \subset \bar{G} \subset \Omega$.

Proof. We suppose first that $1 \leq q \leq p$. The case $q = p$ was studied in [M5]. Without loss of generality we can assume that $q < p$. We choose some bounded open sets g and G such that $\bar{g} \subset G \subset \bar{G} \subset \Omega$ and $f \in W(\bar{g}, G)$ with $0 \leq f \leq 1$. We have

$$\mu(g) \leq C(r, p) \|f\|_{L^{r,p}(\Omega,\mu)}^r$$

and

$$\|f\|_{L^{s,p}(\Omega,\nu)}^s \leq C(s, p) \nu(G)$$

for every $f \in W(\bar{g}, G)$ with $0 \leq f \leq 1$. The necessity for $1 \leq q < p$ is obtained by taking the infimum over all such functions f that are admissible for the conductor (\bar{g}, G) .

We prove the sufficiency now when $1 \leq q < p$. Let $a \in (1, \infty)$. We have

$$a^p \int_0^\infty \mu(M_{at})^{p/r} d(t^p) \leq a^p K_1 \left(\int_0^\infty (\text{cap}_{p,q}(\overline{M_{at}}, M_t) + \nu(M_t)^{p/s}) d(t^p) \right).$$

This and (2) yield the sufficiency for the case $1 \leq q < p$.

Suppose now that $p < q < \infty$. We choose some bounded open sets g and G such that $\bar{g} \subset G \subset \bar{G} \subset \Omega$ and $f \in W(\bar{g}, G)$ with $0 \leq f \leq 1$. We have

$$\mu(g) \leq C(r, q) \|f\|_{L^{r,q}(\Omega,\mu)}^r$$

and

$$\|f\|_{L^{s,q}(\Omega,\nu)}^s \leq C(s, q) \nu(G)$$

for every $f \in W(\bar{g}, G)$ with $0 \leq f \leq 1$. The necessity for $p < q < \infty$ is obtained by taking the infimum over all such functions f that are admissible for the conductor (\bar{g}, G) .

We prove the sufficiency now when $p < q < \infty$. Let $a \in (1, \infty)$. We have

$$a^q \int_0^\infty \mu(M_{at})^{q/r} d(t^q) \leq a^q K_2 \left(\int_0^\infty (\text{cap}_{p,q}(\overline{M_{at}}, M_t)^{q/p} + \nu(M_t)^{q/s}) d(t^q) \right).$$

This and (3) yield the sufficiency for the case $p < q < \infty$. The proof is finished. \square

We look for a simplified necessary and sufficient two-weight imbedding condition when $n = 1$. Before we state and prove such a condition for the case $n = 1$, we need to obtain sharp estimates for the p, q -capacitance of conductors $([a, b], (A, B))$ with $A < a < b < B$. This is the goal of the following proposition.

Proposition 5.2. *Suppose $n = 1$, $1 < p < \infty$ and $1 \leq q \leq \infty$. There exists a constant $C(p, q) \geq 1$ such that*

$$C(p, q)^{-1}(\sigma_1^{1-p} + \sigma_2^{1-p}) \leq \text{cap}_{p,q}([a, b], (A, B)) \leq C(p, q)(\sigma_1^{1-p} + \sigma_2^{1-p}),$$

where $\sigma_1 = a - A$ and $\sigma_2 = B - b$.

Proof. By the behaviour of the Lorentz p, q -quasinorm in q (see for instance [BS, Proposition IV.4.2]), it suffices to find the upper bound for the $p, 1$ -capacitance and the lower bound for the p, ∞ -capacitance of the conductor $([a, b], (A, B))$. We start with the upper bound for the $p, 1$ -capacitance of this conductor.

We use the function $u : (A, B) \rightarrow \mathbf{R}$ defined by

$$u(x) = \begin{cases} 1 & \text{if } a < x < b \\ \frac{x-A}{\sigma_1} & \text{if } A < x < a \\ \frac{B-x}{\sigma_2} & \text{if } b < x < B. \end{cases}$$

Then from Lemma 3.1 it follows that $u \in W_1([a, b], (A, B))$ with

$$|u'(x)| = \begin{cases} 0 & \text{if } a < x < b \\ \sigma_1^{-1} & \text{if } A < x < a \\ \sigma_2^{-1} & \text{if } b < x < B. \end{cases}$$

We want to compute an upper estimate for $\|u'\|_{L^{p,1}((A,B),m_1)}$. We have

$$(15) \quad \begin{aligned} \|u'\|_{L^{p,1}((A,B),m_1)} &\leq \|\sigma_1^{-1}\|_{L^{p,1}((A,a),m_1)} + \|\sigma_2^{-1}\|_{L^{p,1}((b,B),m_1)} \\ &= p \left(\sigma_1^{-1+1/p} + \sigma_2^{-1+1/p} \right). \end{aligned}$$

Therefore

$$\text{cap}_{p,1}([a, b], (A, B)) \leq C(p)(\sigma_1^{1-p} + \sigma_2^{1-p}).$$

We try to get lower estimates for the p, ∞ -capacitance of this conductor. Let $v \in W([a, b], (A, B))$ be an arbitrary admissible function. We let v_1 be the restriction of v to (A, a) and v_2 be the restriction of v to (b, B) respectively. We notice that v' is supported in $(A, a) \cup (b, B)$. Therefore, since v' coincides with v'_1 on (A, a) and with v'_2 on (b, B) , we have that

$$(16) \quad \|v'\|_{L^{p,\infty}((A,B),m_1)} \geq \max(\|v'_1\|_{L^{p,\infty}((A,a),m_1)}, \|v'_2\|_{L^{p,\infty}((b,B),m_1)}).$$

From ([Cos, Corollary 2.4]) we have

$$\|v'_1\|_{L^{p,\infty}((A,a),m_1)} \geq 1/p' \cdot \sigma_1^{-1/p'} \|v'_1\|_{L^1((A,a),m_1)}.$$

Since

$$\|v'_1\|_{L^1((A,a),m_1)} = \int_A^a |v'_1(x)| dx \geq 1,$$

we obtain

$$(17) \quad \|v'_1\|_{L^{p,\infty}((A,a),m_1)} \geq 1/p' \cdot \sigma_1^{-1/p'}.$$

Similarly, since

$$\|v'_2\|_{L^1((A,a),m_1)} = \int_A^a |v'_2(x)| dx \geq 1,$$

we obtain

$$(18) \quad \|v'_2\|_{L^{p,\infty}((b,B),m_1)} \geq 1/p' \cdot \sigma_2^{-1/p'}.$$

From (16), (17) and (18) we get the desired lower bound for the p, ∞ -capacitance. This finishes the proof. \square

Now we state and prove a necessary and sufficient two-weight imbedding condition for the case $n = 1$.

Theorem 5.3. *Suppose $n = 1$. Let p, q, r, s be chosen such that $1 < p < \infty$, $1 \leq q < \infty$ and $1 < s \leq \max(p, q) \leq r < \infty$. Let Ω be an open set in \mathbf{R} and let μ and ν be two nonnegative locally finite measures on Ω .*

(i) *Suppose that $1 \leq q \leq p$. The inequality*

$$(19) \quad \|f\|_{L^{r,p}(\Omega,\mu)} \leq A (\|f'\|_{L^{p,q}(\Omega,m_1)} + \|f\|_{L^{s,p}(\Omega,\nu)})$$

holds for every $f \in Lip_0(\Omega)$ if and only if there exists a constant $K > 0$ such that the inequality (7) is valid whenever x, d and τ are such that $\overline{\sigma_{d+\tau}(x)} \subset \Omega$.

(ii) *Suppose that $p < q < \infty$. The inequality*

$$(20) \quad \|f\|_{L^{r,q}(\Omega,\mu)} \leq A (\|f'\|_{L^{p,q}(\Omega,m_1)} + \|f\|_{L^{s,q}(\Omega,\nu)})$$

holds for every $f \in Lip_0(\Omega)$ if and only if there exists a constant $K > 0$ such that the inequality (7) is valid whenever x, d and τ are such that $\overline{\sigma_{d+\tau}(x)} \subset \Omega$.

Proof. We only have to prove that the sufficiency condition for intervals implies the sufficiency condition for general bounded and open sets g and G with $\bar{g} \subset G \subset \bar{G} \subset \Omega$. Let G be the union of nonoverlapping intervals G_i and let $g_i = G \cap g_i$. We denote by h_i the smallest interval containing g_i and by τ_i the minimal distance from h_i to $\mathbf{R} \setminus G_i$. We also denote by H_i the open interval concentric with h_i such that the minimal distance from h_i to $\mathbf{R} \setminus H_i$ is τ_i . Then $H_i \subset G_i$. From Remark 3.3 we have that $\text{cap}_{p,q}(\bar{g}_i, G_i) = \text{cap}_{p,q}(\bar{h}_i, G_i)$. Moreover, from Theorem 3.2 (ii) and Proposition 5.2 we have

$$C(p, q)^{-1} \tau_i^{1-p} \leq \text{cap}_{p,q}(\bar{h}_i, G_i) \leq \text{cap}_{p,q}(\bar{h}_i, H_i) \leq 2 C(p, q) \tau_i^{1-p}$$

for some constant $C(p, q) \geq 1$. Since \bar{g} is compact lying in $\cup_{i \geq 1} G_i$, it follows that \bar{g} is covered by only finitely many of the sets G_i . This and Theorem 3.2 (ix) allow us to assume that G is in fact written as a finite union of disjoint intervals G_i . Now the proof splits, depending on whether $1 \leq q \leq p$ or $p < q < \infty$.

We assume first that $1 \leq q \leq p$. We have

$$(21) \quad \text{cap}_{p,q}(\bar{g}, G) \geq \sum_i \text{cap}_{p,q}(\bar{g}_i, G_i) = \sum_i \text{cap}_{p,q}(\bar{h}_i, G_i).$$

Using (7), we obtain

$$\begin{aligned}\mu(g_i)^{p/r} &\leq \mu(h_i)^{p/r} \leq K_1(\tau_i^{1-p} + \nu(H_i)^{p/s}) \\ &\leq K_1 C(p, q)(\text{cap}_{p,q}(\bar{g}_i, G_i) + \nu(G_i)^{p/s})\end{aligned}$$

where K_1 is a positive constant independent of g and G . Since $s \leq p \leq r < \infty$, we have

$$\mu(g)^{p/r} \leq \sum_i \mu(g_i)^{p/r}$$

and

$$\sum_i \nu(G_i)^{p/s} \leq \nu(G)^{p/s}.$$

This and (21) prove the claim when $1 \leq q \leq p$.

We assume now that $p < q < \infty$. We have

$$(22) \quad \text{cap}_{p,q}(\bar{g}, G)^{q/p} \geq \sum_i \text{cap}_{p,q}(\bar{g}_i, G_i)^{q/p} = \sum_i \text{cap}_{p,q}(\bar{h}_i, G_i)^{q/p}.$$

Using (7), we obtain

$$\begin{aligned}\mu(g_i)^{q/r} &\leq \mu(h_i)^{q/r} \leq K_2(\tau_i^{q(1-p)/p} + \nu(H_i)^{q/s}) \\ &\leq K_2 C(p, q)^{q/p}(\text{cap}_{p,q}(\bar{g}_i, G_i)^{q/p} + \nu(G_i)^{q/s})\end{aligned}$$

where K_2 is a positive constant independent of g and G . Since $s \leq q \leq r < \infty$, we have

$$\mu(g)^{q/r} \leq \sum_i \mu(g_i)^{q/r}$$

and

$$\sum_i \nu(G_i)^{q/s} \leq \nu(G)^{q/s}.$$

This and (22) prove the claim when $p < q < \infty$. The theorem is proved. \square

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REFERENCES

- [Ad] D.R. Adams. On the existence of capacity strong type estimates in \mathbb{R}^n . *Ark. Mat.* **14** (1976), 125–140.
- [AH] D.R. Adams, L.I. Hedberg. *Function Spaces and Potential Theory*, Springer Verlag, 1996.
- [AP] D.R. Adams, M. Pierre. Capacity strong type estimates in semilinear problems. *Ann. Inst. Fourier (Grenoble)* **41** (1991), 117–135.
- [AX1] D.R. Adams, J. Xiao. Strong type estimates for homogeneous Besov capacities. *Math. Ann.* **325** (2003), no.4, 695–709.
- [AX2] D.R. Adams, J. Xiao. Nonlinear potential analysis on Morrey spaces and their capacities. *Indiana Univ. Math. J.* **53** (2004), no.6, 1631–1666.
- [Ai] H. Aikawa. Capacity and Hausdorff content of certain enlarged sets. *Mem. Fac. Sci. Eng. Shimane Univ. Series B: Mathematical Science* **30** (1997), 1–21.
- [Alb] A. Alberico. Moser type inequalities for higher-order derivatives in Lorentz spaces. To appear in *Potential Anal.*
- [AFT1] A. Alvino, V. Ferone, G. Trombetti. Moser-type inequalities in Lorentz spaces. *Potential Anal.* **5** (1996), no. 3, 273–299.

- [AFT2] A. Alvino, V. Ferone, G. Trombetti. Estimates for the gradient of solutions of nonlinear elliptic equations with L^1 data. *Ann. Mat. Pura Appl.* **178** (2000), no. 4, 129–142.
- [BBGGPV] P. Bénilan, L. Boccardo, T. Gallouët, R. Gariépy, M. Pierre, J.L. Vázquez. An L^1 -theory of existence and uniqueness of solutions of nonlinear elliptic equations. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **22** (1995), no. 2, 241–273.
- [BS] C. Bennett, R. Sharpley. *Interpolation of Operators*, Academic Press, 1988.
- [CS] Z.-Q. Chen, R. Song. Conditional gauge theorem for non-local Feynman-Kac transforms. *Probab. Theory Related Fields.* **125** (2003), no. 1, 45–72.
- [CHK] H.-M. Chung, R.A. Hunt, and D.S. Kurtz. The Hardy-Littlewood maximal function on $L(p, q)$ spaces with weights. *Indiana Univ. Math. J.* **31** (1982), no. 1, 109–120.
- [Cia] A. Cianchi. Moser-Trudinger inequalities without boundary conditions and isoperimetric problems. *Indiana Univ. Math. J.* **54** (2005), no. 3, 669–705.
- [CP] A. Cianchi and L. Pick. Sobolev embeddings into BMO, VMO, and L_∞ . *Ark. Mat.* **36** (1998), 317–340.
- [Cos] Ş. Costea. Scaling invariant Sobolev-Lorentz capacity on \mathbb{R}^n . *Indiana Univ. Math. J.* **56** (2007), no. 6, 2641–2670.
- [DKX] G. Dafni, G. Karadzhov, J. Xiao. Classes of Carleson-type measures generated by capacities. *Math. Z.* **258** (2008), no.4, 827–844.
- [Dah] B. Dahlberg. Regularity properties of Riesz potentials. *Indiana Univ. Math. J.* **28** (1979), no. 2, 257–268.
- [DHM] G. Dolzmann, N. Hungerbühler, S. Müller. Uniqueness and maximal regularity for non-linear elliptic systems of n -Laplace type with measure valued right hand side. *J. Reine Angew. Math.* **520** (2000), 1–35.
- [Fed] H. Federer. *Geometric Measure Theory*, Springer-Verlag, 1969.
- [Fi] P.J. Fitzsimmons. Hardy’s inequality for Dirichlet forms. *J. Math. Anal. and Appl.* **250** (2000), 548–560.
- [FU1] M. Fukushima, T. Uemura. On Sobolev and capacity inequalities for contractive Besov spaces over d -sets. *Potential Analysis.* **18** (2003), no. 1, 59–77.
- [FU2] M. Fukushima, T. Uemura. Capacity bounds of measures and ultracontractivity of time changed processes. *J. Math. Pures Appl.* **82** (2003), no. 5, 553–572.
- [Gr] A. Grigor’yan. Isoperimetric inequalities and capacities on Riemannian manifolds. *Operator Theory, Advances and Applications*, Vol. 110, The Maz’ya Anniversary Collection, vol. 1, Birkhäuser, 1999, 139–153.
- [Haj] P. Hajlasz. Sobolev inequalities, truncation method, and John domains. *Rep. Univ. Jyväskylä Dep. Math. Stat.*, 83, Jyväskylä, 2001.
- [Han] K. Hansson. Embedding theorems of Sobolev type in potential theory, *Math. Scand.* **45** (1979), 77–102.
- [HMV] K. Hansson, V. Maz’ya, I.E. Verbitsky. Criteria of solvability for multi-dimensional Riccati’s equation. *Arkiv för Matem.* **37** (1999), no. 1, 87–120.
- [HL] S. Hudson, M. Leckband. A sharp exponential inequality for Lorentz-Sobolev spaces on bounded domains. *Proc. Amer. Math. Soc.* **127** (1999), no. 7, 2029–2033.
- [Ka] V. Kaimanovich. Dirichlet norms, capacities and generalized isoperimetric inequalities for Markov operators. *Potential Anal.* **1** (1992), no. 1, 61–82.
- [KKM] J. Kauhanen, P. Koskela, J. Malý. On functions with derivatives in a Lorentz space. *Manuscripta Math.* **100** (1999), no. 1, 87–101.
- [Ko1] T. Kolsrud. Condenser capacities and removable sets in $W^{1,p}$. *Ann. Acad. Sci. Fenn. Ser. A. Math.* **8** (1983), 343–348.
- [Ko2] T. Kolsrud. Capacity integrals in Dirichlet spaces. *Math. Scand.* **55** (1984), 95–120.
- [Mal] J. Malý. Sufficient conditions for change of variables in integral. *Proceedings on Analysis and Geometry (Russian) (Novosibirsk Akademgorodok, 1999)*, Izdat. Ross. Akad. Nauk. Sib. Otdel. Inst. Mat., Novosibirsk, (2000), 370–386.

- [M1] V. G. Maz'ya. On certain integral inequalities for functions of many variables. Problems of Mathematical Analysis, Leningrad Univ. **3** (1972), 33–68 (Russian). English trans. J. Soviet Math. **1** (1973), 205–234.
- [M2] V.G. Maz'ya. Summability with respect to an arbitrary measure of functions from S. L. Sobolev-L. N. Slobodeckii spaces. Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **92** (1979), 192–202.
- [M3] V.G. Maz'ya. Sobolev Spaces, Springer, 1985.
- [M4] V. Maz'ya. Conductor and capacity inequalities for functions on topological spaces and their applications to Sobolev type imbeddings. J. Funct. Anal. **224** (2005), no.2, 408–430.
- [M5] V. Maz'ya. Conductor inequalities and criteria for Sobolev type two-weight imbeddings. J. Comput. Appl. Math. **194** (2006), no. 11, 94–114.
- [MN] V. Maz'ya, Yu. Netrusov. Some counterexamples for the theory of Sobolev spaces on bad domains. Potential Anal. **4** (1995), 47–65.
- [MP] V. Maz'ya, S. Poborchii. Differentiable Functions on Bad Domains. World Scientific, 1997.
- [Ne] Yu.V. Netrusov. Sets of singularities of functions in spaces of Besov and Lizorkin-Triebel type. Trudy Math., Inst. Steklov **187** (1990), 185–203.
- [Ra] M. Rao. Capacity inequalities for energy. Israel J. of Math. **61** (1988), no.1, 179–191.
- [ST] S. Serfaty, I. Tice. Lorentz space estimates for the Ginzburg-Landau energy. J. Funct. Anal. **254** (2008), no. 3, 773–825.
- [Ta] M. Takeda. L^p -independence of the spectral radius of symmetric Markov semigroups. Canadian Math. Soc. Conference Proceedings **29** (2000), 613–623.
- [V1] I.E. Verbitsky. Superlinear equations, potential theory and weighted norm inequalities. Proceedings of the Spring School VI, Prague, May 31-June 6, 1998.
- [V2] I.E. Verbitsky. Nonlinear potentials and trace inequalities. Operator Theory, Advances and Applications, Vol. 110, The Maz'ya Anniversary Collection, vol. 2, Birkhäuser, 1999, 323–343.
- [Vo] Z. Vondraček. An estimate for the L^2 -norm of a quasi continuous function with respect to smooth measure, Arch. Math. **67** (1996), 408–414.

ȘERBAN COSTEA: SECOSTEA@MATH.MCMASTER.CA, MCMMASTER UNIVERSITY, DEPARTMENT OF MATHEMATICS AND STATISTICS, 1280 MAIN STREET WEST, HAMILTON, ONTARIO L8S 4K1, CANADA, AND THE FIELDS INSTITUTE FOR RESEARCH IN MATHEMATICAL SCIENCES, 222 COLLEGE STREET, SECOND FLOOR, TORONTO, ONTARIO, M5T 3J1, CANADA

VLADIMIR MAZ'YA: VLMAZ@MATH.OHIO-STATE.EDU, VLMAZ@LIV.AC.UK, VLMAZ@MAIL.LIU.SE, DEPARTMENT OF MATHEMATICS, THE OHIO STATE UNIVERSITY, 231 W 18TH AVE, COLUMBUS, OH 43210, USA, DEPARTMENT OF MATHEMATICAL SCIENCES, M O BUILDING, UNIVERSITY OF LIVERPOOL, LIVERPOOL L69 3BX, UK, AND DEPARTMENT OF MATHEMATICS, LINKÖPING UNIVERSITY, SE-581 83 LINKÖPING, SWEDEN